A generalisation of the deformation variety

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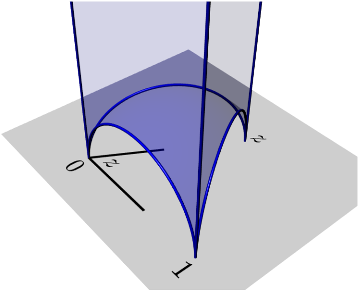
The deformation variety



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Around an edge of the triangulation $\mathcal{T}$ of a 3-manifold M with torus boundary, the product of the complex dihedral angles must be 1 (gluing equation for this edge).

These conditions give us polynomials in the complex angles. The set of such solutions is the deformation variety, $\mathfrak{D}(M; \mathcal{T})$.

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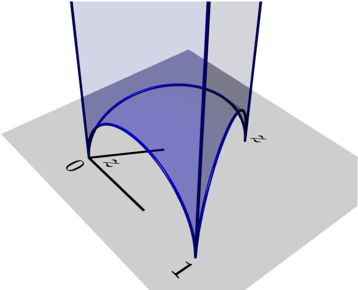

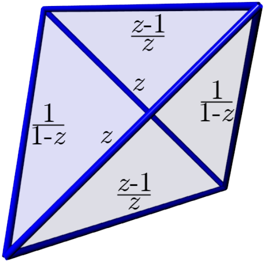

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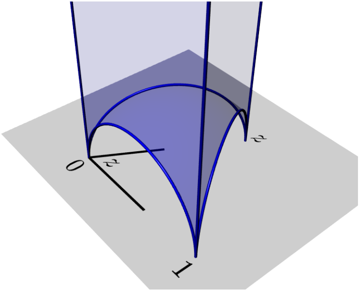

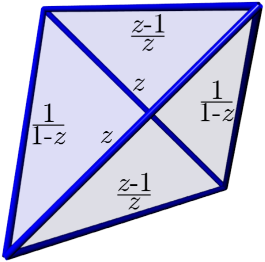

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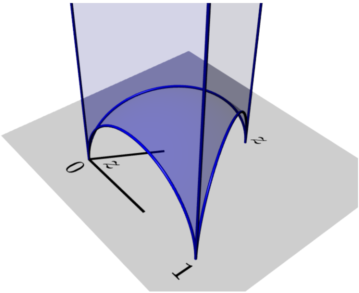

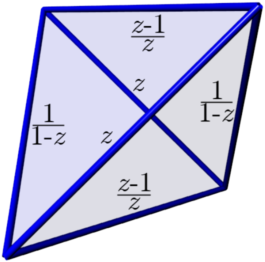

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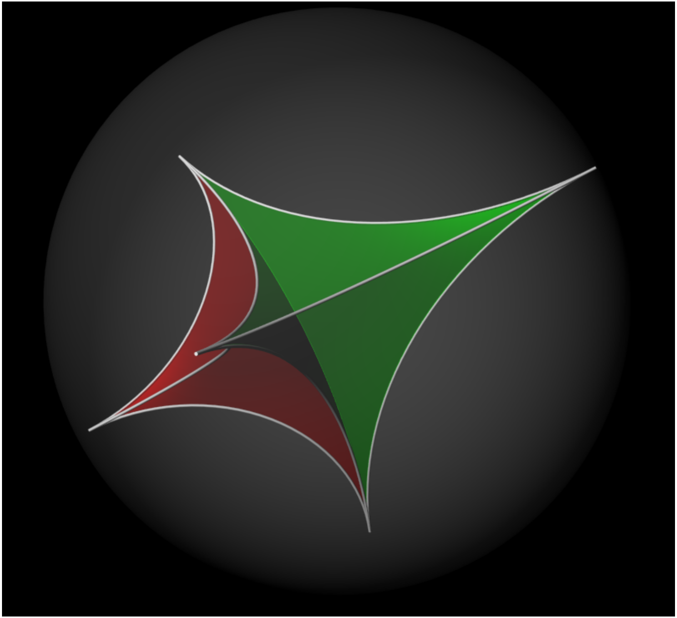

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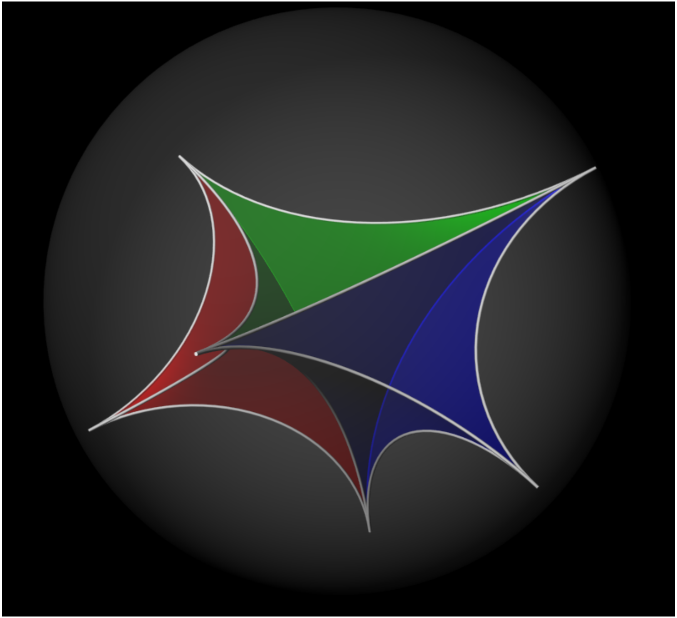

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Let $\widetilde{\mathcal{V}}$ be the set of cusps of $\widetilde{M}$. Given a point $Z \in \mathfrak{D}(M; \mathcal{T})$ the developing map is

$$\Phi_Z : \widetilde{\mathcal{V}} \to \partial\mathbb{H}^3$$

Defined by developing paths of tetrahedra into $\mathbb{H}^3$ and tracking their vertices on $\partial\mathbb{H}^3$, starting from some initial triangle T_0 of $\widetilde{\mathcal{T}}$, the lift of $\mathcal{T}$ to $\widetilde{M}$.

Doesn't depend on the choice of path of tetrahedra because of the gluing equations.

Some tetrahedra in $\mathbb{H}^3$ can end up flat or negatively oriented, but never such that two vertices are in the same place on $\partial\mathbb{H}^3$ (degenerate).

Φ_Z gives us a representation $\pi_1 M \to \mathrm{Isom}(\mathbb{H}^3)$ and so a (generally incomplete) hyperbolic structure.

So we have a map $\mathcal{R}_{\mathcal{T}} : \mathfrak{D}(M; \mathcal{T}) \to \mathcal{R}(M)$.

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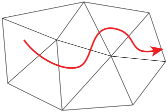

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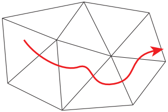

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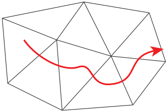

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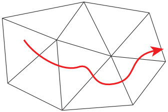

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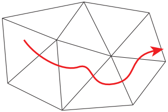

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Question: Given a representation $\rho \in \mathcal{R}(M)$, is there a $Z \in \mathfrak{D}(M; \mathcal{T})$ such that $\mathcal{R}_{\mathcal{T}}(Z) = \rho$ (up to conjugation)?

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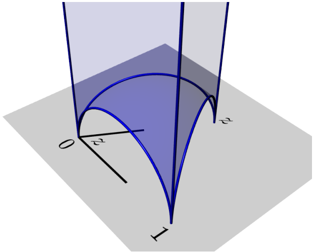
Answer: Not in general.

Given a representation ρ, we can define a map $\Psi_\rho : \widetilde{\mathcal{V}} \to \partial\mathbb{H}^3$ which says where the cusps go on $\partial\mathbb{H}^3$.

We would like to just read off the cross ratios for each tetrahedron from this map. The problem is that edges of the triangulation may connect two vertices that are in the same place on $\partial\mathbb{H}^3$, and this forces the tetrahedra containing that edge to be degenerate.



The problem

There are examples of manifolds for which entire components of the representation variety cannot be "seen" by the deformation variety with one triangulation (but which can be seen with a different triangulation).

Suppose then that we have a map

$$\Psi_\rho : \widetilde{\mathcal{V}} \to \partial\mathbb{H}^3$$

and for a triangulation $\mathcal{T}$, some tetrahedra are degenerate under this map, so that $\mathfrak{D}(M;\mathcal{T})$ will not 'see' ρ.

How can we describe ρ using shapes of tetrahedra?

Why should we care?

We want to use triangulations to talk about arbitrary incomplete hyperbolic structures without worrying about the triangulation.

We can use this for investigating:

- Incompressible surfaces (via spun-normal surfaces)
- The A-polynomial
- Components of the character variety

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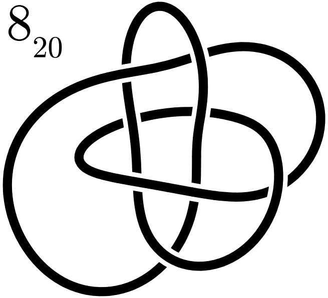
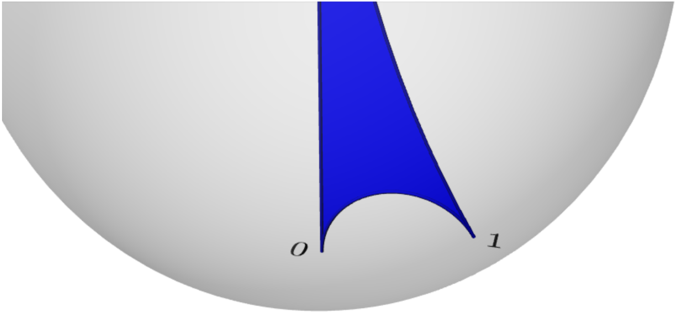


The Solution: Dealing with zero-length edges



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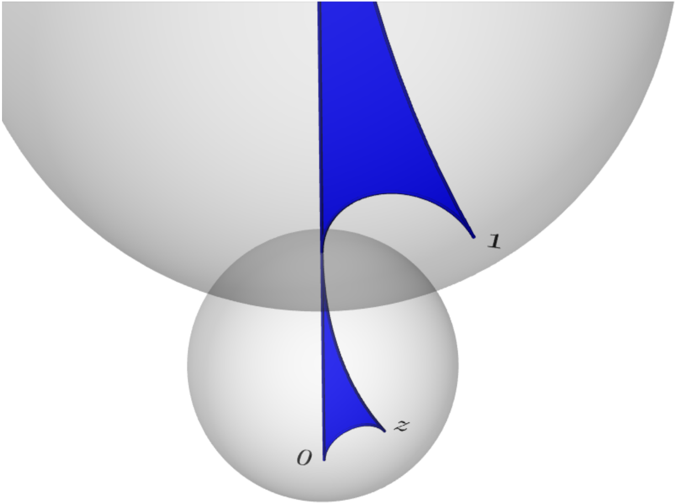

Use 'infinitesimal' cross ratios, or better, work with Laurent series in some variable ζ.

Zero-length edges and horo-normal surfaces

Start with a representation ρ and a choice of Ψ_ρ.

Let $\mathcal{E}$ be the edge set of $\mathcal{T}$. Divide $\mathcal{E}$ into $\mathcal{E}_0$ and $\mathcal{E}_+$, respectively the edges of zero and non-zero length under Ψ_ρ.

We put pieces of normal surface into each tetrahedron in such a way that each edge of $\mathcal{E}_+$ intersects two pieces of surface and each edge of $\mathcal{E}_0$ intersects none.

These connect up to form a *horo-normal* surface S.

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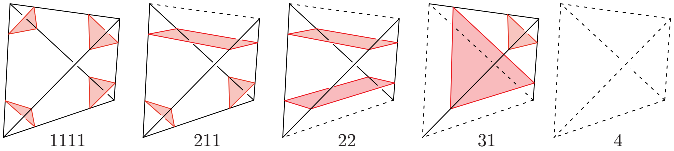

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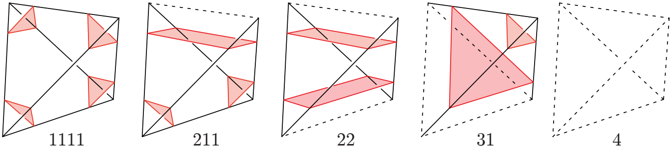

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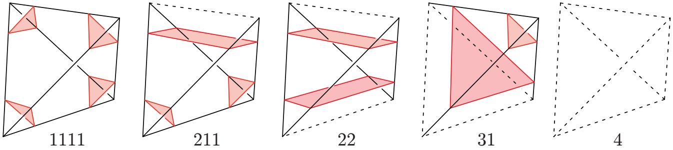

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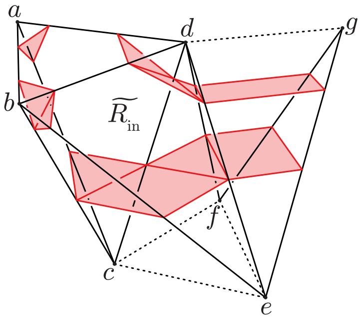
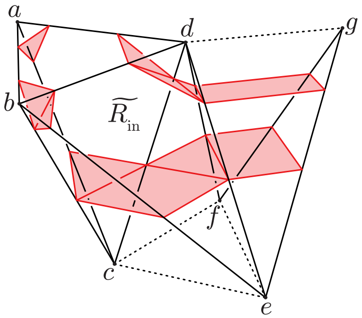
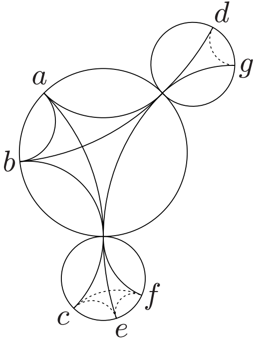


S cuts M into an inside region R_{in} and an outside region R_{out} containing the cusp.

Lift everything to $\widetilde{M}$, then collapse each lift of the surface to a point. We get a central region corresponding to $\widetilde{R_{\text{in}}}$ and many outer regions, assuming $\widetilde{R_{\text{in}}}$ is connected in $\widetilde{M}$.

We want to develop into this extended structure, a tree of $\mathbb{H}^3$s, not just one $\mathbb{H}^3$.





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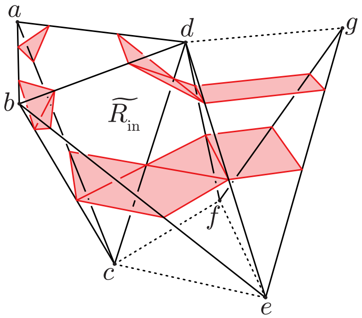

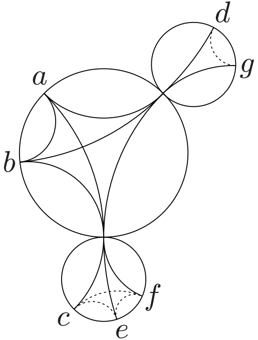

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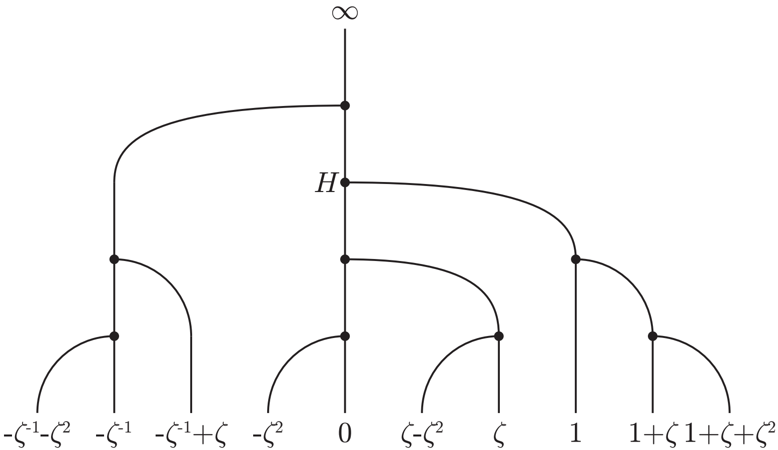

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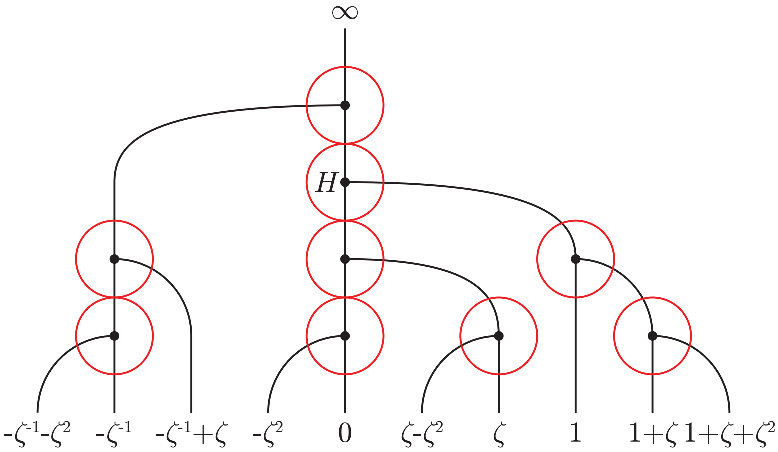

We develop in the same way: given $a, b, c \in \mathbb{C}((\zeta)) \cup \{\infty\}$ and knowing that the cross ratio (dihedral angle) z of (a, b, c, d) is given for this tetrahedron by

$$z = \frac{(a-c)(b-d)}{(a-d)(b-c)} \in \mathbb{C}((\zeta)) \setminus \{0, 1\}$$

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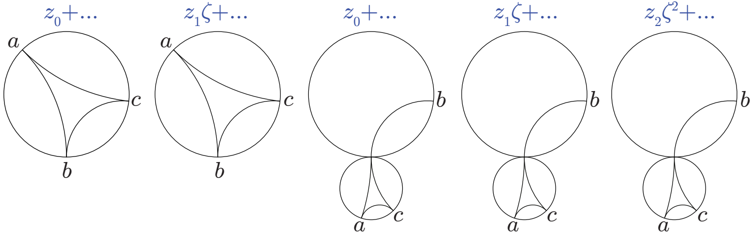

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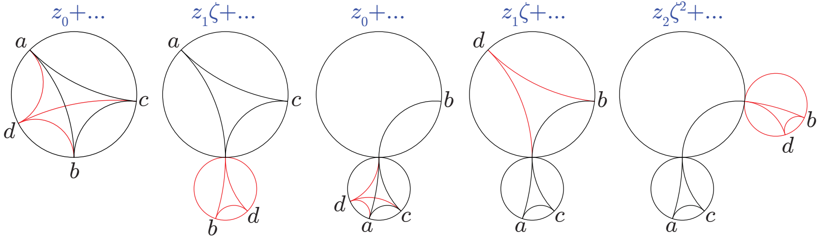

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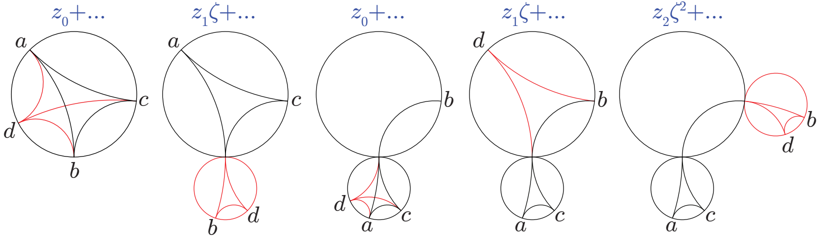

Key technical lemma: We can determine lowest order position data for developed cusps from lowest order cross ratio data, as long as we don't develop through entirely degenerate triangles.

The extended deformation variety

$\widehat{\mathfrak{D}}(M; \mathcal{T}; S)$, the extended deformation variety of M with triangulation $\mathcal{T}$ and horo-normal surface S is an affine algebraic variety whose points encode lowest order cross-ratio data for the dihedral angles of each tetrahedron of $\mathcal{T}$.

The powers of ζ on these cross ratios are given by S.

The cross-ratios must satisfy the consistent development condition, meaning that if we develop through two different paths of tetrahedra that contiguously contain $\widetilde{R_{\text{in}}}$ out to a cusp, then the positions we obtain must match *at the* ζ^0 *term*.

This condition imposes a finite number of polynomial conditions on the cross-ratios, making it a variety.

The $\widetilde{R_{\text{in}}}$ derived from S must be connected (otherwise there would be places we cannot develop to).

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The cross-ratios must satisfy the consistent development condition, meaning that if we develop through two different paths of tetrahedra that contiguously contain $\widetilde{R_{\text{in}}}$ out to a cusp, then the positions we obtain must match *at the* ζ^0 *term*.

This condition imposes a finite number of polynomial conditions on the cross-ratios, making it a variety.

The $\widetilde{R_{\text{in}}}$ derived from S must be connected (otherwise there would be places we cannot develop to).

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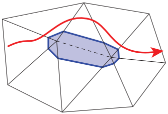

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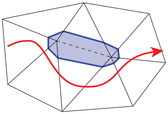

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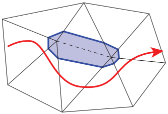

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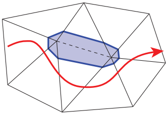

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Once again, given a point $Z \in \widehat{\mathfrak{D}}(M;\mathcal{T};S)$ we have the developing map

$$\Phi_Z : \widetilde{\mathcal{V}} \to \partial\mathbb{H}^3$$

by forgetting all but the ζ^0 information, from which we get $\mathcal{R}_{(\mathcal{T},S)} : \widehat{\mathfrak{D}}(M;\mathcal{T};S) \to \mathcal{R}(M)$.

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Let M be a connected 3-manifold with non-empty boundary consisting of a disjoint union of tori and suppose that M admits an ideal triangulation. Then there exists an ideal triangulation $\mathcal{T}_*$ of M such that for every irreducible $\rho \in \mathfrak{R}(M)$ such that $\rho(\pi_1 M)$ is not a generalised dihedral group, ρ is in the image under $\mathfrak{R}_{\mathcal{T}_*}$ of $\widehat{\mathfrak{D}}(M;\mathcal{T}_*)$, up to conjugation.

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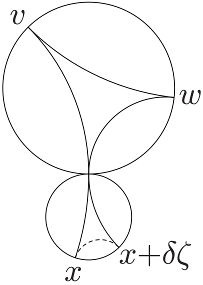
Sketch of proof

Given ρ, we need to construct S and cross-ratios Z_ρ so that $\Phi_{Z_\rho}(\widetilde{\mathcal{V}}) = \Psi_\rho(\widetilde{\mathcal{V}})$, and so translate ρ into dihedral angles.

If Ψ_ρ is 1-1 then no points are in the same place, each quadruple of points gives a non-degenerate tetrahedron and we can read off the cross-ratio.



If not... Ψ_ρ determines the zero$^{\text{th}}$ order positions of points. To determine the higher order information:

For each $e_i \in \mathcal{E}_0$, arbitrarily choose a lift $\widetilde{e}_i \in \widetilde{\mathcal{E}}_0$ and an *offset* $\delta_i \in \mathbb{C} \setminus \{0\}$.

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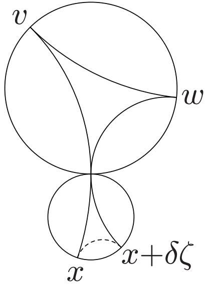

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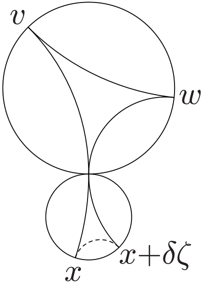

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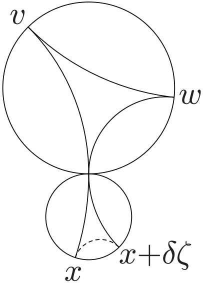


$\pi_1 M$ acts on the edges $\widetilde{\mathcal{E}}_0$ and we extend the assignment of the δ_i everywhere using ρ, so that the lowest order of cross-ratios is preserved.

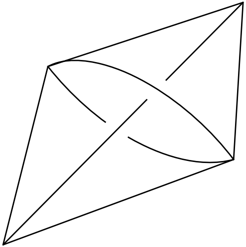
$$z \approx \frac{(\delta\zeta)(v-w)}{(x-w)(v-x)}$$

Now, for each tetrahedron, read off cross ratios for the dihedral angles (some will be degenerate, having ζ's).

- This determines an equivariant set of cross ratios for Z_ρ.
- Φ_{Z_ρ} has the same image as Ψ_ρ...
- ...as long as we can develop through $\widetilde{R_{\text{in}}}$ to every vertex of $\widetilde{\mathcal{V}}$.
- Assuming that Φ_{Z_ρ} has the same image as Ψ_ρ and has at least 3 points, then we can recover ρ from the image.

Retriangulating

Given ρ we get a set of zero length edges $\mathcal{E}_0$, and then a horo-normal surface S. If $\widetilde{R_{\text{in}}}$ is not connected, then we need to change the triangulation in order to make it connected. In fact we can "fix" all of the surfaces at once.



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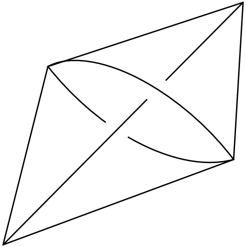

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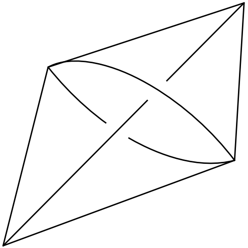

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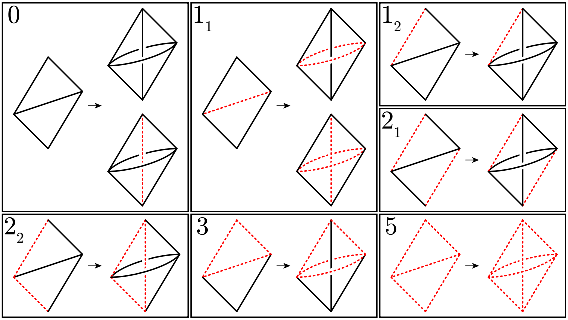
This adds edges, and so alters $\mathcal{E}_0$ and the corresponding surface, increasing the connectivity.



Further directions

- Compactify $\widehat{\mathfrak{D}}(M; \mathcal{T})$. How do we interpret ideal points of this variety?
- How much of the Culler-Shalen machinery can we reproduce in the context of triangulations?
- Generalise other things that people do with triangulations that can now be done more generally, e.g. a version of the A-polynomial.
- New tool for investigating components of the representation variety other than the Dehn surgery component.