DOMAINS OF HOLOMORPHY AND REPRESENTATIONS OF $SL(n, \mathbf{R})$

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Abstract. For $G = SL(n, \mathbf{R})$ and $K = SO(n)$ Akhiezer and Gindikin explicitly determine a G-invariant Stein extension $\mathcal{A}$ of G/K in $G_{\mathbf{C}}/K_{\mathbf{C}}$. We give several other descriptions of $\mathcal{A}$. In terms of the geometry of an arbitrary flag variety for $G_{\mathbf{C}}$, $\mathcal{A}$ is described as the 'polar set' of the closed G-orbit. $\mathcal{A}$ is also the space of 'linear cycles' in an arbitrary open G-orbit. We also see that $\mathcal{A}$ is a domain of holomorphy for Szegö kernels associated to interesting irreducible representations of G.

1. Introduction

A theorem of Grauert states that a real analytic manifold M has a Stein extension $M^{\mathbf{C}}$, that is, M is a totally real submanifold of a Stein manifold $M^{\mathbf{C}}$. If a group G acts on M it is natural to ask for a G-invariant Stein extension. In [1] this question is addressed for $M = G/K$, a Riemanian symmetric space. Given the rich structure and function theory of G/K this is a particularly important example.

For $G = SL(n; \mathbf{R})$ Akhiezer and Gindikin explicitly determine a G-invariant Stein extension $\mathcal{A} \subset G_{\mathbf{C}}/K_{\mathbf{C}}$ (see Def. 2.9 below). In this article we give two different (and in a sense dual) descriptions of $\mathcal{A}$ and explore certain G-invariant spaces of functions on G/K. More precisely, let Z be a flag variety for $G_{\mathbf{C}} = SL(n; \mathbf{C})$. Let D (resp. $\mathcal{O}$) be an open G (resp. $K_{\mathbf{C}} = SO(n; \mathbf{C})$) orbit in Z and Y (resp. X_0) the dual $K_{\mathbf{C}}$ (resp. G) orbit. (See Def. 2.5 for slightly more precise definitions.) We show that $\mathcal{A}$ coincides with the following two open domains in $G_{\mathbf{C}}/K_{\mathbf{C}}$:

$$\mathcal{M}: \text{ connected component of } \{gK_{\mathbf{C}} : gY \subset D\}, \text{ and}$$
$$\widehat{X_0}: \text{ connected component of } \{gK_{\mathbf{C}} : g^{-1}X_0 \subset \mathcal{O}\}.$$

$\mathcal{M}$ is the linear cycle space ([7]), a family of maximal compact complex subvarieties of D. This provides a natural setting for a holomorphic double fibration and corresponding 'Penrose' transform. On the other hand, $\widehat{X_0}$ seems to be closely related to Szegö kernels. This is made explicit in Theorem 5.9 where we consider the Speh representations realized as spaces of smooth sections on G/K via Szegö maps. Our main result is that the Szegö kernels extend holomorphically to $\widehat{X_0}(= \mathcal{A})$, thus providing a realization of the Speh representations in a holomorphic setting. Furthermore, $\widehat{X_0}$ is a domain of holomorphy for the Szegö kernels.

The method of extending representaions is used in [4] to study automorphic forms for $SL(2; \mathbf{R})$. Extensions of Szegö kernels for discrete series representations for $SU(p, q)$ are studied in [2]. The linear cycle space $\mathcal{M}$ is determined in [5] by very different methods.

2. Geometry of the flag variety

Let $G = SL(n; \mathbf{R})$, $K = SO(n)$ and Z an arbitrary flag variety for $G_{\mathbf{C}} = SL(n; \mathbf{C})$. The group $K = SO(n)$ is defined as the group of isometries of $\mathbf{R}^n$ with respect to the standard inner product $(v, w) = \sum_{j=1}^n v_j w_j$, having determinant 1. The complexification of K is $K_{\mathbf{C}} = SO(n; \mathbf{C})$, the isometry group of the nondegenerate symmetric form $(\ ,\ )$ on $\mathbf{C}^n$ which is given by the same formula. We will also assume that $n \geq 2$. Later we will restrict to the case of n an even integer, however the results of this section hold for any $n \geq 3$.

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The flag varieties for $G_{\mathbf{C}}$ may be described as follows. Let $\tilde{m} = (m_1, \dots, m_k)$ with $m_j \in \mathbf{Z}$ and $0 < m_1 < m_2 < \cdots < m_k < n$ and set

$$Z_{\tilde{m}} = \{(z_1, \dots, z_k) : z_1 \subset \cdots \subset z_k \subset \mathbf{C}^n \text{ and } \dim(z_j) = m_j, \text{ for all } j = 1, \dots, k\}.$$

Definition 2.1. For $0 < l, m < n$ set

$$d(l,m) = \begin{cases} 0, & \text{if } l+m \leq n, \\ l+m-n, & \text{if } l+m > n. \end{cases} \tag{2.2}$$

and

$$e(l,m) = \begin{cases} l, & \text{if } l+m \leq n, \\ n-m, & \text{if } l+m > n. \end{cases} \tag{2.3}$$

The following is straightforward.

Lemma 2.4. *Suppose $w, z \subset \mathbf{C}^n$.*

(a) $d(l,m) = \min\{\dim(w \cap \overline{z}) : \dim(w) = l$ *and* $\dim(z) = m\}$,
(b) $e(l,m) = \max\{\dim(w \cap z^{\perp}) : \dim(w) = l$ *and* $\dim(z) = m\}$[1],
(c) $d(l,m) + e(l,m) = l$.

Definition 2.5. We define the following subsets of $Z_{\tilde{m}}$:

$$\begin{aligned}
D &= \{z \in Z_{\tilde{m}} : \dim(z_i \cap \overline{z_j}) = d(m_i, m_j), \text{ all } i,j\}, \text{ (the maximally complex flags)}, \\
X_0 &= \{z \in Z_{\tilde{m}} : z_j = \overline{z_j}, \text{ all } j\}, \text{ (the real flags)}, \\
Y &= \{z \in Z_{\tilde{m}} : \dim(z_i \cap z_j^{\perp}) = e(m_i, m_j), \text{ all } i,j\}, \text{ (the maximally isotropic flags)}, \\
\mathcal{O} &= \{z \in Z_{\tilde{m}} : z_j \cap z_j^{\perp} = \{0\}, \text{ all } j\}, \text{ (the nondegenerate flags)}.
\end{aligned}$$

The following proposition is well known and easily verified.

Proposition 2.6. *With $\tilde{m} = (m_1, \dots, m_k)$ as above, $Z_{\tilde{m}}$ is a flag variety for $G_{\mathbf{C}}$ and all flag varieties are equivalent to some $Z_{\tilde{m}}$.*

(a) *If $m_j \neq \frac{n}{2}$ for all j then, D is the unique open G-orbit in $Z_{\tilde{m}}$. If $m_j = \frac{n}{2}$ for some j then, D splits into two orbits, which we call D_+ and D_-.*
(b) *The unique closed G-orbit is X_0.*
(c) *If $m_j \neq \frac{n}{2}$ for all j then, Y is the unique closed $K_{\mathbf{C}}$-orbit in $Z_{\tilde{m}}$. If $m_j = \frac{n}{2}$ for some j then, Y splits into two orbits, which we call Y_+ and Y_-.*
(d) *The unique open $K_{\mathbf{C}}$-orbit is $\mathcal{O}$.*

The affine space $G_{\mathbf{C}}/K_{\mathbf{C}}$ may be described as the space of all unimodular symmetric bilinear forms on $\mathbf{C}^n$. The action of G is given by $(g \cdot b)(v,w) = b(g^{-1}v, g^{-1}w)$ and the stabilizer of $(\ ,\ )$ is $K_{\mathbf{C}}$. We write b_g for $g \cdot b$. Note that if we choose the standard basis of $\mathbf{C}^n$, then $G_{\mathbf{C}}/K_{\mathbf{C}}$ is identified with the space of complex symmetric $n \times n$ matrices of determinant one. In particular b_g is identified with ${g^t}^{-1} g^{-1}$.

Definition 2.7. For each $Z_{\tilde{m}}$ define subsets of $G_{\mathbf{C}}/K_{\mathbf{C}}$:

(a) $\mathcal{M}$ is the connected component containing eK_C of $\mathcal{M}' = \{gK_{\mathbf{C}} : gY \subset D\}$.
(b) $\widehat{X_0}$ is the connected component containing eK_C of $\widehat{X_0}' = \{gK_{\mathbf{C}} : g^{-1}X_0 \subset \mathcal{O}\}$.

Note that since $Y \subset D$ and $X_0 \subset \mathcal{O}$ the symmetric space $G/K = G \cdot eK_{\mathbf{C}}$ is contained in $\mathcal{M}, \mathcal{M}', \widehat{X_0}$ and $\widehat{X_0}'$. We will see that these four sets are open in $G_{\mathbf{C}}/K_{\mathbf{C}}$, so they are complex extensions of G/K.

There is a (real) parabolic subgroup $P = MAN$ of G so that $X_0 = G/P = K/K \cap M$ and $Z_{\tilde{m}} = G_{\mathbf{C}}/P_{\mathbf{C}}$. From the definition of $\widehat{X_0}$ we have the following lemma.

Lemma 2.8. *$gK_{\mathbf{C}} \in \widehat{X_0}$ if and only if $g^{-1}k \in K_{\mathbf{C}} M_{\mathbf{C}} A_{\mathbf{C}} N_{\mathbf{C}}$, for all $k \in K$.*

[1] $z^{\perp}$ is the subspace of $\mathbf{C}^n$ orthogonal to z with respect to $(\ ,\ )$.

In this case we say that $g^{-1}k$ has a *complex Iwasawa decomposition.* Unlike the Iwasawa decomposition for a real group, not every element of $G_{\mathbf{C}}$ has a complex Iwasawa decomposition, and for those that do the uniqueness which holds in the real case fails.

For the remainder of this section we will determine the structure of $\mathcal{M}$ and $\widehat{X_0}$ for arbitrary flag varieties. We will see that $\mathcal{M}$ and $\widehat{X_0}$ are open and Stein in $G_{\mathbf{C}}/K_{\mathbf{C}}$, are independent of the flag variety for $G_{\mathbf{C}}$ and are equal to each other.

Definition 2.9. Let $\mathcal{A}' \subset G_{\mathbf{C}}/K_{\mathbf{C}}$ be the space of unimodular symmetric forms having no isotropic vector in $\mathbf{R}^n$. Let $\mathcal{A}$ be the connected component containing the form $(\quad,\quad)$.

Remark 2.10. In [1], for an arbitrary simple Lie group G, a complex extension of G/K in $G_{\mathbf{C}}/K_{\mathbf{C}}$ is defined and conjectured to be Stein. This set is defined as the maximal G-invariant domain in $G_{\mathbf{C}}/K_{\mathbf{C}}$ containing G/K on which the action of G is proper. They show that for $G = SL(n;\mathbf{R}), n \geq 3$, it coincides with $\mathcal{A}$.

Theorem 2.11. *For any flag variety $Z_{\tilde{m}}$, $\widehat{X_0} = \mathcal{A}$.*

Proof. It suffices to show that $\widehat{X_0}' = \mathcal{A}'$. Suppose that $gK_{\mathbf{C}} \notin \widehat{X_0}'$, i.e., b_g is degenerate on some real subspace of dimension m_j (some j). Then b_g has a real isotropic vector.

Now suppose that b_g has a real isotropic vector v_0. The subspace of $\mathbf{R}^n$ perpendicular to v_0 with respect to both the real and imaginary parts of b_g has (real) dimension at least $n-2$. Thus for each $m = 1, 2, \dots, n-1$ there is a subspace $E \subset \mathbf{R}^n$ of dimension m containing v_0 and so that $b_g(v_0, v) = 0$ for all $v \in E$. Applying this to $m = m_k$ and setting $z_k = E_{\mathbf{C}}$, we see that z_k is degenerate for b_g. Now z_k belongs to some real flag $z = (z_1 \subset \cdots \subset z_k)$. However, $g^{-1}z \notin \mathcal{O}$. □

Theorem 2.12. *For any flag variety $Z_{\tilde{m}}$, $\mathcal{M} = \mathcal{A}$.*

Proof. It suffices to prove that $\mathcal{M}' = \mathcal{A}'$. In order to prove this for an arbitrary flag variety it is simpler to consider separately the full flag variety ($\tilde{m} = (m_1, \dots, m_{n-1})$) and the flag variety corresponding to maximal parabolic subgroups ($\tilde{m} = (m)$). Denote (temporarily) the two $\mathcal{M}$'s by $\mathcal{M}'_{\text{full}}$ and $\mathcal{M}'_{\text{max}}$. Since there is a fibration of the full flag variety over any other and a fibration of any flag variety over one of the flag varieties corresponding to the maximal parabolic the $\mathcal{M}$ for an arbitrary flag variety Z satisfies $\mathcal{M}'_{\text{full}} \subset \mathcal{M}' \subset \mathcal{M}'_{\text{max}}$. It therefore suffices to show that $\mathcal{A}' \subset \mathcal{M}'_{\text{full}}$ and $\mathcal{M}'_{\text{max}} \subset \mathcal{A}'$.

We first show $\mathcal{A}' \subset \mathcal{M}'_{\text{full}}$. Suppose that $gK_{\mathbf{C}} \notin \mathcal{M}'_{\text{full}}$, so $gY \not\subset D$. There is a flag $z = (z_j) \in Y$ so that for some i_0, j_0, $\dim(gz_{i_0} \cap \overline{gz_{j_0}}) > d(i_0, j_0)$. We may assume $i_0 < j_0$, i.e., $z_{i_0} \subset z_{j_0}$. Now z is maximally isotropic means that gz is maximally isotropic for b_g, i.e., $\dim(gz_{i_0} \cap (gz_{j_0})^{\perp_g}) = e(i_0, j_0)$, where $\perp_g$ refers to orthogonality with respect to b_g. Therefore, by part (c) of Lemma 2.4, $gz_{i_0} \cap \overline{gz_{j_0}}$ meets $gz_{i_0} \cap (gz_{j_0})^{\perp_g}$ in at least one dimension. We have $\{0\} \neq gz_{i_0} \cap \overline{gz_{j_0}} \cap (gz_{j_0})^{\perp_g} \subset gz_{j_0} \cap \overline{gz_{j_0}} \cap (gz_{j_0})^{\perp_g}$, so b_g has a real isotropic vector.

Now consider $Z_{(m)}$, the Grassmannian of all m-planes in $\mathbf{C}^n$. Suppose $b_g \notin \mathcal{A}'$. Let $v_0 \in \mathbf{R}^n$ be a real isotropic vector for b_g. There is a $w_0 \in \mathbf{C}^n$ so that $b_g(v_0, w_0) = 1$. Set $U = (\text{span}\{v_0, w_0\})^{\perp}$. Then b_g is nondegenerate on U and we may choose an $m-1$ dimensional subspace $u \subset U$ which is maximally isotropic. Set $z = g^{-1}(\mathbf{C}v_0 + u)$, a maximally isotropic m-dimensional subspace. Now

$$\dim(gz \cap \overline{gz}) = \dim((\mathbf{C}v_0 + u) \cap (\mathbf{C}v_0 + \overline{u})) \geq 1 + d(m, m).$$

So $gz \not\subset D$, i.e., $gK_{\mathbf{C}} \notin \mathcal{M}'_{\text{max}}$. □

Corollary 2.13. $\widehat{X_0} = \mathcal{M} = \mathcal{A}$ *is a Stein extension of* $G_{\mathbf{C}}/K_{\mathbf{C}}$.

Proof. $\mathcal{A}'$ is an affine space with the hyperplanes $\mathcal{H}_v = \{b_g : b_g(v, v) = 0\}$, $v \in \mathbf{R}^n$, removed. Therefore, as noted in [1], $\mathcal{A}'$ and its connected component $\mathcal{A}$ are Stein. □

As we have defined $\mathcal{M}$ in Definition 2.5, $\mathcal{M}$ is the linear cycle space (see [7] for the definition) in the cases where D is the unique open G-orbit in $Z_{\tilde{m}}$, i.e., when $m_j \neq \frac{n}{2}$ for all j. The following proposition shows that in fact $\mathcal{M}$ is the linear cycle space for all open orbits in all $Z_{\tilde{m}}$. The right-hand side of equation 2.15 is the definition of the linear cycle space.

Proposition 2.14. *Let $n \geq 4$. Suppose $m_j = \frac{n}{2}$ for some j. Let $D_\pm$ be the two open G-orbits and $Y_\pm$ the two closed $K_{\mathbf{C}}$-orbits in $Z_{\tilde{m}}$ as in 2.6. Then*

$$\begin{aligned} &\mathcal{M}' = \mathcal{M}'_\pm \equiv \{gK_{\mathbf{C}} : gY_\pm \subset D_\pm\}, \text{ and,} \\ &\mathcal{M} = \mathcal{M}_\pm \equiv \text{ the connected component of } \mathcal{M}_\pm \text{ containing } eK_{\mathbf{C}}. \end{aligned} \tag{2.15}$$

Proof. Write $Y_\pm = K_{\mathbf{C}}(z_\pm) \subset D_\pm = G(z_\pm)$. The action of G on Z extends to an action of $G' \equiv GL(n;\mathbf{R})$ which is transitive on D. Since D is not connected $\text{stab}_{G'}(z_\pm) \subset G$. In fact (for properly chosen $z_\pm$) there is $w_0 \in O(n)$ so that $z_+ = w_0 \cdot z_-$. If $gY \subset D$ then either $gz_+ \in D_+$ or $gz_+ \in D_-$. In the first case, by the connectedness of Y_+ and the disconnectedness of D, $gY_+ \subset D_+$. In the second case $gY_+ \subset D_-$. However $gY_+ \subset D_-$ cannot happen since $gz_+ = g'w_0z_+$ for some $g' \in G$, i.e., $g^{-1}g'w_0 \notin G$. Therefore $\mathcal{M}' \subset \mathcal{M}'_\pm$.

As in the proof of Theorem 2.12, $\mathcal{M}'_\pm \subset \mathcal{M}'$ will follow from $\mathcal{M}'_{\max,\pm} \subset \mathcal{M}'(=\mathcal{A}')$. Thus we may assume our flag variety is $Z_{(n)}$, the Grassmannian of n planes in $\mathbf{C}^{2n}$. For $gK_{\mathbf{C}} \notin \mathcal{A}'$ there is a real vector v_0 so that $b_g(v_0, v_0) = 0$. One constructs z as in the proof of Theorem 2.12 above, however since $n \geq 2$ we may choose z to be in either of $Y_\pm$. Now $gY_\pm \not\subset D$, $gY_\pm \not\subset D_\pm$. □

3. Parameters for the Speh representations

In this section we set $G = SL(2n;\mathbf{R})$ and describe parameters for the Speh representations of G. It is slightly more convenient to first describe the parameters for certain representations of $G' = GL^+(2n;\mathbf{R})$, the group of invertible linear transformations with positive determinant. The Speh representations will be the restrictions to G.

The maximal compact subgroup of G' is $K' = SO(2n)$. The Lie algebra $\mathfrak{g}'$ contains a fundamental Cartan subalgebra $\mathfrak{t}_0 + \mathfrak{a}'_0$ having blocks

$$\begin{pmatrix} a_j & \theta_j \\ -\theta_j & a_j \end{pmatrix}$$

down the diagonal. Setting

$$e_j \begin{pmatrix} a_1 & \theta_1 & & & \\ -\theta_1 & a_1 & & & \\ & & \ddots & & \\ & & & a_n & \theta_n \\ & & & -\theta_n & a_n \end{pmatrix} = \sqrt{-1}\theta_j \text{ and } f_j \begin{pmatrix} a_1 & \theta_1 & & & \\ -\theta_1 & a_1 & & & \\ & & \ddots & & \\ & & & a_n & \theta_n \\ & & & -\theta_n & a_n \end{pmatrix} = a_j$$

the roots are

$$\Delta(\mathfrak{t} + \mathfrak{a}', \mathfrak{g}') = \{\pm(e_j \pm e_k) \pm (f_j - f_k) : j \neq k\} \cup \{\pm 2e_j\}.$$

A θ-stable parabolic subgroup $Q = HU$ is defined by $\lambda_0 \equiv \sum_{j=1}^n e_j$;

$$\begin{aligned} &\mathfrak{q} = \mathfrak{h} + \mathfrak{u}, \\ &\Delta(\mathfrak{h}, \mathfrak{t} + \mathfrak{a}') = \{\alpha : \langle \alpha, \lambda_0 \rangle = 0\} \text{ and,} \\ &\Delta(\mathfrak{q}, \mathfrak{t} + \mathfrak{a}') = \{\alpha : \langle \alpha, \lambda_0 \rangle > 0\}. \end{aligned}$$

In the terminology of [6], there is a family of representations $A_{\mathfrak{q}}(m\lambda_0)$, $m \in \mathbf{Z}$. We let π_m be the restrictions of the $A_{\mathfrak{q}}(m\lambda_0)$, $m \in \mathbf{Z}$, to $\mathfrak{g}$. Then for $m \geq -n$, π_m is an irreducible, unitarizable representation with lowest K-type E of highest weight $\mu = (m + n + 1)\lambda_0$. See, for example, [6], pages 586-8.

The Langlands parameters of the π_m may be determined as on pages 764-5 of [6]. Write $\mathfrak{a}_0 = \mathfrak{a}_0' \cap \mathfrak{g}$. A real parabolic subgroup $P = MAN$ of G is determined by

$$A = \exp(\mathfrak{a}_0), MA = Z_G(\mathfrak{a}) \text{ and } \Sigma(\mathfrak{n}_0, \mathfrak{a}_0) = \{f_j - f_k : j < k\}.$$

Therefore, MA consists of 2 $times$2 blocks down the diagonal and we may take

$$M = \{g \in (SL^{\pm}(2; \mathbf{R}))^n : \det(g) = 1\} \text{ and,}$$
$$M_e = (SL(2; \mathbf{R}))^n \text{ (the identity component).}$$

Let

$$\nu = \sum_{j=1}^{n} (n - 2j + 1) f_j \in \mathfrak{a}^*.$$

Write χ_m for the character of $(SO(2))^n = M_e \cap K$ with differential $(m+n+1)\lambda_0$ and let δ_{M_e} be the discrete series representation of M_e with minimal $M_e \cap K$-type χ_m. Let

$$\delta_M = \mathrm{Ind}_{M_e}^{M}(\delta_{M_e}).$$

Then π_m occurs as the unique irreducible quotient of the normalized principal series representation

$$\mathrm{Ind}_P^G(\delta_M \otimes \nu) = \{\psi : G \to W : \psi \text{ is smooth and}$$
$$\psi(gman) = \mathrm{e}^{-(\nu+\rho)}(a)\delta_M(m^{-1})\psi(g), \text{ for } man \in MAN, \ g \in G\}.$$

In Section 4 we will use the fact that δ_{M_e} may be realized on a space of smooth sections of the homogeneous vector bundle on $M_e/M_e \cap K$ corresponding to χ_m.

4. The Szegö map

We begin this section with an arbitrary connected semisimple Lie group G and a parabolic subgroup $P = MAN$. We choose a Cartan involution θ, giving us a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ of the Lie algebra of G.

Let (δ_M, W) be a representation of M and $\nu \in \mathfrak{a}^*$. Suppose that (τ, E) is a K-type of a normalized principal series representation $\mathrm{Ind}_P^G(\delta_M \otimes \nu)$. Let $\mathcal{E} \to G/K$ be the homogeneous vector bundle corresponding to E and $C^\infty(G/K, \mathcal{E})$ the space of smooth sections. We construct non-zero G-intertwining maps

$$S : \mathrm{Ind}_P^G(\delta_M \otimes \nu) \to C^\infty(G/K, \mathcal{E}) \tag{4.1}$$

as follows. Since $\mathrm{Ind}_P^G(\delta_M \otimes \nu)|_K \cong \oplus_{\mu \in \widehat{K}} E_\mu \otimes \mathrm{Hom}_{M \cap K}(E_\mu, W)$, E is a K-type if and only if there exists a non-zero $T \in \mathrm{Hom}_{M \cap K}(E_\mu, W)$. Choosing such a T, the adjoint T^* gives an intertwining operator S as in (4.1) defined by

$$(S\psi)(g) = \int_K \tau(k) T^*(\psi(gk)) dk. \tag{4.2}$$

This may be rewritten in terms of a kernel operator by using the Iwasawa decomposition and a standard integration formula. The Iwasawa decomposition with respect to $P = MAN$ is the smooth (unique) decomposition

$$g = \kappa(g) m(g) \exp(H(g)) n(g) \tag{4.3}$$

where $\kappa(g) \in K, m(g) \in \exp(\mathfrak{m} \cap \mathfrak{s}), H(g) \in \mathfrak{a}$ and $n(g) \in N$. The integration formula for the change of variables $k \to \kappa(g^{-1}k)$ gives

$$(S\psi)(g) = \int_K \mathrm{e}^{(\nu-\rho)H(g^{-1}k)} \tau(\kappa(g^{-1}k)) T^*(m(g^{-1}k)\psi(k)) dk. \tag{4.4}$$

Therefore S is defined by integrating against a kernel:

$$s : G \times K \to \mathrm{Hom}_{M\cap K}(W, E)$$
$$s(g,k)(w) = \mathrm{e}^{(\nu-\rho)H(g^{-1}k)}\tau(\kappa(g^{-1}k))T^*(m(g^{-1}k)w).$$

We call S a *Szegö map* and s the *Szegö kernel.*

Remark 4.5. $s(\quad,k)$ is a section of $\mathcal{E} \to G/K$. We will want to extend $s(\quad,k)$ holomorphically to a domain in $G_{\mathbf{C}}/K_{\mathbf{C}}$. For this we note that E extends to a holomorphic representation of $K_{\mathbf{C}}$ defining a holomorphic homogeneous vector bundle $\mathcal{E}^{\mathbf{C}} \to G_{\mathbf{C}}/K_{\mathbf{C}}$. Thus, more precisely, we want to extend sections in $C^\infty(G/K,\mathcal{E})$ to holomorphic sections of $\mathcal{E}^{\mathbf{C}} \to G_{\mathbf{C}}/K_{\mathbf{C}}$ defined on some domain in $G_{\mathbf{C}}/K_{\mathbf{C}}$.

A useful variant of the above construction is as follows. We note that if $\delta_M = \mathrm{Ind}_{M_e}^M(\delta_{M_e})$, as is the case for the representations in Section 3 (and nearly the case for all Langlands parameters) then the Szegö maps may be defined as follows. For $T \in \mathrm{Hom}_{M_e\cap K}(\delta_{M_e}, E)$

$$S : \mathrm{Ind}_{M_eAN}^G(\delta_{M_e}\otimes\nu) \to C^\infty(G/K,\mathcal{E})$$
$$(S\psi)(g) = \int_K \mathrm{e}^{(\nu-\rho)H(g^{-1}k)}\tau(\kappa(g^{-1}k))T^*(m(g^{-1}k)\psi(k))dk.$$

For us this will be useful since the realization of δ_{M_e} is slightly simpler than that of δ_M.

Specialize to $G = SL(2n,\mathbf{R})$ and let π_m, δ_M, ν, etc. be as in Section 3. We assume $m \geq -n$.

Proposition 4.6. *The image of S is an irreducible subrepresentation of $C^\infty(G/K,\mathcal{E})$.*

Proof. This is a consequence of several general facts contained in [3]. First, π_m occurs as a representation in Dolbeault cohomology; π_m is infinitesimally equivalent to $H^s(G/H,\mathcal{L}_\lambda^\sharp)$. There is an intertwining map

$$\mathcal{S} : \mathrm{Ind}_P^G(\delta_M\otimes\nu) \to H^s(G/H,\mathcal{L}_\lambda^\sharp).$$

Furthermore, there is a 'real Penrose transform' P from cohomology to $C^\infty(G/K,\mathcal{E})$ so that $P\cdot\mathcal{S} = S$, the Szegö map defined above. We remark that the conditions necessary here are precisely the conditions on [6], page 764. These hold exactly for $m \geq -n$, as we are assuming. Now the irreducibility of the image of S follows from the irreducibility of $H^s(G/H,\mathcal{L}_\lambda^\sharp)$. $\square$

Proposition 4.7. *If m is sufficiently large then for each $\psi \in Ind_P^G(\delta_M\otimes\nu)$, $S\psi$ extends to a holomorphic section of $\mathcal{E}^{\mathbf{C}} \to \mathcal{M}$.*

Proof. In light of the above proof $S\psi = P(\mathcal{S}\psi)$. In [8] a 'complex Penrose transform'

$$P^{\mathbf{C}} : H^s(G/H,\mathcal{L}_\lambda^\sharp) \to \mathrm{Hol}(\mathcal{M},\mathcal{E}^{\mathbf{C}})$$

is studied. The construction of $P^{\mathbf{C}}$ is in terms of the linear cycle space $\mathcal{M}$. (We remark that G/H is D or $D_\pm$.) If $r : \mathrm{Hol}(\mathcal{M},\mathcal{E}^{\mathbf{C}}) \to C^\infty(G/K,\mathcal{E})$ is the restriction of holomorphic sections to G/K then $r\cdot P^{\mathbf{C}} = P$. In particular $S\psi = r(P^{\mathbf{C}}\mathcal{S}\psi)$, the restriction of the holomorphic section $P^{\mathbf{C}}\mathcal{S}\psi$ from $\mathcal{M}$ to G/K. $\square$

The following section strengthens this proposition considerably. The condition that m be sufficiently large is replaced by $m \geq -n$. More importantly, it is seen that the Szegö kernel extends holomorphically to $\widehat{X_0}(=\mathcal{M})$. In fact, the Szegö kernel is singular on the boundary of $\widehat{X_0}$.

5. Holomorphic extension of the Szegö Kernel

As a first step in the proof of Theorem 5.9 we will give an explicit formula for the Szegö kernel for discrete series representations of $G_1 = SL(2;\mathbf{R})$. We take the upper triangular parabolic subgroup $P_1 = M_1A_1N_1$ with

$$M_1 = \{\pm I\}, A_1 = \{\exp\begin{pmatrix} a & 0 \\ 0 & -a\end{pmatrix}\} \text{ and } N_1 = \{\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\}.$$

The maximal compact subgroup is

$$K_1 = \{k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\}.$$

Let $\chi_d(k_\theta) = e^{id\theta}$. We consider the discrete series representation of G_1 with minimal K_1-type χ_d, for $d \geq 2, d \in \mathbf{Z}$. Let sgn be the sign representation of M_1, α_1 the root of $\mathfrak{a}_1$ in $\mathfrak{n}_1$ and $\epsilon_1 = \frac{1}{2}\alpha_1$. Then $\mathrm{Ind}_{P_1}^{G_1}(sgn^d \otimes (-d+1)\epsilon_1)$ contains the discrete series representation as a quotient and

$$S_1 : \mathrm{Ind}_{P_1}^{G_1}(sgn^d \otimes (-d+1)\epsilon_1) \to C^\infty(G_1/K_1, \chi_d)$$

is given by

$$(S_1 f)(g) = \int_{SO(2)} e^{-d\epsilon_1 H(g^{-1}k)} \chi_d(\kappa(g^{-1}k)) f(k) dk. \tag{5.1}$$

Letting $\widehat{X_0^1}$ be the domain $\widehat{X_0}$ of Def. 2.7 for $G = G_1 = SL(2;\mathbf{R})$ we have the following fact.

Lemma 5.2. *The Szegö kernel of S_1 in formula (5.1) extends holomorphically in g to $\widehat{X_0^1}$.*

Proof. Consider the standard representation of G_1 on $\mathbf{C}^2$. Write

$$v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ the highest weight vector and}$$

$$v_0 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \text{ a vector of } SO(2) \text{ weight } e^{i\theta}.$$

Therefore, for the $K_{1,\mathbf{C}}$-invariant symmetric form $(\ ,\ )$ on $\mathbf{C}^2$

$$e^{d\epsilon_1 H(g^{-1}k)} \chi_d(\kappa(g^{-1}k)) = (g^{-1}kv_+, v_0)^d.$$

Note that this is a holomorphic function of g. Furthermore,

$$e^{2d\epsilon_1 H(g^{-1}k)} = (g^{-1}kv_+, g^{-1}kv_+)^d,$$

a holomorphic function of g. Thus, the Szegö kernel

$$s_1(g,k) = \left(\frac{(g^{-1}kv_+, v_0)}{(g^{-1}kv_+, g^{-1}kv_+)} \right)^d$$

extends holomorphically on any set where the denominator is non-zero. For $g \in G_{1,\mathbf{C}}$ this denominator is non-zero for each $k \in K_1$ if and only if $gK_{1,\mathbf{C}} \in \widehat{X_0^1}$, by Theorem 2.11. □

We now return to $G = SL(2n;\mathbf{R})$ and let $P = MAN, \delta_M, \delta_{M_e}, \nu, \pi_m$ and E as in Section 3. Thus, E is the minimal K-type of π_m and has highest weight $(m+n+1)\lambda_0 \in \mathfrak{t}^*$. We identify E with the K-subrepresentation of $F = Sym^{m+n+1}(\wedge^n \mathbf{C}^{2n})$ generated by

$$\phi = (v_+)^{m+n+1}, v_+ = (e_1 + ie_2) \wedge \cdots \wedge (e_{2n-1} + ie_{2n}),$$

($\{e_j\}$ the standard basis of $\mathbf{C}^{2n}$). Thus, it makes sense to write

$$s(g,k)(w) = g^{-1}\left(e^{(\nu-\rho)H(g^{-1}k)} g\kappa(g^{-1}k) T^*(m(g^{-1}k)w)\right) \tag{5.3}$$

for $w \in W, g \in G$ and $k \in K$.

Since $M_e = (SL(2;\mathbf{R}))^n$, the discrete series representations δ_{M_e} may be realized as smooth sections on $M_e/M_e \cap K$:

$$W \subset \{w : M_e \to \mathbf{C} \cdot \phi \,|\, w(mk) = k^{-1}w(m), \text{ for } k \in M_e \cap K\}.$$

For the Szegö kernel we must specify an $M_e \cap K$ homomorphism $T^* : W \to E$. Set

$$T^*(w) = w(e) \in \mathbf{C} \cdot \phi \subset E.$$

This allows the following form of the Szegö kernel.

Lemma 5.4. *For $g \in G, k \in K$ and $w \in W$,*

$$s(g,k)(w) = e^{(\nu-\rho)H(g^{-1}k)} g^{-1}kn(g^{-1}k)^{-1}m(g^{-1}k)^{-1}w(m(g^{-1}k)^{-1}).$$

Proof. It follows from (4.3) that

$$\kappa(g^{-1}k) = g^{-1}kn(g^{-1}k)^{-1}m(g^{-1}k)^{-1}\exp(H(g^{-1}k)).$$

So by (5.3)

$$\begin{aligned} s(g,k)(w) &= \mathrm{e}^{(\nu-\rho)H(g^{-1}k)} g^{-1}kn(g^{-1}k)^{-1}m(g^{-1}k)^{-1}\mathrm{e}^{-H(g^{-1}k)}(m(g^{-1}k)\cdot w)(e) \\ &= \mathrm{e}^{(\nu-\rho)H(g^{-1}k)} g^{-1}kn(g^{-1}k)^{-1}m(g^{-1}k)^{-1}w(m(g^{-1}k)^{-1}). \end{aligned}$$

Since $\exp(-H(g^{-1}k))\phi = \mathrm{e}^{-(m+n+1)(\sum_{j=1}^{n} f_j)(H(g^{-1}k))} \cdot \phi = \phi$, as $\sum f_j = 0$ on $\mathfrak{sl}(2n;\mathbf{R})$. □

Lemma 5.5. *As a function of $gK_{\mathbf{C}}$, $m(g^{-1}k)^{-1}w(m(g^{-1}k)^{-1})$ extends holomorphically on $\widehat{X_0}$, for each $k \in K$ and $w \in W$.*

Proof. By (4.3) $m(k'g') = m(g')$ for all $g' \in G$ and $k' \in K$. Therefore $m(g^{-1}k)^{-1}w(m(g^{-1}k)^{-1})$ is well defined on G/K. Since $mw(m)$ is a function on $M_0/M_0 \cap K_0$, we need to check two things:

(a) If $gK_{\mathbf{C}} \in \widehat{X_0}$ then $m(g^{-1}k)^{-1} \in \left(\widehat{X_0^1}\right)^n$ and,
(b) $gK_{\mathbf{C}} \to m(g^{-1}k)^{-1}(M_{\mathbf{C}} \cap K_{\mathbf{C}})$ is a holomorphic map from $G_{\mathbf{C}}/K_{\mathbf{C}} \to M_{\mathbf{C}}/M_{\mathbf{C}} \cap K_{\mathbf{C}}$.

Consider $gK_{\mathbf{C}} \in \widehat{X_0}$. By Theorem 2.11, $\widehat{X_0}$ is independent of the flag variety Z. We compare $\widehat{X_0}$ for the flag varieties $Z = G_{\mathbf{C}}/P_{\mathbf{C}}$ and $G_{\mathbf{C}}/B_{\mathbf{C}}$, $B_{\mathbf{C}}$ the Borel subgroup of upper triangular matrices. Thus $g^{-1}k \in K_{\mathbf{C}}B_{\mathbf{C}}$ for all $k \in K$.

To show (a) it suffices to show $m(g^{-1}k)k_1 \in (M_{\mathbf{C}} \cap K_{\mathbf{C}})(B_{\mathbf{C}} \cap M_{\mathbf{C}})$, for all $k_1 \in M \cap K$.
Since k_1 normalizes $K_{\mathbf{C}}, M_{\mathbf{C}}, A_{\mathbf{C}}$ and $N_{\mathbf{C}}$, $m(g^{-1}k)k_1 = k_1 m(k_1^{-1}g^{-1}kk_1)$. It is clear that $g^{-1}k \in K_{\mathbf{C}}P_{\mathbf{C}}$ exactly when $k_1^{-1}g^{-1}kk_1 \in K_{\mathbf{C}}P_{\mathbf{C}}$, therefore it is enough to show that for $x \in K_{\mathbf{C}}P_{\mathbf{C}}$, $m(x) \in (M_{\mathbf{C}} \cap K_{\mathbf{C}})(B_{\mathbf{C}} \cap M_{\mathbf{C}})$. Write

$$\begin{aligned} x &= k' \begin{pmatrix} m_1(x) & & \\ & \ddots & \\ & & m_n(x) \end{pmatrix} a'n' \in K_{\mathbf{C}}M_{\mathbf{C}}A_{\mathbf{C}}N_{\mathbf{C}} \\ &= k''a''n'' \in K_{\mathbf{C}}\widetilde{A}_{\mathbf{C}}\widetilde{N}_{\mathbf{C}}, \end{aligned}$$

where $B_{\mathbf{C}} = \widetilde{A}_{\mathbf{C}}\widetilde{N}_{\mathbf{C}}$. Since $\widetilde{N}_{\mathbf{C}} = (M_{\mathbf{C}} \cap \widetilde{N}_{\mathbf{C}})N_{\mathbf{C}}$ we may write

$$k''a''n'' = k'' \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix} a'''n''' \in K_{\mathbf{C}}(B_{\mathbf{C}} \cap M_{\mathbf{C}})A_{\mathbf{C}}N_{\mathbf{C}}.$$

Since the $N_{\mathbf{C}}$ part is unique (see below) $n' = n''$ and

$$k'''^{-1}k' = \begin{pmatrix} m_1b_1 & & \\ & \ddots & \\ & & m_nb_n \end{pmatrix} a'^{-1}a''' \in M_{\mathbf{C}}A_{\mathbf{C}}.$$

In particular $k'''^{-1}k' \in M \cap K$ and $m_j(x) \in (K_{\mathbf{C}} \cap M_{\mathbf{C}})(B_{\mathbf{C}} \cap M_{\mathbf{C}})$.

Write $L_{\mathbf{C}} = M_{\mathbf{C}}A_{\mathbf{C}}$, so $P_{\mathbf{C}} = L_{\mathbf{C}}N_{\mathbf{C}}$. The expression $g = \kappa(g)\ell(g)n(g) \in K_{\mathbf{C}}L_{\mathbf{C}}N_{\mathbf{C}}$ is not unique. However, since $K_{\mathbf{C}} \cap L_{\mathbf{C}}N_{\mathbf{C}} = K_{\mathbf{C}} \cap L_{\mathbf{C}}$, $n(g)$ is unique. Next we show that $n(g^{-1})$ and $\ell(g^{-1})^t\ell(g^{-1})$ are holomorphic on $\widehat{X_0}$. For this we write the matrices as an array of 2×2 blocks. The notation is:

n_{ij} is the 2×2 block of $n(g^{-1})$ in the ij^{th} position,

ℓ_i is the 2×2 block of $\ell(g^{-1})$ in the i^{th} diagonal position,

b_{ij} is the 2×2 block of ${g^t}^{-1}g^{-1}$ in the ij^{th} position.

Then matrix multiplication gives the following recursive formulas.

$$\begin{aligned} n_{jk} &= (\ell_j^t\ell_j)^{-1}\big(b_{jk} - \sum_{i=1}^{j-1} n_{ij}^t\ell_i^t\ell_i n_{ik}\big),\ (j<k) \\ \ell_k^t\ell_k &= b_{kk} - \sum_{i=1}^{k-1} n_{ik}^t\ell_i^t\ell_i n_{ik}. \end{aligned} \tag{5.6}$$

Note that $\ell_k^t\ell_k$ and n_{jk} are holomorphic in $\ell_i^t\ell_i$ and n_{im} with $i<k$ and $m\le k$. Since $\ell_1^t\ell_1 = b_{11}$ is holomorphic in $gK_{\mathbf{C}}$ the formulas (5.6) show that each n_{jk} and $\ell_k^t\ell_k$ is holomorphic, so also $n(g^{-1})$ and $\ell(g^{-1})^t\ell(g^{-1})$.

The identification $g\to {g^t}^{-1}g^{-1}$ of $G_{\mathbf{C}}/K_{\mathbf{C}}$ with the space of symmetric $n\times n$ matrices of determinant one is biholomorphic, and similarly for $L_{\mathbf{C}}/K_{\mathbf{C}}\cap L_{\mathbf{C}}$. Therefore $gK_{\mathbf{C}}\to \ell(g^{-1})^{-1}L_{\mathbf{C}}\cap K_{\mathbf{C}}$ is a well-defined holomorphic map on $\widehat{X_0}$. Since $M_{\mathbf{C}}\cap A_{\mathbf{C}}\subset L_{\mathbf{C}}\cap K_{\mathbf{C}} = M_{\mathbf{C}}\cap K_{\mathbf{C}}$, $M_{\mathbf{C}}\cap(L_{\mathbf{C}}\cap K_{\mathbf{C}})A_{\mathbf{C}} = M_{\mathbf{C}}\cap K_{\mathbf{C}}$ and so $L_{\mathbf{C}}/(L_{\mathbf{C}}\cap K_{\mathbf{C}})A_{\mathbf{C}}\simeq M_{\mathbf{C}}/M_{\mathbf{C}}\cap K_{\mathbf{C}}$. Thus the quotient map $\pi: L_{\mathbf{C}}/L_{\mathbf{C}}\cap K_{\mathbf{C}}\to M_{\mathbf{C}}/M_{\mathbf{C}}\cap K_{\mathbf{C}}$ given by $\pi(maL_{\mathbf{C}}\cap K_{\mathbf{C}}) = mM_{\mathbf{C}}\cap K_{\mathbf{C}}$ is holomorphic. In particular, $gK_{\mathbf{C}}\to m(g^{-1})^{-1}M_{\mathbf{C}}\cap K_{\mathbf{C}}$ is a well-defined and holomorphic map $\widehat{X_0}\to M_{\mathbf{C}}/K_{\mathbf{C}}\cap M_{\mathbf{C}}$, for all $k\in K$.

We may now conclude that $m(g^{-1}k)^{-1}w(m(g^{-1}k)^{-1})$ is holomorphic on $\widehat{X_0}$.

□

From the above proof it follows that both $\ell(g^{-1})^t\ell(g^{-1})$ and $m(g^{-1})^tm(g^{-1})$ are holomorphic. However $\ell(g^{-1})^t\ell(g^{-1}) = \exp(2H(g^{-1}))m(g^{-1})^tm(g^{-1})$, therefore $gK_{\mathbf{C}}\to\exp(2H(g^{-1}))$ is a holomorphic map $\widehat{X_0}\to G_{\mathbf{C}}$. We use this fact in the following Lemma.

We now consider the scalar part of the kernel which may be written in terms of principal minors. Therefore we let $\Delta_\ell(B)$ denote the ℓ^{th} principal minor of the complex matrix B.

Lemma 5.7. *For each $k\in K$ and $g\in G_{\mathbf{C}}$*

$$e^{(\nu-\rho)H(g^{-1}k)} = \prod_{j=1}^{n}\frac{1}{\Delta_{2j}((g^{-1}k)^t(g^{-1}k))^{\frac12}},$$

is a meromorphic function on $G_{\mathbf{C}}/K_{\mathbf{C}}$.

Proof. Set $\Lambda_\ell = 2\sum_{j=1}^{\ell} f_j$ and compute $\mathrm{e}^{\Lambda_\ell(H(g^{-1}k))}$. Consider the $G_{\mathbf{C}}$ representation $\wedge^{2\ell}\mathbf{C}^{2n}$, $\ell=1,2,\dots,n$. Then $v_{+,\ell} = e_1\wedge\cdots\wedge e_{2\ell}$ is a highest $\mathfrak{a}$-weight vector of weight Λ_ℓ which is fixed by each $m\in(SL(2;\mathbf{R}))^n$. Therefore

$$\begin{aligned} \mathrm{e}^{(2\Lambda_\ell)H(g)} &= (gv_{+,\ell}, gv_{+,\ell}) \\ &= \det\big((ge_i, ge_j)_{1\le i,j\le 2\ell}\big) \\ &= \Delta_{2\ell}(g^tg). \end{aligned}$$

In particular, $\Delta_{2j}((g^{-1}k)^t(g^{-1}k)$ has holomorphic square root by the comment preceding the Lemma. The lemma follows since $\nu-\rho = -\sum_{j=1}^n(n-2j+1)f_j = -\sum_{\ell=1}^n\Lambda_\ell + (n+1)\sum_{j=1}^n f_j = -\sum_{\ell=1}^n\Lambda_\ell$ (as $\sum_{j=1}^n f_j = 0$). □

Corollary 5.8. *The function $gK\to e^{(\nu-\rho)H(g^{-1}k)}$ is holomorphic on $\widehat{X_0}$.*

Proof. This follows from the definition of $\widehat{X_0}$ applied to the flag variety $Z = G_{\mathbf{C}}/P_{\mathbf{C}}$ as follows. For the flag $x_0 = (z_1 \subset z_2 \subset \cdots \subset z_{n-1})$, $z_\ell = \text{span}\{e_1, \ldots, e_{2\ell}\}$, $X_0 = G \cdot x_0$. Now $gK_{\mathbf{C}} \in \widehat{X_0}'$ if and only if $g^{-1}kx_0 \in \mathcal{O}$ for all $k \in K$. Thus $b_{k^{-1}g}$ is nondegenerate on all z_ℓ, i.e., $\Delta_{2\ell}\big((g^{-1}k)^t(g^{-1}k)\big) \neq 0$, for all $k \in K$ and all $\ell = 1, \ldots, n$. □

We have proved the following theorem.

Theorem 5.9. *The Szegö kernel extends holomorphically to $\widehat{X_0}$. Thus, the Speh representations occur as a space of holomorphic sections of the restriction of $\mathcal{E}^{\mathbf{C}} \to G_{\mathbf{C}}/K_{\mathbf{C}}$ to $\widehat{X_0}$.*

References

[1] D. Akhiezer and S. Gindikin, *On the Stein extensions of real symmetric spaces*, Math. Ann. **286** (1990), pp. 1-12.

[2] L. Barchini, S. Gindikin and H.-W. Wong, *The geometry of flag manifolds and holomorphic extensions of Szegö kernels for $U(p,q)$*, Pacific Journal of Math. **179** (1997), pp. 201–220.

[3] L. Barchini, *Szegö mappings, harmonic forms and Dolbeault cohomology*, J. of Functional Analysis **118** (1993), pp. 351–406.

[4] J. Bernstein and A. Reznikov, *Analytic continuation of representations and estimates of automorphic forms*, Ann. of Math. (2) **150** (1999), no. 1, pp. 329–352.

[5] A. Huckleberry and A. Simon, *On cycle spaces of flag domains of $SL_n(\mathbf{R})$*, preprint, 2000.

[6] A. W. Knapp and D. A. Vogan, *Cohomological Induction and Unitary Representations*, Princeton University press, Princeton, N. J., 1995.

[7] R.O. Wells and J. A. Wolf, *Poincare series and automorphic cohomology on flag domains*, Ann. of Math. **105** (1977), pp. 397–448.

[8] J. A. Wolf and R. Zierau, *Holomorphic double fibration transforms* in The Mathematical Legacy of Harish-Chandra, PSPM vol. 68, Amer. Math. Soc., Providence, RI, 2000, pp. 527–551.

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