CONFORMALLY INVARIANT SYSTEMS OF DIFFERENTIAL OPERATORS

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Abstract. Kostant's theory of conformally invariant differential operators on certain homogeneous spaces is generalized to cover conformally invariant systems of endomorphism-valued differential operators. In particular, the connection discovered by Kostant between conformally invariant operators and highest weight vectors in generalized Verma modules is extended.

1. Introduction

This work concerns itself with conformally invariant systems of differential operators. We begin by giving a rough account of the meaning that we assign to this term. A more precise and more general definition may be found in Section 2. Suppose that M is a manifold and $\mathfrak{g}$ is a Lie algebra of first order linear differential operators on M. We call a list $D_1, \ldots, D_m$ of linear differential operators on M *conformally invariant* if we have

$$[X, D_i] = \sum_{j=1}^{m} C(X)_{ji} D_j$$

for all $1 \leq i \leq m$ and $X \in \mathfrak{g}$, where $C(X)$ is an m-by-m matrix of smooth functions on M. This definition corresponds to that used by Kostant [6] in the case where $m = 1$ and M is an open subset of a generalized flag manifold. It is also consistent with the general definition given by Ehrenpreis [3]. In this latter work one may find an interesting discussion of the philosophical significance of conformal invariance and its relation to other themes in analysis.

In [1], the authors constructed many examples of conformally invariant systems of differential operators. In particular, for any complex simple Lie algebra $\mathfrak{g}$ of rank greater than one, a system of many second-order differential operators and a system consisting of a single fourth-order differential operator that are both conformally invariant under $\mathfrak{g}$ were constructed. Whereas [1] was focused on the construction of examples, the present work is focused on establishing some general properties of such systems. One of our aims is to justify a number of claims that were made without proof in [1], and to answer a question that was left open there. A second aim is to show that the main result of Kostant's original theory (Theorem 4.7 in [6]) generalizes satisfactorily to conformally invariant systems provided that one replaces highest weight vectors with what we call *leading types* (the reader may find the definition of this term in Section 6). A third aim is to place various arguments in their natural settings, with a resulting gain in both simplicity and generality. In particular, parts of the theory presented here apply to conformally invariant

2000 *Mathematics Subject Classification.* Primary: 22E47.

systems on manifolds, and with respect to Lie algebras, beyond those originally considered by Kostant.

In keeping with the third aim identified above, the paper is structured by progressive specialization. Sections 2 and 3 take place on a manifold equipped with a suitable Lie algebra of first order operators, and contain the basic definitions and results on conformally invariant systems. In the brief Section 4, the consequences of an additional assumption on the manifold are investigated. Section 5 moves somewhat closer to Kostant's original setting by focusing on the case where the manifold and Lie algebra arise from a dense open double coset in a connected real Lie group. In this setting, it is already possible to obtain a purely algebraic description of conformally invariant systems, which is given in Theorem 5.5. The process of specialization continues in Section 6, where we adopt Kostant's original framework and prove a generalization of his main result; this generalization appears as Theorem 6.3. In this section, we also justify the outstanding claims made in [1]. Finally, in Section 7, we apply the theory that has been developed up to that point to answer a question concerning the existence of certain conformally invariant systems that was left open in [1]. The answer is given as Theorem 7.1, whose is proof is merely sketched, since it is somewhat tangential to the main development. It is included as an example of how the present theory may be used to settle concrete questions about conformally invariant systems.

The reader should be aware of a previous extension of Kostant's theory, which may be found in [4]. This extension uses differential intertwining operators instead of conformally invariant systems. By comparing the discussion in Section 6 with Lemma 2.4 in [2], the reader will see that there is a close connection between the two theories. In particular, every straight homogeneous L-stable conformally invariant system gives rise to a differential intertwining operator between suitable smooth induced representations, and conversely. The reader may thus be puzzled to encounter Huang's assertion, in Section 5 of [4], that quasi-invariant systems are inadequate to detect the reducibility of generalized Verma modules in this setting. The resolution of this puzzle is that Huang's quasi-invariant systems are defined differently from our conformally invariant systems, and have different properties. We shall postpone the task of making the connection between the two theories more explicit until we require it in future work.

2. Conformally Invariant Systems

The goal of this section is to define the notion of a conformally invariant system of differential operators, and to note some of the basic properties of such systems. Before we can do so, we require a number of preliminary definitions. For definiteness, we phrase the theory with real coefficients; with trivial adjustments, the discussion applies equally well with complex coefficients. We shall take advantage of this observation beginning in Section 5.

We always work in the smooth category. Let M be a manifold. If $\mathcal{V} \to M$ is a vector bundle over M and $U \subset M$ is an open set then we denote by $\Gamma(U, \mathcal{V})$ the space of sections of $\mathcal{V}$ over U, and abbreviate $\Gamma(M, \mathcal{V})$ to $\Gamma(\mathcal{V})$. We write $T(M)$ for the tangent bundle of M, $T_p(M)$ for the tangent space at p, $C^\infty(M)$ for the space of smooth functions on M, and $\mathcal{X}(M)$ for $\Gamma(T(M))$, the space of smooth vector fields on M. For any vector bundle $\mathcal{V} \to M$ and open set U, the space $\Gamma(U, \mathcal{V})$ is understood to carry its usual smooth topology.

If $\mathcal{V}_1 \to M$ and $\mathcal{V}_2 \to M$ are two vector bundles then we shall have to consider the space $\mathbb{D}(\mathcal{V}_1, \mathcal{V}_2)$ of (finite-order) differential operators from $\mathcal{V}_1$ to $\mathcal{V}_2$. In order to describe this space, suppose first that M is an open subset of $\mathbb{R}^n$ and that both $\mathcal{V}_1$ and $\mathcal{V}_2$ are trivial bundles. We may then identify $\Gamma(\mathcal{V}_j)$ with the space $C^\infty(M; V_j)$ of smooth maps from M to a vector space V_j. An operator $D : \Gamma(\mathcal{V}_1) \to \Gamma(\mathcal{V}_2)$ is said to be a *differential operator* if it may be written in the form

$$D\bullet\sigma = \sum_{|\alpha|\leq k} T_\alpha(\partial^\alpha\bullet\sigma) \tag{2.1}$$

for some $k \geq 0$ and all $\sigma \in \Gamma(\mathcal{V}_1)$, where each T_α is a smooth map from M to $\mathrm{Hom}(V_1, V_2)$ and we are using multi-index notation for partial derivatives. Here we write $\bullet$ to denote the action of differential operators on sections. This notation, common in the algebraic literature, allows us to preserve juxtaposition for multiplication. In the general case, an operator $D : \Gamma(\mathcal{V}_1) \to \Gamma(\mathcal{V}_2)$ is a *differential operator* if it is so everywhere locally. This implies, in particular, that D is continuous and support non-increasing. A differential operator D induces a differential operator $D : \Gamma(U, \mathcal{V}_1) \to \Gamma(U, \mathcal{V}_2)$ for any open set U in M. We abbreviate $\mathbb{D}(\mathcal{V}, \mathcal{V})$ to $\mathbb{D}(\mathcal{V})$.

Let $\mathfrak{g}$ be a real Lie algebra. We say that M is a $\mathfrak{g}$-*manifold* if we are given an $\mathbb{R}$-linear map $\Pi : \mathfrak{g} \to C^\infty(M) \oplus \mathfrak{X}(M)$ such that $\Pi([X,Y]) = [\Pi(X), \Pi(Y)]$ for all $X, Y \in \mathfrak{g}$. Intuitively, M is a $\mathfrak{g}$-manifold if $\mathfrak{g}$ has been realized as an algebra of first-order differential operators on M. We shall assume henceforth that all the manifolds we consider are equipped with the structure of a $\mathfrak{g}$-manifold. For $X \in \mathfrak{g}$ we shall write $\Pi(X) = \Pi_0(X) + \Pi_1(X)$ with $\Pi_0(X) \in C^\infty(M)$ and $\Pi_1(X) \in \mathfrak{X}(M)$. With this notation, the condition on Π is equivalent to the identities

$$\Pi_0([X,Y]) = \Pi_1(X)\bullet\Pi_0(Y) - \Pi_1(Y)\bullet\Pi_0(X)$$

and

$$\Pi_1([X,Y]) = [\Pi_1(X), \Pi_1(Y)]$$

for all $X, Y \in \mathfrak{g}$.

Suppose now that $\mathcal{V} \to M$ is a vector bundle. Note that any smooth function on M corresponds to an element of $\mathbb{D}(\mathcal{V})$ given by multiplication by that function. We say that $\mathcal{V} \to M$ is a $\mathfrak{g}$-*bundle* if we are given a linear map $\Pi_{\mathcal{V}} : \mathfrak{g} \to \mathbb{D}(\mathcal{V})$ that has the following properties:

(B1) We have $\Pi_{\mathcal{V}}([X,Y]) = [\Pi_{\mathcal{V}}(X), \Pi_{\mathcal{V}}(Y)]$ for all $X, Y \in \mathfrak{g}$.
(B2) In $\mathbb{D}(\mathcal{V})$, $[\Pi_{\mathcal{V}}(X), f] = \Pi_1(X)\bullet f$ for all $X \in \mathfrak{g}$ and all $f \in C^\infty(M)$.

We note one source of $\mathfrak{g}$-bundles on M. Suppose that we are given a bundle $\mathcal{V} \to M$ together with a flat connection ∇ on $\mathcal{V}$. Then we may define a $\mathfrak{g}$-bundle structure on $\mathcal{V}$ by setting $\Pi_{\mathcal{V}}(X)\bullet\sigma = \nabla_{\Pi_1(X)}(\sigma) + \Pi_0(X)\sigma$ for $\sigma \in \Gamma(\mathcal{V})$ and $X \in \mathfrak{g}$. However, by no means all $\mathfrak{g}$-bundle structures arise in this way.

Let $\mathcal{V} \to M$ be a $\mathfrak{g}$-bundle. A list $D_1, \ldots, D_n$ of elements of $\mathbb{D}(\mathcal{V})$ is said to constitute a *conformally invariant system* on $\mathcal{V}$ if the following two conditions are satisfied:

(S1) For all $p \in M$, the list $D_1, \ldots, D_n$ is linearly independent at p.
(S2) For each $X \in \mathfrak{g}$ there is a matrix $C(X)$ with entries in $C^\infty(M)$ such that

$$[\Pi_{\mathcal{V}}(X), D_i] = \sum_{j=1}^{n} C(X)_{ji} D_j$$

in $\mathbb{D}(\mathcal{V})$.

We call $C : \mathfrak{g} \to \mathfrak{gl}(n, C^\infty(M))$ the *structure operator* of the conformally invariant system. Note that, because of the non-degeneracy assumption (S1), the system $D_1, \ldots, D_n$ uniquely determines C. In the following lemma, we extend the action of vector fields on functions to the entry-wise action on matrices of functions. We continue to use this notation below.

Lemma 2.1. *Let C be the structure operator of a conformally invariant system. Then, for all $X, Y \in \mathfrak{g}$, we have*

$$C([X,Y]) = \Pi_1(X)\bullet C(Y) - \Pi_1(Y)\bullet C(X) + [C(X), C(Y)].$$

Proof. A calculation. □

Two conformally invariant systems $D_1, \ldots, D_n$ and $D'_1, \ldots, D'_n$ are *equivalent* if there is a matrix $A \in \mathrm{GL}(n, C^\infty(M))$ such that

$$D'_i = \sum_{j=1}^{n} A_{ji} D_j$$

for $1 \le i \le n$. The matrix A is said to *realize* the equivalence. Note that by (S1) A is uniquely determined by the systems $D_1, \ldots, D_n$ and $D'_1, \ldots, D'_n$.

Lemma 2.2. *Let $D_1, \ldots, D_n$ and $D'_1, \ldots, D'_n$ be equivalent conformally invariant systems with structure operators C and C'. Let A be the matrix realizing the equivalence. Then*

$$C'(X) = A^{-1}(\Pi_1(X)\bullet A) + A^{-1}C(X)A$$

for all $X \in \mathfrak{g}$.

Proof. A calculation. □

We say that a conformally invariant system $D_1, \ldots, D_n$ is *reducible* if there is an equivalent system $D'_1, \ldots, D'_n$ and an $m < n$ such that the system $D'_1, \ldots, D'_m$ is conformally invariant; otherwise, we say that $D_1, \ldots, D_n$ is *irreducible.* We may similarly formulate the notion of a direct sum of two conformally invariant systems, use of which will be made below.

We shall use the abbreviation D in place of $D_1, \ldots, D_n$ when convenient. If D is a conformally invariant system on $\mathcal{V}$ then let $\mathcal{E}(D)$ be the trivial vector bundle over M that is spanned by $D_1, \ldots, D_n$. Note that these operators are linearly independent at each p and so $\mathcal{E}(D)$ is indeed a rank n vector bundle over M. Observe also that the bundle $\mathcal{E}(D)$ depends only on the equivalence class of D. If we choose any other ordered basis of global sections of $\mathcal{E}(D)$ then it will correspond to a conformally invariant system equivalent to D.

We wish to define a $\mathfrak{g}$-bundle structure on $\mathcal{E}(D)$. We begin by setting

$$\Pi_{\mathcal{E}(D)}(X)\bullet D_i = \sum_{j=1}^{n} C(X)_{ji} D_j.$$

Property (B2) and linearity then specify a unique extension of this action to all sections. In order that this extension yield a $\mathfrak{g}$-bundle structure it is sufficient that we have $\Pi_{\mathcal{E}(D)}([X,Y])\bullet D_i = [\Pi_{\mathcal{E}(D)}(X), \Pi_{\mathcal{E}(D)}(Y)]\bullet D_i$ for all $X, Y \in \mathfrak{g}$ and $1 \le i \le n$. A calculation gives

$$\Pi_{\mathcal{E}(D)}(X)\bullet\big(\Pi_{\mathcal{E}(D)}(Y)\bullet D_i\big) = \sum_{j=1}^{n} \big(\Pi_1(X)\bullet C(Y)_{ji} + (C(X)C(Y))_{ji}\big) D_j$$

and this and Lemma 2.1 imply the required conclusion.

The bundle $\mathcal{E}(D)$ gives a convenient setting in which to discuss certain properties of conformally invariant systems. For instance, a system D is irreducible if and only if $\mathcal{E}(D)$ is an irreducible $\mathfrak{g}$-bundle.

3. Abstract Verma Modules

We expect there to be a connection between the existence of conformally invariant systems and the reducibility of Verma modules. In order to formulate this relationship in the setting of $\mathfrak{g}$-manifolds, we must first explain what is to play the role of a Verma module in this generality.

Let $\mathcal{V} \to M$ be a $\mathfrak{g}$-bundle, $p \in M$, and W a finite-dimensional real vector space. We define $D'_p(\mathcal{V}; W)$ to be the set of all $\mathbb{R}$-linear maps $\Lambda : \Gamma(\mathcal{V}) \to W$ that have the following properties:

- (D1) The map Λ is continuous with respect to the smooth topology on $\Gamma(\mathcal{V})$ and the usual topology on W.
- (D2) If $U \subset M$ is an open set containing p and $\sigma \in \Gamma(\mathcal{V})$ restricts to zero on U then $\Lambda(\sigma) = 0$.

We abbreviate $D'_p(\mathcal{V}; \mathbb{R})$ to $D'_p(\mathcal{V})$. In effect, $D'_p(\mathcal{V}; W)$ is the space of W-valued distributions on $\Gamma(\mathcal{V})$ that are supported at p. Note that if $\Lambda \in D'_p(\mathcal{V}; W)$, $U \subset M$ is an open set containing p and $\sigma \in \Gamma(U, \mathcal{V})$ then (D2) implies that $\Lambda(\sigma)$ may be defined simply by choosing some $\tau \in \Gamma(\mathcal{V})$ such that $\tau|_U = \sigma$ and setting $\Lambda(\sigma) = \Lambda(\tau)$. This extension will be taken for granted below.

Define an action of $\mathfrak{g}$ on $D'_p(\mathcal{V}; W)$ by

$$\big(\Pi'_{\mathcal{V}}(X)\bullet\Lambda\big)(\sigma) = -\Lambda(\Pi_{\mathcal{V}}(X)\bullet\sigma)$$

for $X \in \mathfrak{g}$, $\Lambda \in D'_p(\mathcal{V}; W)$, and $\sigma \in \Gamma(\mathcal{V})$. Since $\Pi_{\mathcal{V}}(X)$ is continuous and support non-increasing, the action is well-defined. One verifies that $\Pi'_{\mathcal{V}}([X, Y]) = [\Pi'_{\mathcal{V}}(X), \Pi'_{\mathcal{V}}(Y)]$ for all $X, Y \in \mathfrak{g}$. Let $\mathcal{U}_0(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. (The subscript is included to emphasize that we are considering real coefficients; as mentioned before, everything we say applies equally well with complex coefficients.) The action of $\mathfrak{g}$ on $D'_p(\mathcal{V}; W)$ via $\Pi'_{\mathcal{V}}$ may be extended to give $D'_p(\mathcal{V}; W)$ the structure of a left $\mathcal{U}_0(\mathfrak{g})$-module. It is easy to see that $D'_p(\mathcal{V}; W) \cong D'_p(\mathcal{V}) \otimes_{\mathbb{R}} W$ as $\mathcal{U}_0(\mathfrak{g})$-modules, where $\mathcal{U}_0(\mathfrak{g})$ acts on the first factor in the tensor product.

If $T \in \mathrm{Hom}(\mathcal{V}_p, W)$ then we may associate to T the element of $D'_p(\mathcal{V}; W)$ given by $\sigma \mapsto T(\sigma(p))$. This identifies $\mathrm{Hom}(\mathcal{V}_p, W)$ as a subspace of $D'_p(\mathcal{V}; W)$, and we shall regard this identification as an identity henceforth. Note that $\mathrm{Hom}(\mathcal{V}_p, W)$ may be characterized as the set of all $\Lambda \in D'_p(\mathcal{V}; W)$ such that $\Lambda(f\sigma) = f(p)\Lambda(\sigma)$ for all $f \in C^\infty(M)$. If $D \in \mathbb{D}(\mathcal{V})$ and $\Lambda \in D'_p(\mathcal{V}; W)$ then we define $D\Lambda \in D'_p(\mathcal{V}; W)$ by $(D\Lambda)(\sigma) = \Lambda(D\bullet\sigma)$. Since D is continuous and support non-increasing, $D\Lambda$ is well-defined.

Theorem 3.1. *Let $D_1, \ldots, D_n$ be a conformally invariant system on the $\mathfrak{g}$-bundle $\mathcal{V}$. Take $p \in M$ and suppose that there is some $\sigma_0 \in \Gamma(\mathcal{V})$ such that $D_i\bullet\sigma_0 = 0$ for all $1 \le i \le n$ and $\sigma_0(p) \ne 0$. Let F be the $\mathbb{R}$-span of the set*

$$\{D_i\lambda \mid 1 \le i \le n, \lambda \in \mathcal{V}_p^*\}$$

in $D'_p(\mathcal{V})$. Then $\mathcal{U}_0(\mathfrak{g})F$ is a non-zero proper submodule of the module $D'_p(\mathcal{V})$. In particular, $D'_p(\mathcal{V})$ is reducible.

Proof. We first show that $F \neq \{0\}$. By hypothesis, $D_1, \ldots, D_n$ is linearly independent at p. In particular, D_1 does not vanish identically at p and so there is some $\sigma \in \Gamma(\mathcal{V})$ such that $(D_1 \bullet \sigma)(p) \neq 0$. We may then find some $\lambda \in \mathcal{V}_p^*$ such that $\lambda(D_1 \bullet \sigma(p)) \neq 0$. That is, $(D_1\lambda)(\sigma) \neq 0$ and so $D_1\lambda \neq 0$. It follows from this that $F \neq \{0\}$, and hence that $\mathcal{U}_0(\mathfrak{g})F \neq \{0\}$.

Next we claim that if $\Lambda \in \mathcal{U}_0(\mathfrak{g})F$ then $\Lambda(\sigma_0) = 0$. The conformal invariance of the system $D_1, \ldots, D_n$ implies that the common solution space of these operators is stable under the action of $\mathfrak{g}$. The Lie algebra homomorphism $\Pi_{\mathcal{V}} : \mathfrak{g} \to \mathbb{D}(\mathcal{V})$ extends to an algebra homomorphism $\Pi_{\mathcal{V}} : \mathcal{U}_0(\mathfrak{g}) \to \mathbb{D}(\mathcal{V})$ and it follows from the preceding observation that the common solution space of $D_1, \ldots, D_n$ is stable under the action of $\mathcal{U}_0(\mathfrak{g})$. Now let $b \in \mathcal{U}_0(\mathfrak{g})$, $\lambda \in \mathcal{V}_p^*$, and $1 \leq i \leq n$. Then $\big(\Pi_{\mathcal{V}}(b) \bullet (D_i\lambda)\big)(\sigma_0)$ is a sum of terms of the form $\lambda\big(D_i \bullet (\Pi_{\mathcal{V}}(a) \bullet \sigma_0)\big)$ for various $a \in \mathcal{U}_0(\mathfrak{g})$. Since $D_i \bullet (\Pi_{\mathcal{V}}(a) \bullet \sigma_0) = 0$, we have $\big(\Pi_{\mathcal{V}}(b) \bullet (D_i\lambda)\big)(\sigma_0) = 0$, and the claim follows from this and the $\mathbb{R}$-linearity of the action.

By hypothesis, $\sigma_0(p) \neq 0$ and so there is some $\lambda \in \mathcal{V}_p^*$ such that $\lambda(\sigma_0(p)) \neq 0$. This provides an element of $D_p'(\mathcal{V})$ that does not annihilate σ_0 and we conclude that $\mathcal{U}_0(\mathfrak{g})F \neq D_p'(\mathcal{V})$. This completes the proof. □

Some hypothesis is certainly necessary in Theorem 3.1, because the identity operator by itself constitutes a conformally invariant system on any $\mathfrak{g}$-bundle, but does not give rise to reducibility in the associated modules. We next compute how $\mathfrak{g}$ acts via $\Pi_{\mathcal{V}}'$ on an element of the space F defined in the statement of Theorem 3.1.

Lemma 3.2. *Let $\mathcal{V}$ be a $\mathfrak{g}$-bundle on M and $D_1, \ldots, D_n$ a conformally invariant system on $\mathcal{V}$. If $\lambda \in \mathcal{V}_p^*$ and $Y \in \mathfrak{g}$ then*

$$\Pi_{\mathcal{V}}'(Y) \bullet (D_i\lambda) = D_i(\Pi_{\mathcal{V}}'(Y) \bullet \lambda) + \sum_{j=1}^{n} C(Y)_{ji}(p) D_j\lambda.$$

Proof. Let $\sigma \in \Gamma(\mathcal{V})$. Then

$$\begin{aligned}
\big(\Pi_{\mathcal{V}}'(Y) \bullet (D_i\lambda)\big)(\sigma) &= -(D_i\lambda)\big(\Pi_{\mathcal{V}}(Y) \bullet \sigma\big) \\
&= -\lambda\big(D_i \bullet (\Pi_{\mathcal{V}}(Y) \bullet \sigma)\big) \\
&= -\lambda\big(\Pi_{\mathcal{V}}(Y) \bullet (D_i \bullet \sigma) - [\Pi_{\mathcal{V}}(Y), D_i] \bullet \sigma\big) \\
&= -\lambda\big(\Pi_{\mathcal{V}}(Y) \bullet (D_i \bullet \sigma)\big) + \sum_{j=1}^{n} \lambda\big(C(Y)_{ji} D_j \bullet \sigma\big) \\
&= \big(\Pi_{\mathcal{V}}'(Y) \bullet \lambda\big)(D_i \bullet \sigma) + \sum_{j=1}^{n} C(Y)_{ji}(p) \lambda\big(D_j \bullet \sigma\big) \\
&= \big(D_i(\Pi_{\mathcal{V}}'(Y) \bullet \lambda)\big)(\sigma) + \sum_{j=1}^{n} C(Y)_{ji}(p)(D_j\lambda)(\sigma).
\end{aligned}$$

The asserted identity follows. □

We can be more precise about the structure of the space F in the presence of additional structure on the underlying $\mathfrak{g}$-manifold. Suppose that $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}$. We shall call a point $p \in M$ a *polar point* for $\mathfrak{m}$ if $(\Pi_1(Z) \bullet f)(p) = 0$ for all $Z \in \mathfrak{m}$ and $f \in C^\infty(M)$. In cases where the $\mathfrak{g}$-structure on M arises from a group action, saying that p is a polar point for $\mathfrak{m}$ will be essentially the same as saying that $\mathfrak{m}$ is contained in the isotropy subalgebra of p. This may give some intuition for the concept in general.

Lemma 3.3. *Let $\mathfrak{m}$ be a subalgebra of $\mathfrak{g}$, $p \in M$ a polar point for $\mathfrak{m}$ and $\mathcal{V}$ a $\mathfrak{g}$-bundle. Then $\mathfrak{m}$ preserves the subspace $\mathcal{V}_p^*$ of $D_p'(\mathcal{V})$.*

Proof. Let $\lambda \in \mathcal{V}_p^*$, $Z \in \mathfrak{m}$, $\sigma \in \Gamma(\mathcal{V})$ and $f \in C^\infty(M)$. Then

$$\begin{aligned}\big(\Pi_{\mathcal{V}}'(Z)\bullet\lambda\big)(f\sigma) &= -\lambda\big(\Pi_{\mathcal{V}}(Z)\bullet(f\sigma)\big)\\ &= -\lambda\big((\Pi_1(Z)\bullet f)\sigma + f\Pi_{\mathcal{V}}(Z)\bullet\sigma\big)\\ &= -(\Pi_1(Z)\bullet f)(p)\lambda(\sigma) - f(p)\lambda\big(\Pi_{\mathcal{V}}(Z)\bullet\sigma\big)\\ &= f(p)\big(\Pi_{\mathcal{V}}'(Z)\bullet\lambda\big)(\sigma).\end{aligned}$$

It follows that $\Pi_{\mathcal{V}}'(Z)\bullet\lambda \in \mathcal{V}_p^*$. □

Lemma 3.4. *Let $\mathfrak{m}$ be a subalgebra of $\mathfrak{g}$ and $p \in M$ a polar point for $\mathfrak{m}$. Take $\mathcal{V}$ a $\mathfrak{g}$-bundle on M and $D = D_1, \ldots, D_n$ a conformally invariant system on $\mathcal{V}$. Then the map $\mathfrak{m} \to \mathfrak{gl}(n, \mathbb{R})$ given by $Z \mapsto C(Z)(p)$ is a representation of $\mathfrak{m}$.*

Proof. Let $Z_1, Z_2 \in \mathfrak{m}$. By Lemma 2.1, we have

$$C([Z_1, Z_2]) = \Pi_1(Z_1)\bullet C(Z_2) - \Pi_1(Z_2)\bullet C(Z_1) + [C(Z_1), C(Z_2)].$$

By evaluating both sides of this equation at p and using the definition of a polar point, we obtain

$$C([Z_1, Z_2])(p) = [C(Z_1)(p), C(Z_2)(p)],$$

as required. □

If p is a polar point for a subalgebra $\mathfrak{m}$ then Lemma 3.2 says that, as a representation of $\mathfrak{m}$, the space F defined in Theorem 3.1 is a quotient of the tensor product $\mathbb{R}^n \otimes \mathcal{V}_p^*$. Here $\mathfrak{m}$ acts on $\mathbb{R}^n$ via the map $Z \mapsto C(Z)(p)$ identified in Lemma 3.4 and on $\mathcal{V}_p^*$ via $\Pi_{\mathcal{V}}'$. If we identify the standard basis in $\mathbb{R}^n$ with $D_1, \ldots, D_n$ then the quotient map $\mathbb{R}^n \otimes \mathcal{V}_p^* \to F$ is given on simple tensors by $D_i \otimes \lambda \mapsto D_i\lambda$.

4. Straight Manifolds

In this section, we consider a restricted class of $\mathfrak{g}$-manifolds on which we may prove further properties of conformally invariant systems. The additional restrictions placed on manifolds in this class may not seem particularly natural; they are motivated by the context in which we intend to apply the results.

We say that the $\mathfrak{g}$-manifold M is *straight* if there is a subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ such that

(N1) $\Pi(\mathfrak{n}) \subset \mathfrak{X}(M)$.

(N2) For all $p \in M$, the map $X \mapsto \Pi(X)(p)$ is a linear isomorphism between $\mathfrak{n}$ and $T_p(M)$.

Suppose that M is straight and let $X_1, \ldots, X_r$ be a basis for $\mathfrak{n}$ and $V_j = \Pi(X_j)$. The vector fields $V_1, \ldots, V_r$ are a global ordered basis of smooth sections of $T(M)$. In particular, a straight $\mathfrak{g}$-manifold necessarily has trivial tangent bundle. If M is straight and $\mathcal{V} \to M$ is a $\mathfrak{g}$-bundle then there is a unique connection ∇ on $\mathcal{V}$ that satisfies $\nabla_{V_j}(\sigma) = \Pi_{\mathcal{V}}(X_j)\bullet\sigma$ for $\sigma \in \Gamma(\mathcal{V})$ and $1 \le j \le r$. A calculation, relying on property (B1), shows that this connection is flat.

Lemma 4.1. *Let M be a straight $\mathfrak{g}$-manifold, $\mathcal{V} \to M$ a $\mathfrak{g}$-bundle of rank m and $U \subset M$ a simply-connected open set. Then there are $\sigma_1, \ldots, \sigma_m \in \Gamma(U, \mathcal{V})$ such that $\sigma_1(p), \ldots, \sigma_m(p)$ is an ordered basis of $\mathcal{V}_p$ for all $p \in U$ and $\Pi_{\mathcal{V}}(X)\bullet\sigma_i = 0$ for all $X \in \mathfrak{n}$ and $1 \leq i \leq m$.*

Proof. Fix $q \in U$. The holonomy group of ∇ at q with respect to the manifold U is trivial. It follows that parallel transportation taking any ordered basis of $\mathcal{V}_q$ as initial data produces a list of sections of $\mathcal{V}|_U$ with the required properties. □

We next identify a particularly nice set of representatives in each equivalence class of conformally invariant systems when M is straight. To this end, we shall call a conformally invariant system $D_1, \ldots, D_n$ on a straight manifold *straight* if its structure operator vanishes on $\mathfrak{n}$.

Proposition 4.2. *Let M be simply-connected and straight, and $\mathcal{V} \to M$ be a $\mathfrak{g}$-bundle on M. Then every conformally invariant system on $\mathcal{V}$ is equivalent to a straight conformally invariant system.*

Proof. This follows at once by applying Lemma 4.1 to the $\mathfrak{g}$-bundle $\mathcal{E}(D)$. □

Note that this result may be applied to the restriction of a conformally invariant system to a simply-connected open set $U \subset M$. Thus every conformally invariant system on a straight manifold is everywhere locally equivalent to a straight conformally invariant system. Although each equivalence class of conformally invariant systems contains an infinite number of straight conformally invariant systems, the equivalences between these systems are of a very restricted type.

Lemma 4.3. *Let $D_1, \ldots, D_n$ and $D'_1, \ldots, D'_n$ be equivalent, straight, conformally invariant systems on $\mathcal{V} \to M$. Then the matrix realizing the equivalence between the two systems is locally constant.*

Proof. Let $A \in \mathrm{GL}(n, C^\infty(M))$ be the matrix realizing the equivalence. It follows from Lemma 2.2 that $\Pi(X)\bullet A = 0$ for all $X \in \mathfrak{n}$. Since the image of $\mathfrak{n}$ under Π is the full tangent space at every point, we conclude that A is locally constant. □

Assume that M is simply-connected and that the conformally invariant system $D_1, \ldots, D_n$ is straight and reducible. Then there is an equivalent system $D'_1, \ldots, D'_n$ and an $m < n$ such that $D'_1, \ldots, D'_m$ is conformally invariant. The system $D'_1, \ldots, D'_m$ may be straightened to $D''_1, \ldots, D''_m$ and the system of operators $D''_1, \ldots, D''_m, D'_{m+1}, \ldots, D'_n$ is conformally invariant. This system may be straightened to obtain the system $D''_1, \ldots, D''_n$, which is equivalent to the original system. From this and the argument of Lemma 4.3, we conclude that we may take

$$D'_i = \sum_{j=1}^{n} a_{ji} D_j$$

with a_{ji} constant.

5. A Specialization of the Theory

The purpose of this section is to specialize the theory to a setting which encompasses a number of interesting examples and in which more precise results can be obtained. Recall that, as we remarked above, the theory developed so far applies equally well with complex scalars. In keeping with the usual conventions for generalized Verma modules, we shall henceforth take the scalar field to be $\mathbb{C}$.

Let G be a connected real Lie group with (complex) Lie algebra $\mathfrak{g}$ and real Lie algebra $\mathfrak{g}_0$. Suppose that N and H are closed subgroups of G such that $N \cap H = \{e\}$ and NH is a dense open subset of G. We also require that N be connected. Let $\mathfrak{n}$ and $\mathfrak{h}$ denote the Lie algebras of N and H, respectively, and note that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$. For $Y \in \mathfrak{g}$, we write $Y = Y_{\mathfrak{n}} + Y_{\mathfrak{h}}$ for the decomposition of Y in this direct sum. If $g \in NH$ then there is a unique factorization $g = \mathbf{n}(g)\mathbf{h}(g)$ with $\mathbf{n}(g) \in N$ and $\mathbf{h}(g) \in H$, and the maps $g \mapsto \mathbf{n}(g)$ and $g \mapsto \mathbf{h}(g)$ are smooth. For $Y \in \mathfrak{g}_0$ we have

$$Y_{\mathfrak{n}} = \frac{d}{dt}\mathbf{n}\big(\exp(tY)\big)\big|_{t=0},$$

and similarly with $Y_{\mathfrak{h}}$.

Let (η, V) be a finite-dimensional smooth representation of H. For any manifold M, denote by $C^\infty(M; V)$ the space of smooth maps from M to V. The group G acts on the space

$$C^\infty_\eta(G; V) = \{\Phi \in C^\infty(G; V) \mid \Phi(gh) = \eta(h^{-1})\Phi(g) \text{ for all } h \in H \text{ and } g \in G\}$$

by left translation. The derived action Π_η of $\mathfrak{g}$ on $C^\infty_\eta(G; V)$ is given by

$$\big(\Pi_\eta(Y)\bullet\Phi\big)(g) = \frac{d}{dt}\Phi\big(\exp(-tY)g\big)\big|_{t=0}$$

for $Y \in \mathfrak{g}_0$. This action extends by $\mathbb{C}$-linearity to $\mathfrak{g}$ and then by universality to $\mathcal{U}(\mathfrak{g})$. We denote the extended actions by the same symbol and make such extensions silently in future.

The restriction map $C^\infty_\eta(G; V) \to C^\infty(N; V)$ is injective and its image is dense in $C^\infty(N; V)$ in the smooth topology. We may transport Π_η to the image of this map by defining $\Pi_\eta(u)\bullet\varphi = (\Pi_\eta(u)\bullet\Phi)|_N$ for $u \in \mathcal{U}(\mathfrak{g})$ and $\Phi \in C^\infty_\eta(G; V)$ such that $\varphi = \Phi|_N$. Let R denote the action of $\mathcal{U}(\mathfrak{n})$ on $C^\infty(N; V)$ defined by

$$\big(R(X)\bullet\varphi\big)(n) = \frac{d}{dt}\varphi(n\exp(tX))\big|_{t=0}$$

for $X \in \mathfrak{n}_0$ and $\varphi \in C^\infty(N; V)$. A calculation shows that

$$\big(\Pi_\eta(Y)\bullet\varphi\big)(n) = d\eta\big((\mathrm{Ad}(n^{-1})Y)_{\mathfrak{h}}\big)\varphi(n) - \big(R\big((\mathrm{Ad}(n^{-1})Y)_{\mathfrak{n}}\big)\bullet\varphi\big)(n)$$

for φ in the image of the restriction map from $C^\infty_\eta(G; V)$. This expression may then be used to extend $\Pi_\eta(Y)$ to the whole space $C^\infty(N; V)$. When η is the trivial representation we write $\Pi(Y)$ for $\Pi_\eta(Y)$. For $n_0 \in N$, we define $(\ell_{n_0}\varphi)(n) = \varphi(n_0^{-1}n)$.

Lemma 5.1. *For $X, X' \in \mathfrak{n}$, $n_0 \in N$, and $Y \in \mathfrak{g}$, we have*
1. $\ell_{n_0} \circ R(X) = R(X) \circ \ell_{n_0}$,
2. $\Pi_\eta(X') \circ R(X) = R(X) \circ \Pi_\eta(X')$,
3. $\Pi_\eta(Y) \circ \ell_{n_0} = \ell_{n_0} \circ \Pi_\eta(\mathrm{Ad}(n_0^{-1})Y)$.

Proof. Computation. □

With (η, V) as above, let $\mathcal{V}_\eta$ be the trivial bundle over N with fiber V. Then $\Gamma(\mathcal{V}_\eta)$ may be identified with $C^\infty(N; V)$. It follows from the construction sketched above that Π gives N the structure of a $\mathfrak{g}$-manifold, and Π_η gives $\mathcal{V}_\eta$ the structure of a $\mathfrak{g}$-bundle over this $\mathfrak{g}$-manifold. The $\mathfrak{g}$-manifold N is straight with respect to the subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, and $e \in N$ is a polar point for the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In fact, $(\Pi_\eta(Y)\bullet\varphi)(e) = d\eta(Y)\varphi(e)$ for all $Y \in \mathfrak{h}$ and $\varphi \in C^\infty(N; V)$.

In order to study conformally invariant systems on N, we consider the space

$$\mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}} = \{D \in \mathbb{D}(\mathcal{V}_\eta) \mid [\Pi_\eta(X), D] = 0 \text{ for all } X \in \mathfrak{n}\}.$$

Lemma 5.2. *If $D \in \mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$ then $\ell_{n_0} \circ D = D \circ \ell_{n_0}$ for all $n_0 \in N$.*

Proof. The tangent space to N at any point is spanned by the vector fields $R(X)$ for $X \in \mathfrak{n}$. Thus there is an expression of the form

$$D = \sum_{|\alpha| \leq k} T_\alpha R(X)^\alpha,$$

where we are using a slight variation of multi-index notation, and each T_α is a smooth map from N to $\operatorname{End}(V)$. It follows from the construction of Π_η, or from direct calculation, that if $Z \in \mathfrak{n}$ then $[\Pi_\eta(Z), R(X)] = 0$. Thus the condition $[\Pi_\eta(Z), D] = 0$ for all $Z \in \mathfrak{n}$ implies that $\Pi(Z) \bullet T_\alpha = 0$ for all α. Since N is connected, it follows that T_α is constant for all α. Given this, D visibly commutes with ℓ_{n_0} for all $n_0 \in N$. □

Proposition 5.3. *Let $D_1, \ldots, D_m$ be a list of operators in $\mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$. Suppose that the list is linearly independent at e and that there is a map $b : \mathfrak{g} \to \mathfrak{gl}(m, \mathbb{C})$ such that*

$$\big([\Pi_\eta(Y), D_i] \bullet \varphi\big)(e) = \sum_{j=1}^m b(Y)_{ji} (D_j \bullet \varphi)(e)$$

for all $Y \in \mathfrak{g}$, $\varphi \in C^\infty(N; V)$, and $1 \leq i \leq m$. Then $D_1, \ldots, D_m$ is a conformally invariant system on $\mathcal{V}_\eta$. The structure operator of the system is given by $C(Y)(n) = b(\operatorname{Ad}(n^{-1})Y)$ for all $n \in N$ and $Y \in \mathfrak{g}$.

Proof. The linear independence of $D_1, \ldots, D_m$ at e combines with their translation invariance to imply that they are independent at all $n \in N$. Let $n \in N$. By using the commutation relations for $\ell_{n^{-1}}$, $\Pi_\eta(Y)$ and D_i, we have

$$\begin{aligned}
\big([\Pi_\eta(Y), D_i] \bullet \varphi\big)(n) &= \ell_{n^{-1}} \big([\Pi_\eta(Y), D_i] \bullet \varphi\big)(e) \\
&= \big([\Pi_\eta(\operatorname{Ad}(n^{-1})Y), D_i] \bullet \ell_{n^{-1}} \varphi\big)(e) \\
&= \sum_{j=1}^m b(\operatorname{Ad}(n^{-1})Y)_{ji} \big(D_j \bullet \ell_{n^{-1}} \varphi\big)(e) \\
&= \sum_{j=1}^m b(\operatorname{Ad}(n^{-1})Y)_{ji} \big(\ell_{n^{-1}} (D_j \bullet \varphi)\big)(e) \\
&= \sum_{j=1}^m b(\operatorname{Ad}(n^{-1})Y)_{ji} (D_j \bullet \varphi)(n).
\end{aligned}$$

This relationship expresses the conformal invariance of the system $D_1, \ldots, D_m$, and shows that the structure operator is as claimed. □

Note that if Proposition 5.3 is applied to a system that is already known to be conformally invariant and straight then it gives a formula for the structure operator in terms of its value at the identity.

Corollary 5.4. *Let $D = D_1, \ldots, D_m$ be a straight conformally invariant system on the bundle $\mathcal{V}_\eta$ with structure operator C. Then D is irreducible if and only if the representation $Z \mapsto C(Z)(e)$ of $\mathfrak{h}$ is irreducible.*

Proof. Choose a simply-connected neighborhood U of e in N. If D is reducible then the restriction of D to U is reducible and hence we may find a straight conformally invariant system $D' = D'_1, \ldots, D'_m$ on U that is equivalent to D and such that $D'_1, \ldots, D'_k$ is conformally invariant for some $k < m$. Let C' be the structure operator of the system D'. The representation $Z \mapsto C'(Z)(e)$ of $\mathfrak{h}$ is isomorphic to the representation $Z \mapsto C(Z)(e)$. The first k standard basis vectors in $\mathbb{C}^m$ span a proper subrepresentation of $Z \mapsto C'(Z)(e)$. It follows that $Z \mapsto C(Z)(e)$ is reducible.

Suppose now that $Z \mapsto C(Z)(e)$ is reducible. After replacing D with a system equivalent to it by a constant change-of-basis matrix, we may suppose that the first k standard basis vectors in $\mathbb{C}^m$ span a proper subrepresentation of $Z \mapsto C(Z)(e)$ for some $k < m$. Then we will have $C(Y)_{ji}(e) = 0$ for all $Y \in \mathfrak{g}$, $i \le k$ and $j > k$. From Proposition 5.3, if $i \le k$, $j > k$, $Y \in \mathfrak{g}$, and $n \in N$ then $C(Y)_{ji}(n) = C(\mathrm{Ad}(n^{-1})Y)_{ji}(e) = 0$. It follows that the system $D_1, \ldots, D_k$ is conformally invariant and hence that D is reducible. □

Let W be a finite-dimensional complex vector space and define an action of $\mathfrak{g}$ on $\mathrm{Hom}(V, W)$ by $Y \cdot T = -T \circ d\eta(Y)$ for $Y \in \mathfrak{g}$ and $T \in \mathrm{Hom}(V, W)$. For $Y \in \mathfrak{h}$, $T \in \mathrm{Hom}(V, W)$, and $\varphi \in C^\infty(N; V)$,

$$\big(\Pi'_\eta(Y)\bullet T\big)(\varphi) = -T\big(\Pi_\eta(Y)\bullet\varphi\big) = -T\big(d\eta(Y)\varphi(e)\big) = (Y \cdot T)(\varphi).$$

It follows that there is a map from $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{Hom}(V, W)$ to $D'_e(\mathcal{V}_\eta; W)$ given on simple tensors by $u \otimes T \mapsto \Pi'_\eta(u)\bullet T$. It is well-known that this map is an isomorphism of $\mathcal{U}(\mathfrak{g})$-modules, and we shall henceforth identify the two spaces via this isomorphism. We may also identify these spaces with $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^* \otimes_{\mathbb{C}} W$ as convenient.

If we apply the observations of the preceding paragraph with $W = V$ then we obtain an isomorphism of $\mathcal{U}(\mathfrak{g})$-modules between $D'_e(\mathcal{V}_\eta; V)$ and $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V)$. However, some caution is necessary with the notation here, since $\mathfrak{h}$ is acting on $\mathrm{End}(V)$ by $Y \cdot T = -T \circ d\eta(Y)$. This is not the usual action, which is rather $(Y, T) \mapsto d\eta(Y) \circ T - T \circ d\eta(Y)$. Unfortunately, both actions will play a role later on. To avoid confusion, we shall henceforth take $\mathrm{End}(V)$ to have the non-standard action by default, and explicitly specify whenever the standard action is being considered instead.

There is a vector space isomorphism between the spaces $\mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$ and $D'_e(\mathcal{V}_\eta; V)$ which provides a convenient description of the former. If $D \in \mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$ then the map $\varphi \mapsto (D\bullet\varphi)(e)$ lies in $D'_e(\mathcal{V}_\eta; V)$. Since D commutes with $\ell_{n^{-1}}$ for all $n \in N$, this map completely determines D. This gives the isomorphism in one direction. If $\Lambda \in D'_e(\mathcal{V}_\eta; V)$ then define D_Λ by $\big(D_\Lambda\bullet\varphi\big)(n) = \Lambda\big(\ell_{n^{-1}}\varphi\big)$. Routine calculations show that $D_\Lambda \in \mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$ and that these maps are mutually inverse.

We have already seen that if $D_1, \ldots, D_m$ is a conformally invariant system on $\mathcal{V}_\eta$ then $F = \mathrm{span}_{\mathbb{C}}\{D_i\lambda \mid 1 \le i \le m,\ \lambda \in \mathcal{V}^*_{\eta,e}\}$ is a finite-dimensional $\mathfrak{h}$-submodule of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*$. Indeed, this follows on applying Lemmas 3.2, 3.3, and 3.4 to the present situation. We are now in a position to obtain a converse statement.

Theorem 5.5. *Suppose that F is a finite-dimensional $\mathfrak{h}$-submodule of the module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*$. Let $f_1, \ldots, f_k$ be a basis of F and define constants $a_{ri}(Y)$ by*

$$Y f_i = \sum_{r=1}^{k} a_{ri}(Y) f_r$$

for $1 \le i \le k$ *and* $Y \in \mathfrak{h}$*. Let* $\xi_1, \dots, \xi_l$ *be a basis of* V *and define constants* $b_{tj}(Y)$ *by*

$$d\eta(Y)\xi_j = \sum_{t=1}^{l} b_{tj}(Y)\xi_t$$

for $1 \le j \le l$ *and* $Y \in \mathfrak{h}$*. Define* $\Lambda_{ij} = f_i \otimes \xi_j \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^* \otimes_{\mathbb{C}} V$ *and* $D_{ij} = D_{\Lambda_{ij}} \in \mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$ *for* $1 \le i \le k$ *and* $1 \le j \le l$*. Then*

$$\begin{aligned}\big([\Pi_\eta(Z), D_{ij}]\bullet\varphi\big)(n) \\ = \sum_{r=1}^{k} a_{ri}\big((\mathrm{Ad}(n^{-1})Z)_{\mathfrak{h}}\big)(D_{rj}\bullet\varphi)(n) + \sum_{t=1}^{l} b_{tj}\big((\mathrm{Ad}(n^{-1})Z)_{\mathfrak{h}}\big)(D_{it}\bullet\varphi)(n)\end{aligned} \tag{5.1}$$

for all $Z \in \mathfrak{g}$*,* $\varphi \in C^\infty(N;V)$*,* $n \in N$*,* $1 \le i \le k$*, and* $1 \le j \le l$*. In particular, the system* $D_{11}, \dots, D_{kl}$ *is conformally invariant. Moreover,*

$$\mathrm{span}_{\mathbb{C}}\{D_{ij}\lambda \mid 1 \le i \le k,\ 1 \le j \le l,\ \lambda \in \mathcal{V}^*_{\eta,e}\} = F.$$

Proof. By Proposition 5.3, the relation (5.1) will follow in general if we can obtain it for $n = e$. By writing $Z = Z_{\mathfrak{h}} + Z_{\mathfrak{n}}$, and using the linearity of both sides of (5.1) and the fact that $\mathfrak{n}$ is stable under $\mathrm{Ad}(n^{-1})$ for all $n \in N$, we conclude that we are also free to assume that $Z \in \mathfrak{h}$. If $u \otimes T \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V)$, $Y \in \mathfrak{h}$, and $\varphi \in C^\infty(N;V)$ then

$$\begin{aligned}\big([\Pi_\eta(Y), D_{u\otimes T}]\bullet\varphi\big)(e) &= \big(\Pi_\eta(Y)\bullet(D_{u\otimes T}\bullet\varphi)\big)(e) - \big(D_{u\otimes T}\bullet(\Pi_\eta(Y)\bullet\varphi)\big)(e) \\ &= d\eta(Y)\big((D_{u\otimes T}\bullet\varphi)(e)\big) - (\Pi'_\eta(u)\bullet T)(\Pi_\eta(Y)\bullet\varphi) \\ &= d\eta(Y)\big((D_{u\otimes T}\bullet\varphi)(e)\big) + (\Pi'_\eta(Yu)\bullet T)(\varphi) \\ &= d\eta(Y)\big((D_{u\otimes T}\bullet\varphi)(e)\big) + \big(D_{Y(u\otimes T)}\bullet\varphi\big)(e).\end{aligned}$$

It follows from this that for all $\Lambda \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V)$, $\varphi \in C^\infty(N;V)$, and $Y \in \mathfrak{h}$, we have

$$\begin{aligned}\big([\Pi_\eta(Y), D_\Lambda]\bullet\varphi\big)(e) &= (D_{Y\Lambda}\bullet\varphi)(e) + d\eta(Y)(D_\Lambda\bullet\varphi)(e) \\ &= (Y\Lambda)(\varphi) + d\eta(Y)\Lambda(\varphi),\end{aligned}$$

where Λ and $Y\Lambda$ are interpreted as $\mathrm{End}(V)$-valued distributions in the last line. By introducing the definition of D_{ij} into this identity, we obtain

$$\big([\Pi_\eta(Y), D_{ij}]\bullet\varphi\big)(e) = (Yf_i)(\varphi)\xi_j + f_i(\varphi)d\eta(Y)\xi_j.$$

This equation and the definitions of the constants $a_{ri}(Y)$ and $b_{tj}(Y)$ now imply the required relation. This establishes the first two claims. The last claim follows from the fact that $D_{ij}\lambda = \lambda(\xi_j)f_i$. □

If we begin with a finite-dimensional $\mathfrak{h}$-submodule F of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*$, construct the conformally invariant system described in Theorem 5.5, and then take the $\mathbb{C}$-span of $D\lambda$ for all D in the system and $\lambda \in \mathcal{V}^*_{\eta,e}$, we recover the space F. On the other hand, if we begin with a conformally invariant system $D_1, \dots, D_m$, let $F \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*$ be the $\mathbb{C}$-span of $D_i\lambda$ for $1 \le i \le m$ and $\lambda \in \mathcal{V}^*_{\eta,e}$, and then construct the conformally invariant system described in Theorem 5.5 for F, we generally do not recover the original system. For a start, the system so constructed will be straight and so equivalence classes of conformally invariant systems that cannot be straightened cannot be built via this construction. If we assume that the original system was straight then we obtain a conformally invariant system that has a subsystem equivalent to the original system. This enlargement of the original system is canonical, in the sense that repeating the process leads back to the same result.

For later applications, it is useful to obtain a second description of the operator $D_\Lambda \in \mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$ associated to an element $\Lambda \in D'_e(\mathcal{V}_\eta; V)$. The PBW theorem implies that the natural map

$$\mathcal{U}(\mathfrak{n}) \otimes_{\mathbb{C}} \mathrm{End}(V) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V)$$

induced by the inclusion $\mathcal{U}(\mathfrak{n}) \hookrightarrow \mathcal{U}(\mathfrak{g})$ is an isomorphism. Thus we may always assume that $\Lambda \in \mathcal{U}(\mathfrak{n}) \otimes_{\mathbb{C}} \mathrm{End}(V)$.

Proposition 5.6. *Let* $\Lambda = \sum_{i=1}^k u_i \otimes T_i \in \mathcal{U}(\mathfrak{n}) \otimes_{\mathbb{C}} \mathrm{End}(V)$, $\varphi \in C^\infty(N; V)$, *and* $n \in N$. *Then*

$$\big(D_\Lambda \bullet \varphi\big)(n) = \sum_{i=1}^k T_i\big((R(u_i)\bullet\varphi)(n)\big).$$

Proof. By Lemma 5.1, the map that sends $\varphi \in C^\infty(N; V)$ to the function

$$n \mapsto \sum_{i=1}^k T_i\big((R(u_i)\bullet\varphi)(n)\big)$$

is an element of $\mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$. We already know that $D_\Lambda \in \mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$, and so it suffices to verify the proposed identity for $n = e$. By linearity, we may assume that $\Lambda = X_1X_2\cdots X_r \otimes T$ with $X_1, \ldots, X_r \in \mathfrak{n}$ and $T \in \mathrm{End}(V)$. The key observation is that for $X \in \mathfrak{n}$ we have $\big(\Pi_\eta(X)\bullet\varphi\big)(e) = -\big(R(X)\bullet\varphi\big)(e)$. This follows at once from the definitions of the two actions. Thus

$$\begin{aligned}
\big(D_{X_1\cdots X_r\otimes T}\bullet\varphi\big)(e) &= \big(\Pi'_\eta(X_1\cdots X_r)\bullet T\big)(\varphi)\\
&= (-1)^r T\big((\Pi_\eta(X_r\cdots X_1)\bullet\varphi)(e)\big)\\
&= (-1)^{r+1} T\big((R(X_r)\bullet\Pi_\eta(X_{r-1}\cdots X_1)\bullet\varphi)(e)\big)\\
&= (-1)^{r+1} T\big((\Pi_\eta(X_{r-1}\cdots X_1)\bullet R(X_r)\bullet\varphi)(e)\big)\\
&= \cdots\\
&= (-1)^{r+r} T\big((R(X_1\cdots X_r)\bullet\varphi)(e)\big)\\
&= T\big((R(X_1\cdots X_r)\bullet\varphi)(e)\big),
\end{aligned}$$

as required. □

The representation of elements of $\mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$ in the form given in Proposition 5.6 was used by Kostant [6] in the case where $\mathcal{V}_\eta$ is a line bundle.

6. Systems Arising from Generalized Verma Modules

In this section, we apply the preceding theory to the original motivating example. This will, in particular, allow us to justify a number of claims made without proof in [1].

Let G be a connected real reductive Lie group with Lie algebra $\mathfrak{g}$. Let Q be a closed subgroup of G whose Lie algebra is a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. We may then choose a Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ and a corresponding decomposition $Q = LN$ of Q. Let $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{n}}$ be the opposite subalgebra and $\bar{Q} = L\bar{N}$ the corresponding group. Note that we are not assuming that Q is the full normalizer of $\mathfrak{q}$, nor that it is connected; in general, Q will be intermediate between these extremes. In this choice, we are following the framework established in [6]. The set $N\bar{Q}$ is open and dense in G, $N \cap \bar{Q} = \{e\}$, and the group N is simply-connected. Thus we are in the situation of Section 5. In particular, N is a straight $\mathfrak{g}$-manifold on which every conformally invariant system is equivalent to a straight system. Whenever we consider a conformally invariant system on N, we shall assume that it is straight. Moreover, we shall only consider conformally invariant systems on $\mathfrak{g}$-bundles of the form $\mathcal{V}_\eta \to N$, where (η, V) is a representation of $\bar{Q}$.

Let $D_1, \ldots, D_m$ be a conformally invariant system. For $Z \in \mathfrak{l}$ and $X \in \mathfrak{n}$, we have $[Z, X] \in \mathfrak{n}$. By applying Lemma 2.1 to this bracket, we conclude that $\Pi(X) \bullet C(Z) = 0$, so that $C(Z)$ is constant. It then follows from Lemma 3.4 and the fact that e is a polar point for $\bar{\mathfrak{q}}$ that $Z \mapsto C(Z)$ is a representation of $\mathfrak{l}$ on $\mathbb{C}^m$.

Given a semisimple element $H_0 \in \mathfrak{g}$, let $\mathfrak{g}(j)$ be the j-eigenspace of $\mathrm{ad}(H_0)$. It is known that we may find a semisimple element $H_0 \in \mathfrak{l}$ such that all its eigenvalues are integers, $\mathfrak{g}(1) \neq \{0\}$, $\mathfrak{l} = \mathfrak{g}(0)$, $\mathfrak{n} = \oplus_{j>0}\mathfrak{g}(j)$, and $\bar{\mathfrak{n}} = \oplus_{j<0}\mathfrak{g}(j)$. Let us fix a choice of H_0 with these properties. We call a conformally invariant system $D_1, \ldots, D_m$ *homogeneous* if $C(H_0)$ is a scalar matrix. Since we are assuming that all systems are straight, the only allowable equivalences are realized by constant matrices. This implies that homogeneity is invariant under equivalence.

Proposition 6.1. *An irreducible conformally invariant system is homogeneous. If $D = D_1, \ldots, D_m$ is a homogeneous conformally invariant system with structure operator C then $C(Y)(e) = 0$ for all $Y \in \bar{\mathfrak{n}}$. A homogeneous conformally invariant system is a direct sum of irreducible conformally invariant systems.*

Proof. Suppose first that $D = D_1, \ldots, D_m$ is irreducible. It follows from Corollary 5.4 that the representation $Z \mapsto C(Z)(e)$ of $\bar{\mathfrak{q}}$ is irreducible. As is well known, this implies that the representation $Z \mapsto C(Z)(e)$ of $\mathfrak{l}$ is irreducible. Since H_0 lies in the center of $\mathfrak{l}$, $C(H_0)$ is a scalar matrix by Schur's Lemma.

Next suppose that D is a homogeneous system and let $Y \in \mathfrak{g}(j)$ for some $j < 0$. By Lemma 2.1 and the hypothesis that $C(H_0)$ is a scalar matrix, we have

$$jC(Y) = C([H_0, Y]) = \Pi_1(H_0) \bullet C(Y).$$

Now e is a polar point for $\bar{\mathfrak{q}}$, and so evaluating this identity at e gives $jC(Y)(e) = 0$. We have assumed that $j < 0$ and it follows that $C(Y)(e) = 0$ for all $Y \in \mathfrak{g}(j)$. The general case follows from the linearity of C.

We continue with the assumption that D is a homogeneous system. Since $C(Y)(e) = 0$ for all $Y \in \bar{\mathfrak{n}}$, the representation $Z \mapsto C(Z)(e)$ of $\bar{\mathfrak{q}}$ factors through $\mathfrak{l}$. Because $\mathfrak{l}$ is reductive, it follows that this representation is a direct sum of irreducible subrepresentations. As in Proposition 5.3 and Corollary 5.4, each of these subrepresentations corresponds to an irreducible conformally invariant system, and D is the direct sum of these systems. □

We remark that the last conclusion of Proposition 6.1 fails if the system is not assumed to be homogeneous; that is, there are non-homogeneous, reducible, indecomposable conformally invariant systems. Indeed, such systems are very common. To sketch an example, let us temporarily adopt the setting and notation of [1]. We take $\mathfrak{g}$ to be a simple complex Lie algebra not of type A, $\mathfrak{q}$ to be a Heisenberg parabolic subalgebra of $\mathfrak{g}$, $\{Y_1, \ldots, Y_k\}$ to be a basis for the unique $\mathfrak{l}$-invariant complement to the center of $\mathfrak{n}$ in $\mathfrak{n}$, and $\mathcal{V}$ to be the restriction to N of the homogeneous line bundle corresponding to the character $s\gamma$ of $\mathfrak{l}$, where γ is the highest root in $\mathfrak{g}$. It follows from the first theorem in Section 5 of [1] that the system $D = 1, \Omega_1(Y_1), \ldots, \Omega_1(Y_k)$ is conformally invariant, as is the subsystem consisting of 1 alone. The same result easily implies that D is indecomposable unless $s = 0$. Note that D is inhomogeneous and does not annihilate any non-zero section of $\mathcal{V}$.

Let (η, V) be a finite-dimensional irreducible representation of $\bar{Q}$ and denote by $\mathcal{M}(V) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} V$ the associated generalized Verma module. Since L acts on V, it also acts on $\mathcal{M}(V)$. The action satisfies $g(u \otimes v) = \mathrm{Ad}(g)u \otimes \eta(g)v$ for $g \in L$, $u \in \mathcal{U}(\mathfrak{g})$, and $v \in V$. Note that this action is by twisted $\mathcal{U}(\mathfrak{g})$-module automorphisms; that is, if $u \in \mathcal{U}(\mathfrak{g})$, $g \in L$, and $\xi \in \mathcal{M}(V)$ then $g(u\xi) = (\mathrm{Ad}(g)u)(g\xi)$. Moreover, the action is locally finite and its derived action is simply the standard action of $\mathfrak{l} \subset \mathfrak{g}$ on $\mathcal{M}(V)$.

The space

$$\mathcal{M}(V)^{\bar{\mathfrak{n}}} = \{\xi \in \mathcal{M}(V) \mid Y\xi = 0 \text{ for all } Y \in \bar{\mathfrak{n}}\}$$

is known to be finite-dimensional. Because L normalizes $\bar{\mathfrak{n}}$ under the adjoint action, $\mathcal{M}(V)^{\bar{\mathfrak{n}}}$ is stable under the action of L. By a *leading L-type for V* we shall mean a non-zero irreducible L-submodule of $\mathcal{M}(V)^{\bar{\mathfrak{n}}}$. Since L is reductive, $\mathcal{M}(V)^{\bar{\mathfrak{n}}}$ may be decomposed as a direct sum of a finite number of leading L-types.

The subspace $\{1 \otimes v \mid v \in V\}$ of $\mathcal{M}(V)^{\bar{\mathfrak{n}}}$ is always a leading L-type, which we shall call the *canonical leading L-type.* Any leading L-type may be viewed as a representation of $\bar{Q}$ by making $\bar{N}$ act trivially on it. To each leading L-type F for V there is associated a non-zero element of the space

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g}),L}\big(\mathcal{M}(F), \mathcal{M}(V)\big) \tag{6.1}$$

given on simple tensors by $u \otimes f \mapsto uf$. The canonical leading L-type corresponds in this way to the identity map in $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g}),L}(\mathcal{M}(V), \mathcal{M}(V))$. Conversely, if F is an irreducible $\bar{Q}$-module and we are given a non-zero element of (6.1) then this gives rise to a leading L-type for V, namely the image under the given homomorphism of the canonical leading L-type in $\mathcal{M}(F)$.

In order to relate conformally invariant systems and leading L-types, we require a notion of L-invariant system. For this purpose, we begin by defining an action of L on $\mathbb{D}(\mathcal{V}_\eta)$. For $g \in L$ and $\varphi \in C^\infty(N; V)$, let $g * \varphi \in C^\infty(N; V)$ be defined by $(g * \varphi)(n) = \eta(g)\varphi(g^{-1}ng)$ for $n \in N$. One checks that this defines an action of L on $C^\infty(N; V)$. The restriction of this action to the image of $C^\infty_\eta(G; V)$ under restriction to N is simply the action of L in the smooth induced representation. In particular, the associated infinitesimal action is Π_η. For $D \in \mathbb{D}(\mathcal{V}_\eta)$, define $g * D$ by $(g * D)\bullet\varphi = g * (D\bullet(g^{-1} * \varphi))$.

Lemma 6.2. *The map $(g, D) \mapsto g * D$ is an action of L on $\mathbb{D}(\mathcal{V}_\eta)$. It enjoys the following properties:*

(1) *$g * (D_1D_2) = (g * D_1)(g * D_2)$ for all $g \in L$ and $D_1, D_2 \in \mathbb{D}(\mathcal{V}_\eta)$,*
(2) *$g * T = \eta(g) \circ T \circ \eta(g^{-1})$ for all $g \in L$ and $T \in \mathrm{End}(V)$,*
(3) *$g * R(X) = R(\mathrm{Ad}(g)X)$ for all $g \in L$ and $X \in \mathfrak{n}$.*

The subspace $\mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$ is stable under L, and the isomorphism $\Lambda \mapsto D_\Lambda$ between $\mathcal{M}(\mathrm{End}(V))$ and $\mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}}$ becomes L-intertwining if we let L act on $\mathcal{M}(\mathrm{End}(V))$ by

$$g * (u \otimes T) = (\mathrm{Ad}(g)u) \otimes \big(\eta(g) \circ T \circ \eta(g^{-1})\big).$$

Proof. The fact that $(g, D) \mapsto g * D$ is an action and identities (1), (2), and (3) follow by calculation. For $\Lambda \in \mathcal{M}(\mathrm{End}(V))$, we may represent D_Λ in the form given in Proposition 5.6. By using the properties of the action that have already been established, we conclude that $g * D_\Lambda = D_{g*\Lambda}$ for all $\Lambda \in \mathcal{M}(\mathrm{End}(V))$ and $g \in L$. This demonstrates the truth of the remaining claims. □

Note that the infinitesimal action of $\mathfrak{l}$ on $\mathcal{M}(\mathrm{End}(V))$ derived from the action of L on $\mathcal{M}(\mathrm{End}(V))$ via $*$ is *not* the one that might be expected naively. Indeed, a calculation reveals that

$$Z * (u \otimes T) = Zu \otimes T + u \otimes \big(d\eta(Z) \circ T\big)$$

for $Z \in \mathfrak{l}$. The additional term compensates for the non-standard action of $\mathfrak{l}$ on $\mathrm{End}(V)$ that we have used to define $\mathcal{M}(\mathrm{End}(V))$. As we remarked above, it seems that both the standard and non-standard actions of $\mathfrak{l}$ on $\mathrm{End}(V)$ play an unavoidable role.

Let $D_1, \ldots, D_m$ be a conformally invariant system on the bundle $\mathcal{V}_\eta \to N$. We shall say that the system is *L-stable* if there is a map $c : L \to \mathrm{GL}(n, C^\infty(N))$ such that

$$g * D_i = \sum_{j=1}^{m} c(g)_{ji} D_j \tag{6.2}$$

for all $g \in L$ and $1 \le i \le m$. As before, the map c is unique. A calculation shows that $g * (\ell_{n_0}\varphi) = \ell_{gn_0g^{-1}}(g * \varphi)$ for all $g \in L$, $n_0 \in N$, and $\varphi \in C^\infty(N; V)$. Since we are assuming that the system $D_1, \ldots, D_n$ is straight, each D_i commutes with ℓ_{n_0} for all $n_0 \in N$. By following the calculation in the proof of Proposition 5.3, we conclude that $c(g)(n) = c(g)(e)$ for all $n \in N$; that is, the entries in the matrix $c(g)$ are in fact constants. From this, we conclude that $c(gh) = c(g)c(h)$ for all $g, h \in L$, so that c is a representation of L. There are elements $\Lambda_1, \ldots, \Lambda_m$ in $\mathcal{M}(\mathrm{End}(V))$ such that $D_i = D_{\Lambda_i}$ for $1 \le i \le m$. Because the functions $c(g)_{ji}$ are constant, (6.2) is equivalent to

$$g * \Lambda_i = \sum_{j=1}^{m} c(g)_{ji} \Lambda_j \tag{6.3}$$

for $1 \le i \le m$. In particular, c is smooth and we may define

$$Z \cdot c = \frac{d}{dt} c(\exp(tZ))\Big|_{t=0}$$

for $Z \in \mathfrak{l}$, as usual. By taking the derivative on both sides of (6.3), we obtain

$$Z\Lambda_i + d\eta(Z)\Lambda_i = \sum_{j=1}^{m} (Z \cdot c)_{ji} \Lambda_j.$$

In light of the calculation made in the proof of Theorem 5.5, this is equivalent to

$$[\Pi_\eta(Z), D_i] = \sum_{j=1}^{m} (Z \cdot c)_{ji} D_j,$$

so that $Z \cdot c = C(Z)$ for all $Z \in \mathfrak{l}$. That is, the representation of L afforded by the space spanned by $D_1, \ldots, D_m$ is a globalization of the representation $Z \mapsto C(Z)$ of $\mathfrak{l}$. We may similarly show that if $D_1, \ldots, D_m$ is a conformally invariant system corresponding to $\Lambda_1, \ldots, \Lambda_m \in \mathcal{M}(\mathrm{End}(V))$, and the space spanned by the Λ_i is L-invariant under the $*$ action, then the system is L-stable.

Consideration of L-stability is necessary for a reason familiar in Lie theory: the group L need not be connected and so infinitesimal data alone fail to capture all the essential information. As usual, one could just as well replace L by any subgroup meeting each of the connected components of L.

Theorem 6.3. *Let $D_1, \ldots, D_m$ be a straight L-stable homogeneous conformally invariant system on the bundle $\mathcal{V}_\eta$, and let*

$$F = \operatorname{span}_{\mathbb{C}}\{D_i\lambda \mid \lambda \in \mathcal{V}^*_{\eta,e},\ 1 \le i \le m\}.$$

Then F is an L-invariant subspace of $\mathcal{M}(V^*)^{\bar{\mathfrak{n}}}$. Conversely, if $F \subset \mathcal{M}(V^*)^{\bar{\mathfrak{n}}}$ is a leading L-type for V^* then the conformally invariant system on $\mathcal{V}_\eta$ associated to F as in Theorem 5.5 is L-stable and homogeneous.

Proof. We first show that the homogeneity of the system $D_1, \ldots, D_m$ alone implies that $F \subset \mathcal{M}(V^*)^{\bar{\mathfrak{n}}}$. To this end, let $Y \in \bar{\mathfrak{n}}$. By Lemma 3.2, we have

$$\Pi'_\eta(Y)\bullet(D_i\lambda) = D_i\big(\Pi'_\eta(Y)\bullet\lambda\big) + \sum_{j=1}^{m} C(Y)_{ji}(e)D_j\lambda.$$

By Proposition 6.1, $C(Y)(e) = 0$, and so the second term on the right-hand side vanishes. Since $\bar{\mathfrak{n}}$ acts trivially on V, it also acts trivially on V^*, and this implies that the first term on the right-hand side vanishes also. Thus $\bar{\mathfrak{n}}$ annihilates $D_i\lambda$, as required.

Let $X_1, \ldots, X_r \in \mathfrak{g}$, $\lambda \in V^*$, $T \in \operatorname{End}(V)$, and $\varphi \in C^\infty(N; V)$. Then

$$\begin{aligned} \lambda\big((\Pi'_\eta(X_1\cdots X_r)\bullet T)(\varphi)\big) &= (-1)^r\lambda\big(T(\Pi_\eta(X_r\cdots X_1)\bullet\varphi)\big) \\ &= (-1)^r(\lambda\circ T)\big(\Pi_\eta(X_r\cdots X_1)\bullet\varphi\big) \\ &= \big(\Pi'_\eta(X_1\cdots X_r)\bullet(\lambda\circ T)\big)(\varphi) \end{aligned}$$

and hence if $u \in \mathcal{U}(\mathfrak{g})$ then

$$\lambda\big((\Pi'_\eta(u)\bullet T)(\varphi)\big) = \big(\Pi'_\eta(u)\bullet(\lambda\circ T)\big)(\varphi).$$

This identity in turn gives

$$\begin{aligned} (D_{u\otimes T}\lambda)(\varphi) &= \lambda\big((D_{u\otimes T}\bullet\varphi)(e)\big) \\ &= \lambda\big((\Pi'_\eta(u)\bullet T)(\varphi)\big) \\ &= \big(\Pi'_\eta(u)\bullet(\lambda\circ T)\big)(\varphi). \end{aligned}$$

Consequently, when each object is identified with the corresponding generalized Verma module, the map $\mathbb{D}(\mathcal{V}_\eta)^{\mathfrak{n}} \otimes_{\mathbb{C}} \mathcal{V}^*_{\eta,e} \to D'_e(\mathcal{V}_\eta)$ given by $D\otimes\lambda \mapsto D\lambda$ is identified with the map $\mathcal{M}(\operatorname{End}(V)) \otimes_{\mathbb{C}} V^* \to \mathcal{M}(V^*)$ given on simple tensors by $(u\otimes T)\otimes\lambda \mapsto u\otimes(\lambda\circ T)$. With L acting on $\mathcal{M}(\operatorname{End}(V))$ via $*$ and on V^* and $\mathcal{M}(V^*)$ in the natural way, this map is L-intertwining. It follows that if $D_1, \ldots, D_m$ is L-stable then F is L-invariant.

Now suppose that $F \subset \mathcal{M}(V^*)^{\bar{\mathfrak{n}}}$ is a leading L-type, D is the conformally invariant system associated to F as in Theorem 5.5, and C is its structure operator. Observe that since V^* and F are irreducible representations of L, H_0 acts on each by a scalar. It follows from the formula given in Theorem 5.5 that $C(H_0)$ is a scalar matrix whose diagonal entries equal the sum of these two scalars. Thus D is homogeneous. Regarded as elements of $\mathcal{M}(\operatorname{End}(V))$, the span of the operators comprising D is equal to $F \otimes_{\mathbb{C}} V \subset \mathcal{M}(\operatorname{End}(V))$. Since both F and V are L-invariant, this subspace is also L-invariant, and hence D is L-stable. $\square$

It follows from the above discussion that if we have an L-stable homogeneous conformally invariant system $D_1, \dots, D_m$ on the bundle $\mathcal{V}_\eta$ such that all D_i annihilate the constants sections of $\mathcal{V}_\eta$ then we obtain a leading L-type in $\mathcal{M}(V^*)$ other than the canonical one. In particular, we may then conclude that $\mathcal{M}(V^*)$ is reducible. All the conformally invariant systems constructed in [1] satisfy these hypotheses, and this justifies the claims regarding reducibility of generalized Verma modules and existence of leading L-types made in the introduction to [1].

We remark that the results obtained above relating leading L-types and L-stable homogeneous systems could also be obtained purely on the Lie algebra level. That is, one could define *leading* $\mathfrak{l}$*-types* and show that they are related to homogeneous systems in a similar way. No additional hypothesis of $\mathfrak{l}$*-stability* would be necessary in this development, because the requisite stability is already implied by conformality. We have chosen the above framework because it appears to be the most appropriate for applications. However, we shall make use of this remark in Section 7.

In Section 6 of [1], we constructed several differential operators, denoted by $\Omega_4(\mathcal{C})$, for each simple algebra $\mathfrak{g}$. These operators are all conformally invariant on a certain line bundle $\mathcal{L} \to N$. Let us choose a representation (η, V) of $\bar{Q}$ such that $\mathcal{V}_\eta = \mathcal{L}$, and note that the dimension of V is one. Fix a particular $\mathfrak{g}$. Then, as may be deduced from the results given in Section 6 of [1], the leading L-types corresponding to the various $\Omega_4(\mathcal{C})$ are all isomorphic to one another as L-representations. Let F be a representative of the isomorphism class containing these leading L-types, and note that the dimension of F is one. In accordance with the theory developed above, from each $\Omega_4(\mathcal{C})$ we obtain a non-zero element of the space

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g}),L}\big(\mathcal{M}(F), \mathcal{M}(V^*)\big).$$

However, because F and V^* both have dimension one, it is known that this space of homomorphisms has dimension at most one. Consequently, as claimed at the end of Section 6 of [1], the various $\Omega_4(\mathcal{C})$ associated to a particular $\mathfrak{g}$ are all proportional to one another. This verifies the last outstanding claim made in [1].

7. Scalar Generalized Verma Modules

If, in the setting established in Section 6, we additionally assume that the representation (η, V) is one-dimensional then significant simplifications occur. The purpose of this section is to obtain further information under this additional hypothesis, and to apply it to resolve a question left open in [1]. We continue with the notation and assumptions made in Section 6. For convenience, we also assume that G is semisimple.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$, R be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and W be the Weyl group of R. Let $\langle \cdot\,, \cdot \rangle$ be a positive multiple of the inner product induced on $\mathfrak{h}^*$ by the Killing form. Choose a positive system $R^+ \subset R$ and assume that the parabolic subalgebra $\mathfrak{q}$ and Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ are standard with respect to R^+. Let ρ denote half the sum of the positive roots in $\mathfrak{g}$. Let $\rho(\mathfrak{n})$ denote half the sum of the positive roots in $\mathfrak{n}$, and similarly for other $\mathrm{ad}(\mathfrak{h})$-stable subalgebras of $\mathfrak{g}$. Note that $\rho(\mathfrak{l}) = \rho - \rho(\mathfrak{n})$.

Fix a $\gamma \in \mathfrak{h}^*$ that is orthogonal to all the simple roots in $\mathfrak{l}$. Then γ is trivial on $[\mathfrak{l}, \mathfrak{l}]$ and so it extends to a character of $\mathfrak{l}$. For $s \in \mathbb{C}$, we denote by $\mathbb{C}_{s\gamma}$ the one-dimensional representation of $\mathfrak{l}$ associated to the character $s\gamma$. In keeping with the conventions of [1], we consider an L-stable irreducible conformally invariant system D on the line bundle $\mathcal{L}_{-s\gamma} \to N$ associated to the representation $V = \mathbb{C}_{-s\gamma}$. Let C be the structure operator of D. Then $Z \mapsto C(Z)$ is a representation of $\mathfrak{l}$ on $\mathbb{C}^m$, where m denotes the number of operators in the system, and by Corollary 5.4 this representation is irreducible. Let us denote this irreducible representation by E and by $\varpi \in \mathfrak{h}^*$ the highest weight of E. The system D gives rise to a leading $\mathfrak{l}$-type $F \subset \mathcal{M}(\mathbb{C}_{s\gamma})^{\bar{\mathfrak{n}}}$ and, by the observation made at the end of Section 3, we have $F \cong E \otimes \mathbb{C}_{s\gamma}$. Thus the highest weight of F is $\varpi + s\gamma$. Now, as we observed in Section 6, the existence of this leading $\mathfrak{l}$-type implies that there is a non-zero homomorphism from $\mathcal{M}(F)$ to $\mathcal{M}(\mathbb{C}_{s\gamma})$, and it follows that these two generalized Verma modules must have the same infinitesimal character. We regard the infinitesimal characters as elements of $\mathfrak{h}^*/W$. Since we form our Verma modules by tensoring over $\mathcal{U}(\bar{\mathfrak{q}})$, the infinitesimal character of $\mathcal{M}(F)$ is represented by

$$\varpi + s\gamma + \rho(\mathfrak{l}) - \rho(\mathfrak{n}) = \varpi + s\gamma + \rho - 2\rho(\mathfrak{n})$$

and the infinitesimal character of $\mathcal{M}(\mathbb{C}_{s\gamma})$ is represented by

$$s\gamma + \rho(\mathfrak{l}) - \rho(\mathfrak{n}) = s\gamma + \rho - 2\rho(\mathfrak{n}).$$

It follows that we must have

$$\varpi + s\gamma + \rho - 2\rho(\mathfrak{n}) = w\big(s\gamma + \rho - 2\rho(\mathfrak{n})\big) \tag{7.1}$$

for some $w \in W$. By computing the squared length of both sides of (7.1), we obtain

$$s\langle \varpi, \gamma\rangle = 2\langle \varpi, \rho(\mathfrak{n})\rangle - \langle \varpi, \rho\rangle - \frac{1}{2}\|\varpi\|^2. \tag{7.2}$$

To summarize, if D is an irreducible conformally invariant system on the line bundle $\mathcal{L}_{-s\gamma} \to N$, and the irreducible representation of $\mathfrak{l}$ arising from the structure operator of D has highest weight ϖ then (7.1) and (7.2) must hold.

We wish to apply the above conclusions to resolve a question left open in [1]. As before, the reader will need to be familiar with the notation used in [1] in order to follow the details of the argument, but the general idea can be easily stated. In [1], we constructed a conformally invariant system called Ω_3 on the line bundle $\mathcal{L}_{-s\gamma} \to N$ for each exceptional simple algebra and a suitable value of s, the parabolic subalgebra being the Heisenberg parabolic in all cases. The corresponding system on the symplectic algebras is known to be identically zero, but the question of whether an analogue of Ω_3 exists on the non-symplectic classical algebras was left open. This accounts for the question marks that disfigure the table of conformally invariant systems given at the end of [1]. Now the conformally invariant systems studied in [1] are constructed by beginning with a specific representation of L that should serve as the representation E, and so the highest weight ϖ that should correspond to Ω_3 is known. This may be used to apply the conditions (7.1) and (7.2) to study whether or not Ω_3 exists. We state the conclusion of this investigation as a theorem, although we shall not provide a complete proof.

Theorem 7.1. *The system Ω_3 exists for the simple algebras of types A_2 and D_4. It does not exist for the simple algebras of types A_r with $r \geq 3$, B_r with $r \geq 3$, nor D_r with $r \geq 5$.*

Proof. We discuss the case of D_r with $r \geq 4$. The other cases that have to be considered are similar, and all the necessary data may be found in [1], particularly in the compendium that appears as Section 8 of that work. We use the standard model for the root system D_r inside $\mathbb{R}^r$, in which the positive roots are $\varepsilon_i \pm \varepsilon_j$ with $1 \leq i < j \leq r$. For the situation considered in [1] and the system Ω_3, the essential data have the following values:

$$\begin{aligned} \gamma &= \varepsilon_1 + \varepsilon_2, \\ \varpi &= 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ \rho &= \sum_{i=1}^{r} (r-i)\varepsilon_i, \\ \rho(\mathfrak{n}) &= (r - 3/2)\gamma. \end{aligned}$$

From this data and (7.2), one finds that $s = (2r-5)/3$. Thus

$$\varpi + s\gamma + \rho - 2\rho(\mathfrak{n}) = \big(\frac{7-r}{3}, \frac{1-r}{3}, r-2, r-4, r-5, \ldots, 1, 0\big)$$

and

$$s\gamma + \rho - 2\rho(\mathfrak{n}) = \big(\frac{1-r}{3}, \frac{-2-r}{3}, r-3, r-4, r-5, \ldots, 1, 0\big).$$

The Weyl group in this instance acts by permutation of the coordinates and an even number of sign changes. It is required that the two displayed vectors are in the same orbit under the action of this group. This condition holds for $r = 4$ and fails for $5 \leq r \leq 7$ by direct verification. Now suppose that $r \geq 8$. After making two sign changes in each displayed vector, we render all the coordinates non-negative with precisely one zero coordinate in each vector. The sets of coordinates in these non-negative vectors must be equal, and this yields

$$\{(r-7)/3, r-2\} = \{(r+2)/3, r-3\}.$$

By solving the resulting linear equations, this condition turns out to be unsatisfiable. The conclusion of the argument so far is that Ω_3 does not exist for the algebra of type D_r with $r \geq 5$. As the reader of [1] might suspect, there does not seem to be a simple way to verify that Ω_3 does exist for the algebra of type D_4. The only way known to the authors at present is to go through the proof given for the existence of Ω_3 for the exceptional algebras in Section 6 of [1] and verify that it also succeeds for D_4. The crucial points are that the constants $c(\mathfrak{g}, \mathcal{C})$ have the same value for all three components of the deleted Dynkin diagram in this case and that the center of $\mathfrak{l}$ has dimension one. □

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