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Extending the deformation variety

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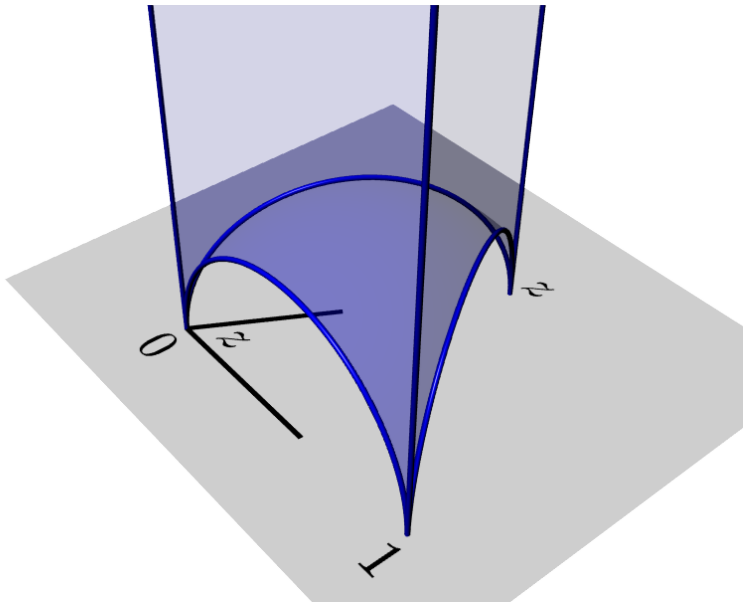

The shape of an ideal tetrahedron embedded in $\mathbb{H}^3$ is determined by a single complex dihedral angle (aka cross ratio) $z \in \mathbb{C} \setminus \{0, 1\}$.

Around an edge of the triangulation $\mathcal{T}$ of M the product of the complex dihedral angles must be 1 (gluing equation for this edge).

These conditions give us polynomials in the complex angles. The set of such solutions is the deformation variety, $\mathcal{D}(M; \mathcal{T})$.

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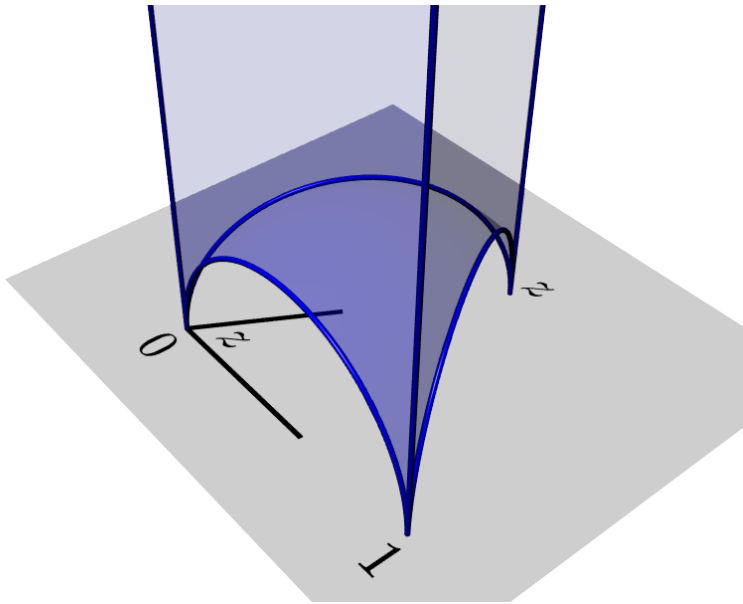

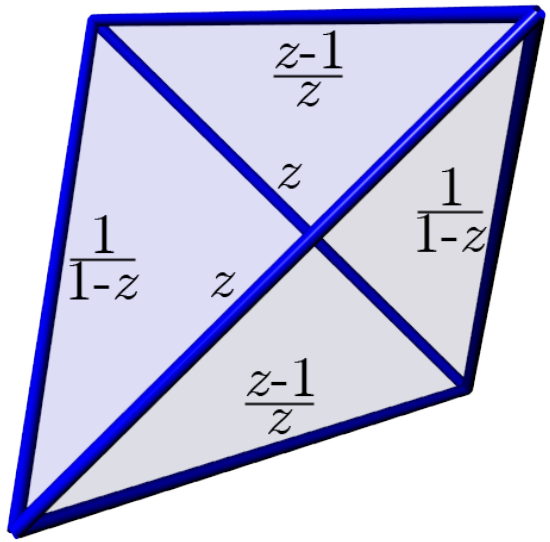

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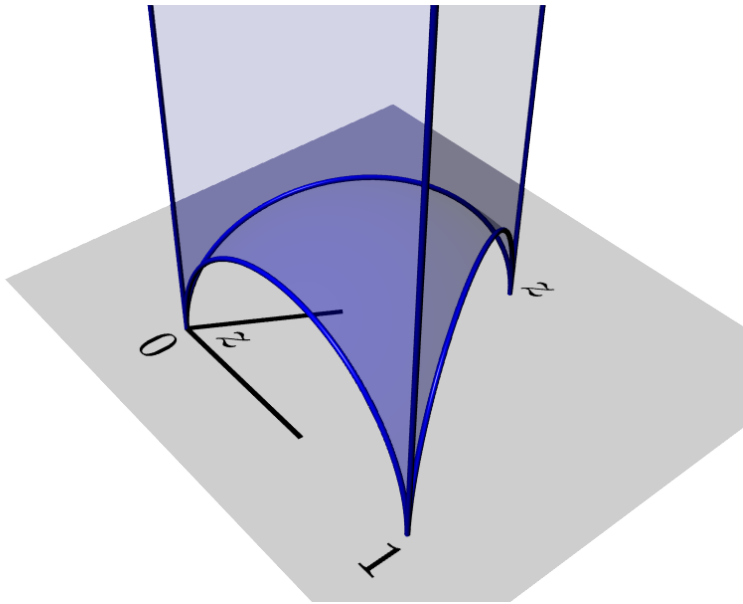

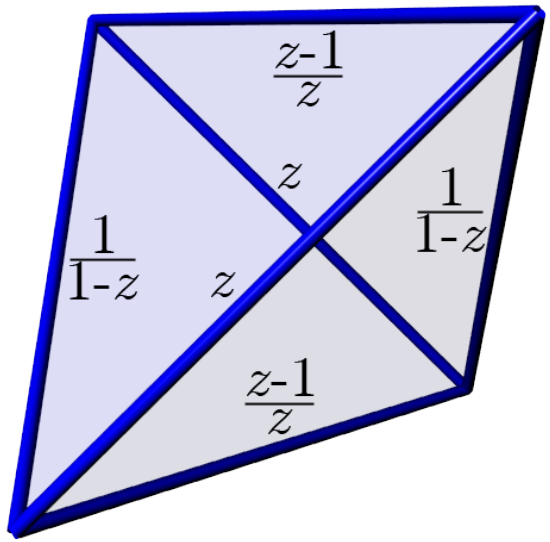

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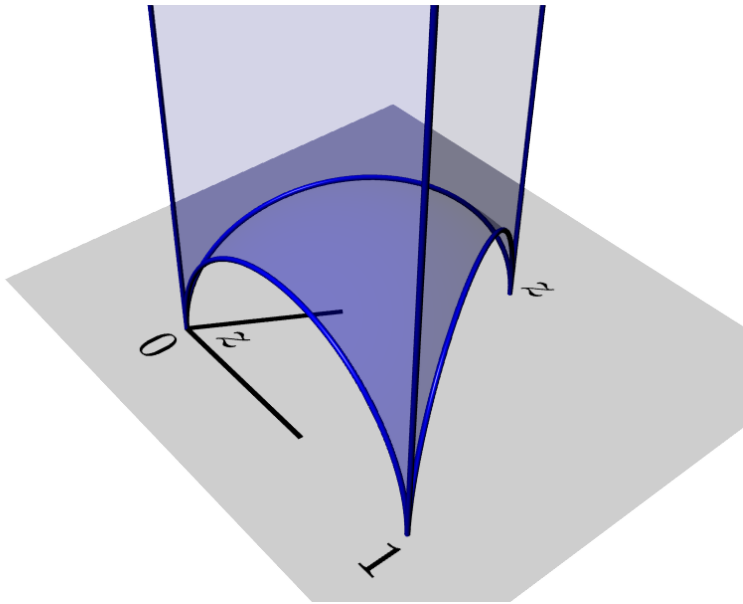

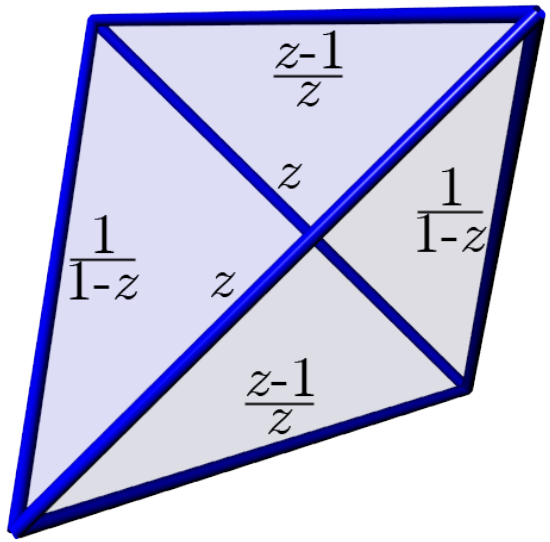

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The Developing Map

Given a point $p \in \mathfrak{D}(M; \mathcal{T})$ the developing map is

$$\Phi_p : \widetilde{M} \to \mathbb{H}^3$$

Defined by developing paths of tetrahedra into $\mathbb{H}^3$ by tracking their vertices on $\partial\mathbb{H}^3$, starting from some initial triangle T_0 of $\widetilde{\mathcal{T}}$ ($\widetilde{\mathcal{T}}$ is the lift of $\mathcal{T}$ to $\widetilde{M}$).

- Defined up to conjugation, both in the choice of images of vertices of T_0, and choice of T_0 among the triangles of $\widetilde{\mathcal{T}}$.
- Doesn't depend on the choice of path of tetrahedra because of the gluing equations.
- Some tetrahedra in $\mathbb{H}^3$ can end up flat or negatively oriented, but never such that two vertices are in the same place on $\partial\mathbb{H}^3$ (degenerate).
- Φ_p gives us a representation into $\mathrm{Isom}(\mathbb{H}^3)$ and so a (generally incomplete) hyperbolic structure.

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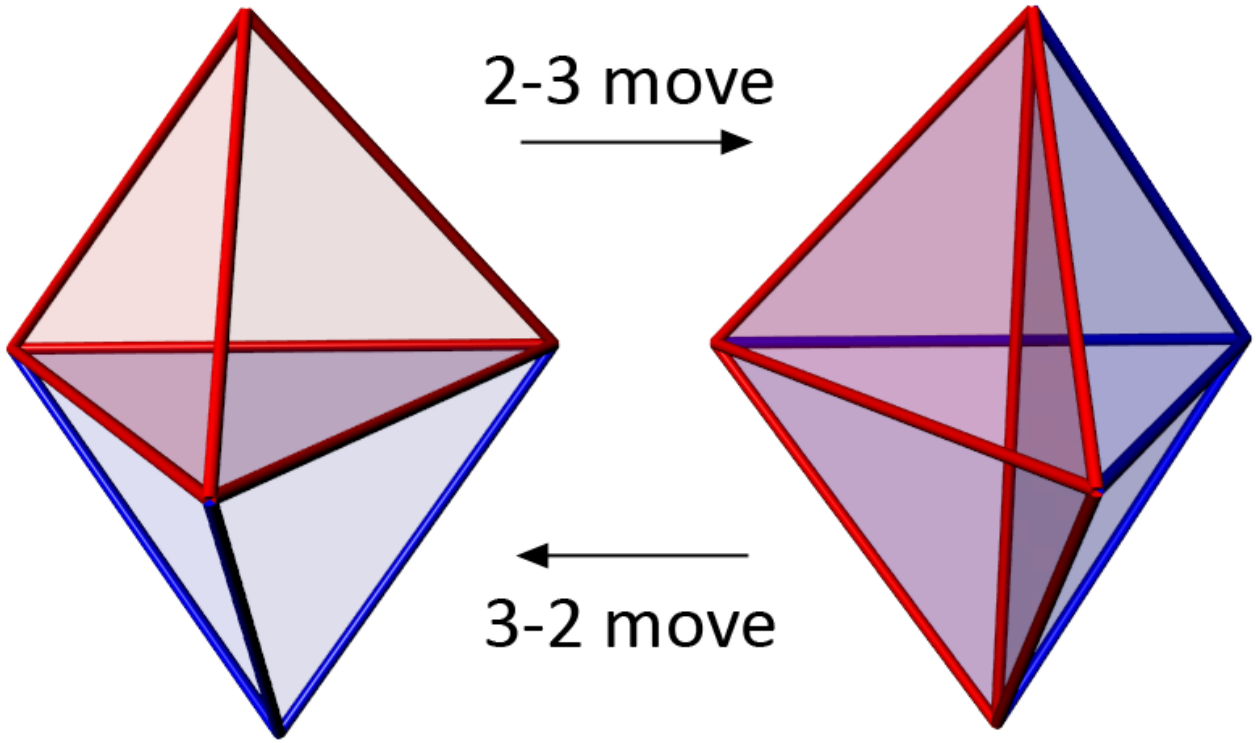

Theorem (Matveev) Any two ideal triangulations of a given manifold M are connected by a sequence of 2-3 and 3-2 moves.

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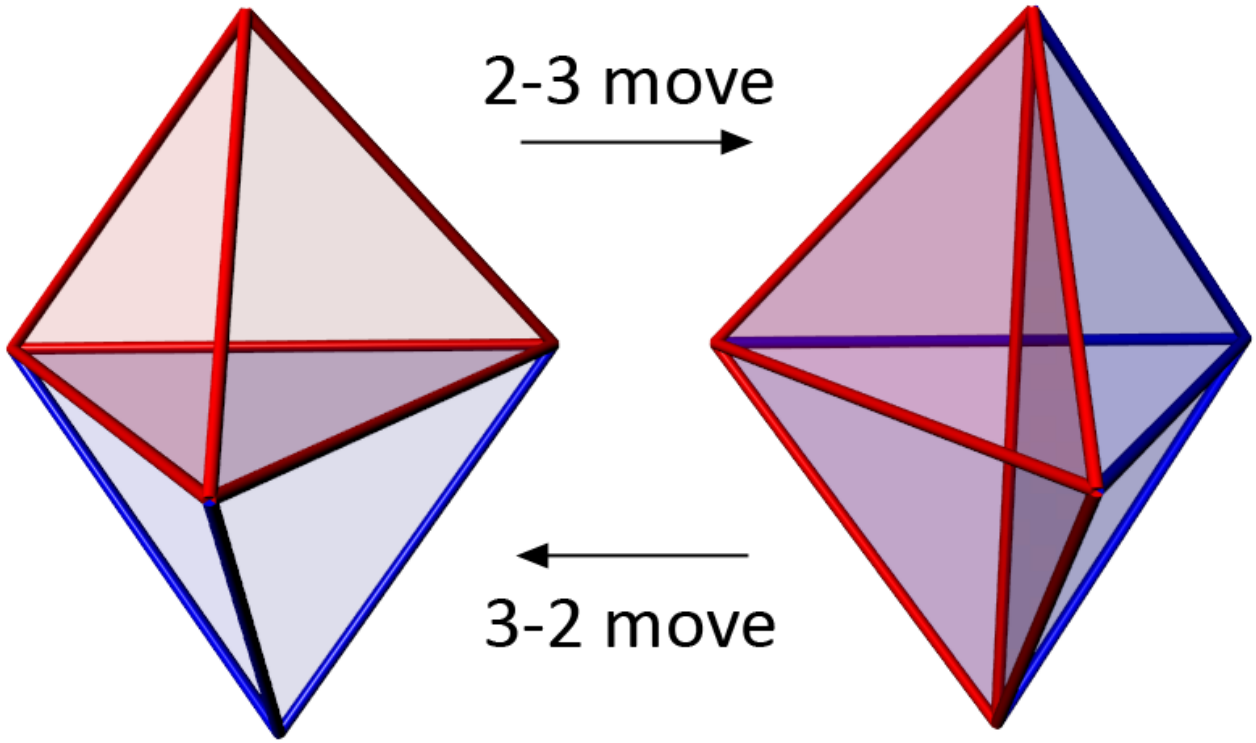

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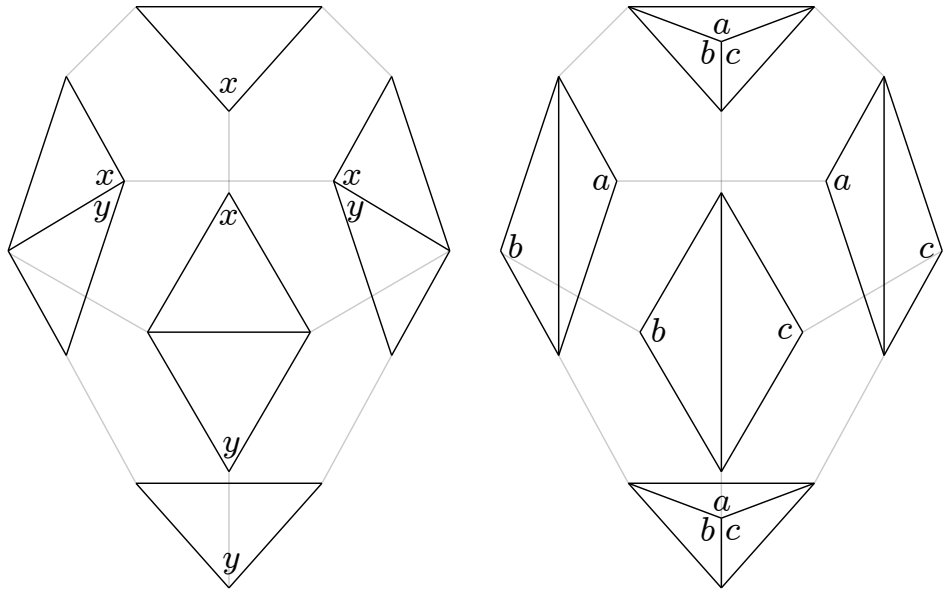
What do these moves do to the deformation variety?

How are $\mathfrak{D}(M;\mathcal{T}_2)$ and $\mathfrak{D}(M;\mathcal{T}_3)$ related?
Outside of the six sided polyhedron, nothing changes...



$$a = \frac{y-1}{y(1-x)}$$
$$b = \frac{x-1}{x(1-y)}$$
$$c = \frac{1}{ab}$$

$$x = \frac{a-1}{a(1-b)}$$
$$y = \frac{b-1}{b(1-a)}$$

Everything translates unless $x = 1/y$, which corresponds to the top and bottom vertices of the two tetrahedra developing to the same place on $\partial\mathbb{H}^3$. The corresponding three tetrahedra are all degenerate.

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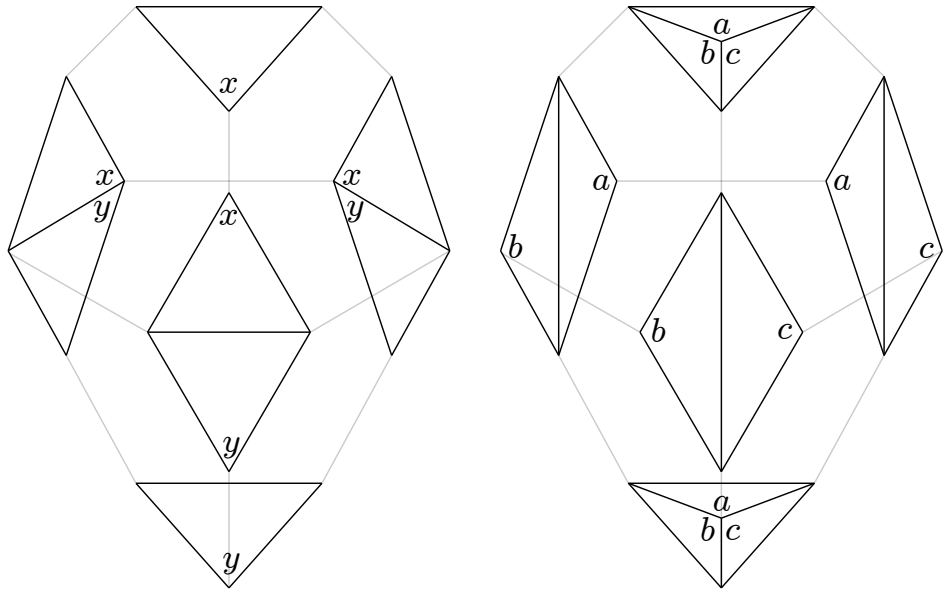

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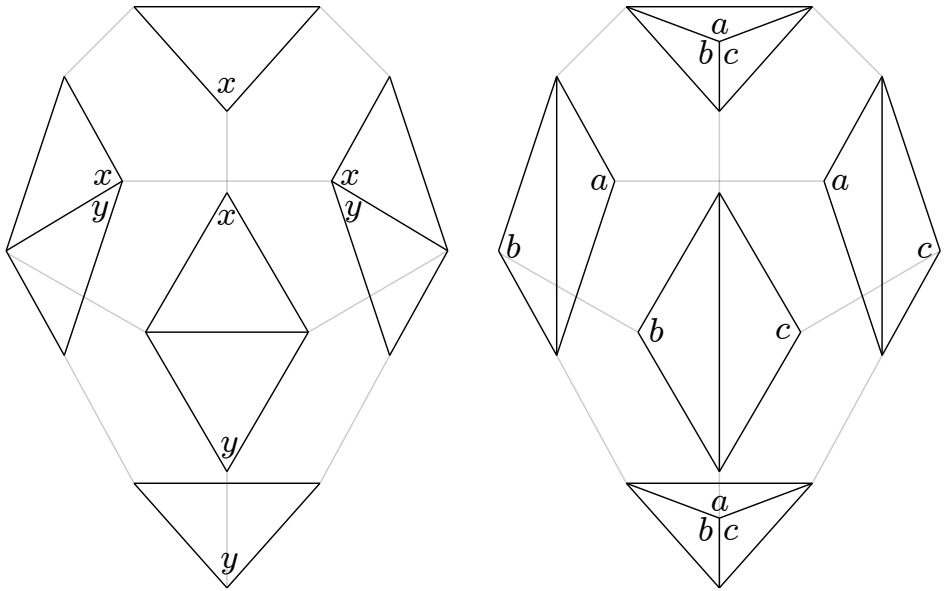

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An ideal point of $\mathfrak{D}(M;\mathcal{T})$ is a limit point of $\mathfrak{D}(M;\mathcal{T})$ such that at least one complex angle of a tetrahedron converges to 0, 1 or ∞.

1. An isolated point of $\mathfrak{D}(M;\mathcal{T}_2)$ for which "$x = 1/y$" corresponds to an ideal point of $\mathfrak{D}(M;\mathcal{T}_3)$.
2. If an entire component of $\mathfrak{D}(M;\mathcal{T}_2)$ has "$x = 1/y$" then there is no corresponding component of $\mathfrak{D}(M;\mathcal{T}_3)$.

The developing map Φ_p gives a representation $\rho_p : \pi_1 M \to \mathrm{Isom}(\mathbb{H}^3)$. $\mathfrak{D}(M;\mathcal{T}_2)$ can give some representations that $\mathfrak{D}(M;\mathcal{T}_3)$ does not.

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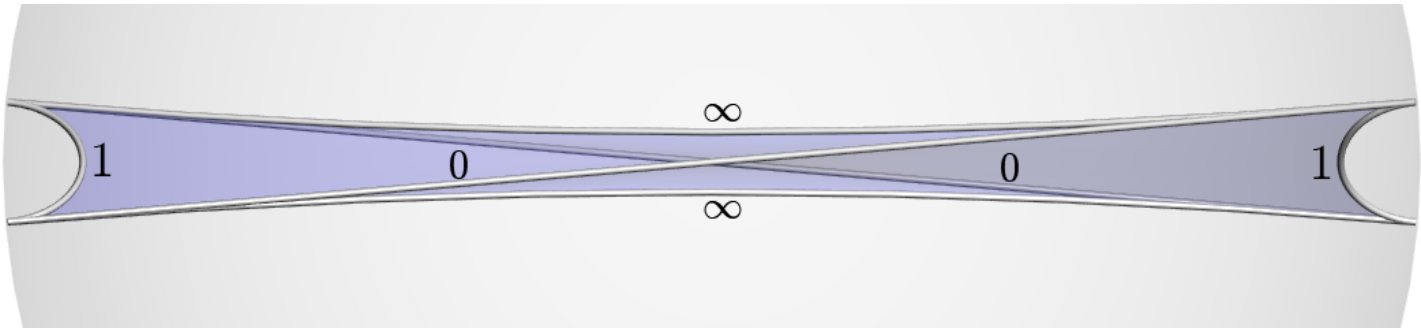

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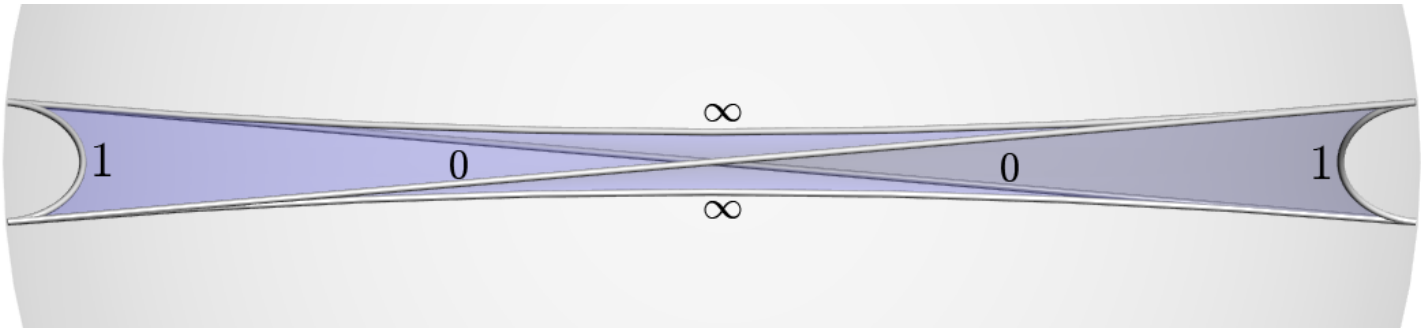

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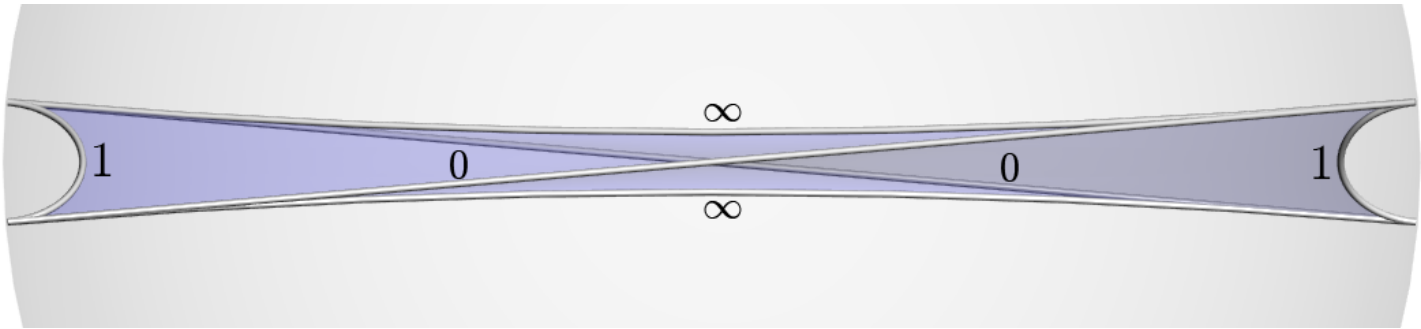

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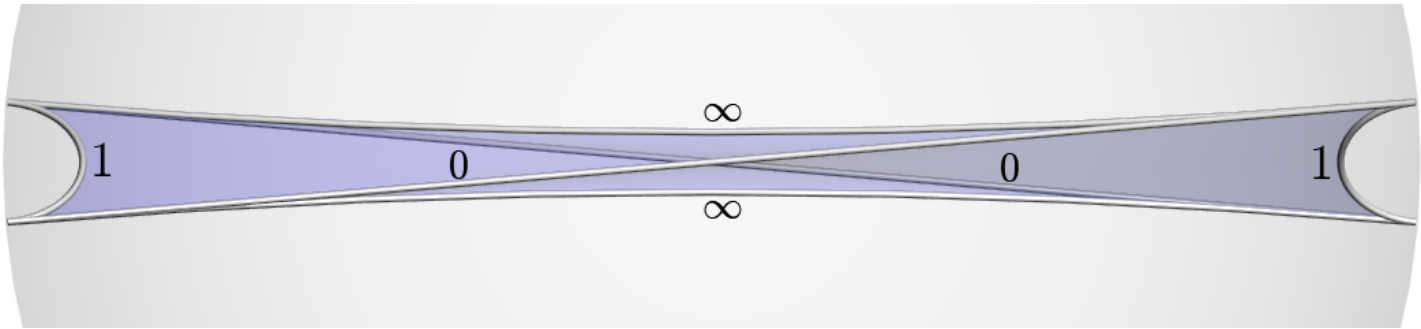

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This behaviour happens with very simple manifolds and triangulations: e.g. the punctured torus bundle with monodromy LLR has triangulations $\mathcal{T}$ and $\mathcal{T}'$ with 4 and 5 tetrahedra respectively such that $\mathcal{T}$ has an entire component with "$x = 1/y$", and to which there is no corresponding component for $\mathcal{T}'$.

We would like to be able to define some extension of the deformation variety which deals with degenerate tetrahedra:

- How do we define a developing map when some tetrahedra are degenerate?
- How do we compactify $\mathfrak{D}(M;\mathcal{T})$? (in order to add ideal points)
- How do we include "ideal components"?

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- How do we compactify $\mathfrak{D}(M;\mathcal{T})$? (in order to add ideal points)
- How do we include "ideal components"?

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The Deligne-Mumford compactification

Consider the space of n distinct points in $\mathbb{CP}^1$ up to conjugation,

$$((\mathbb{CP}^1)^n \setminus \Delta)/\sim$$

The Deligne-Mumford compactification

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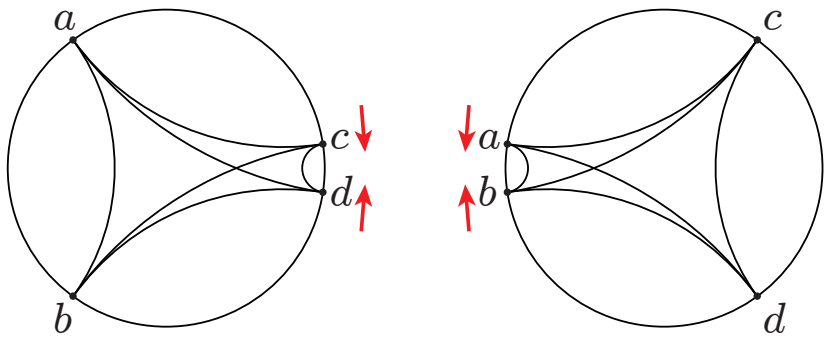

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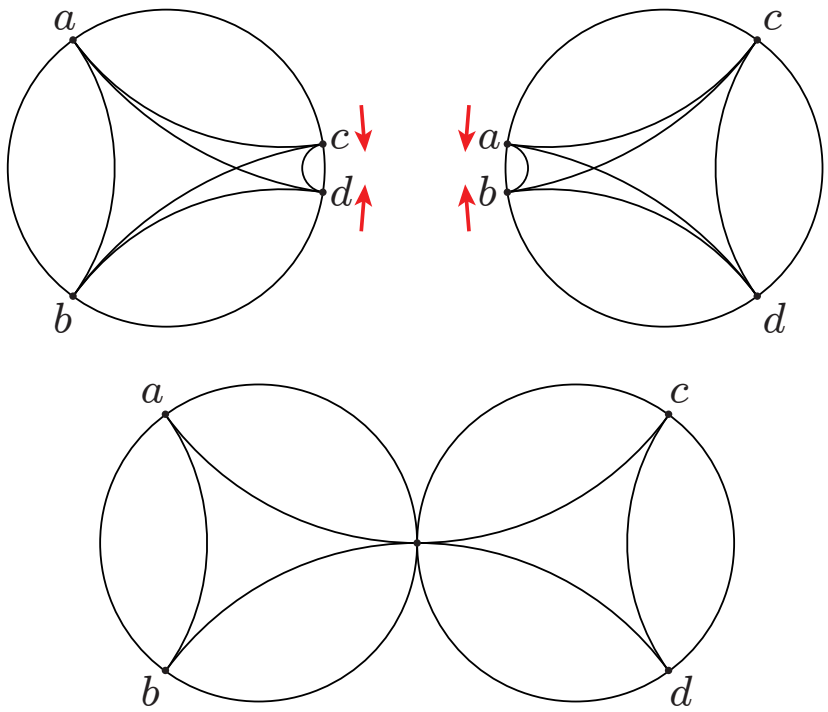

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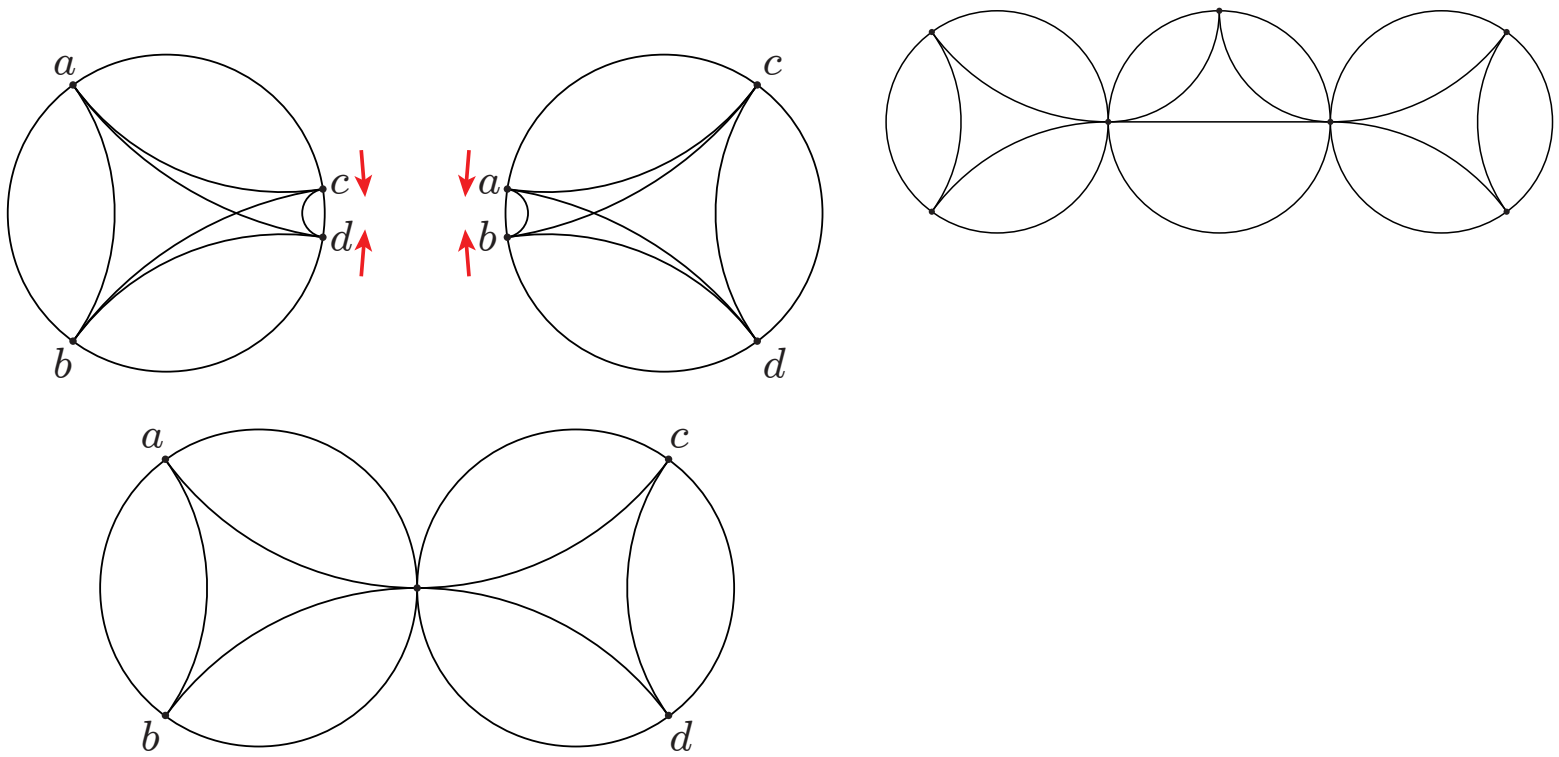

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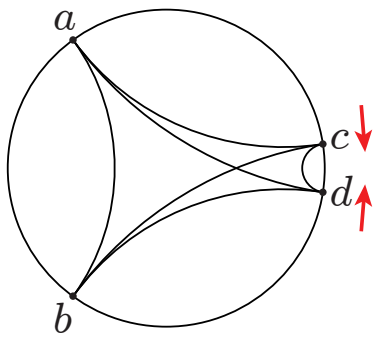

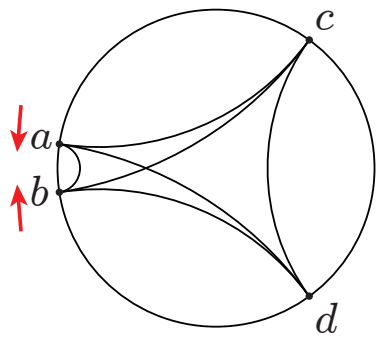

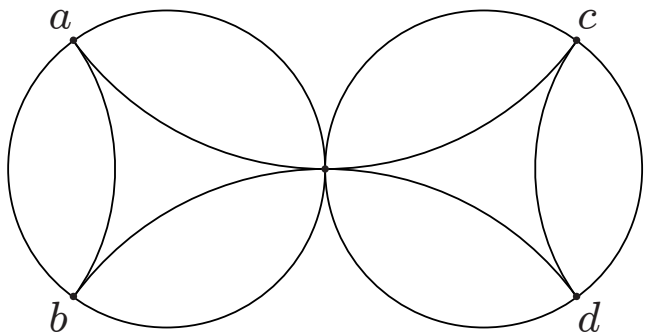

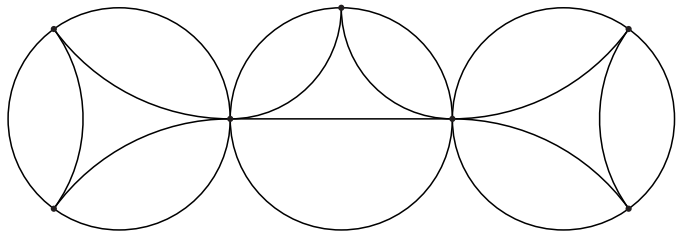

We compactify with arrangements of points on various trees of $\mathbb{CP}^1$'s depending on how and at what rates the points approach each other.

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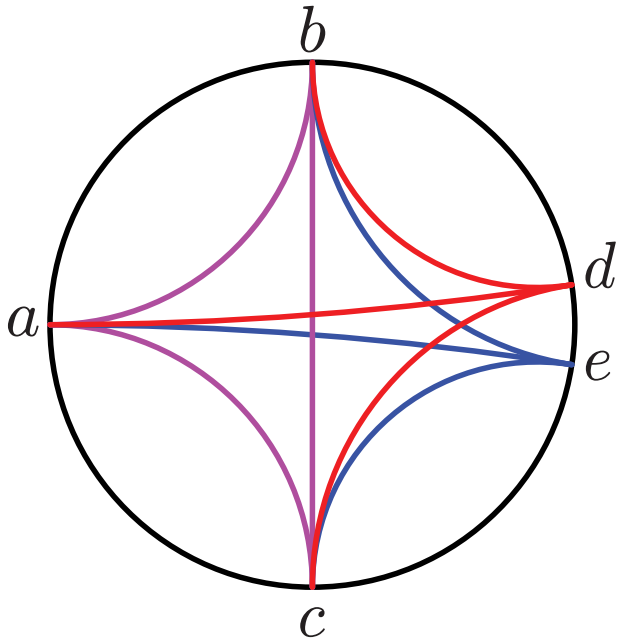

Now do the same with positions of cusps on $\partial\mathbb{H}^3$, with infinitely many points but with a $\pi_1 M$ equivariance condition.

As we arrive at an ideal point $\widetilde{p}$ of $\mathcal{D}(M;\mathcal{T})$ our developing map

$$\Phi_p : \widetilde{M} \to \mathbb{H}^3$$

should become

$$\Phi_{\widetilde{p}} : \widetilde{M} \to (\text{tree of } \mathbb{H}^3\text{'s})$$

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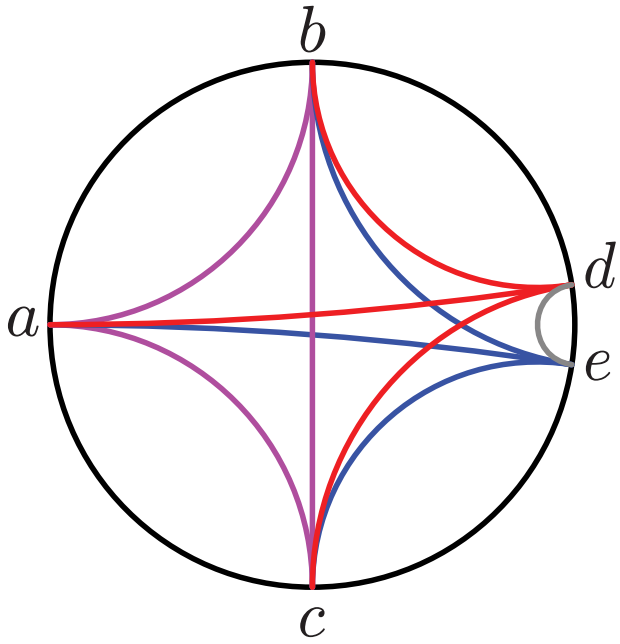

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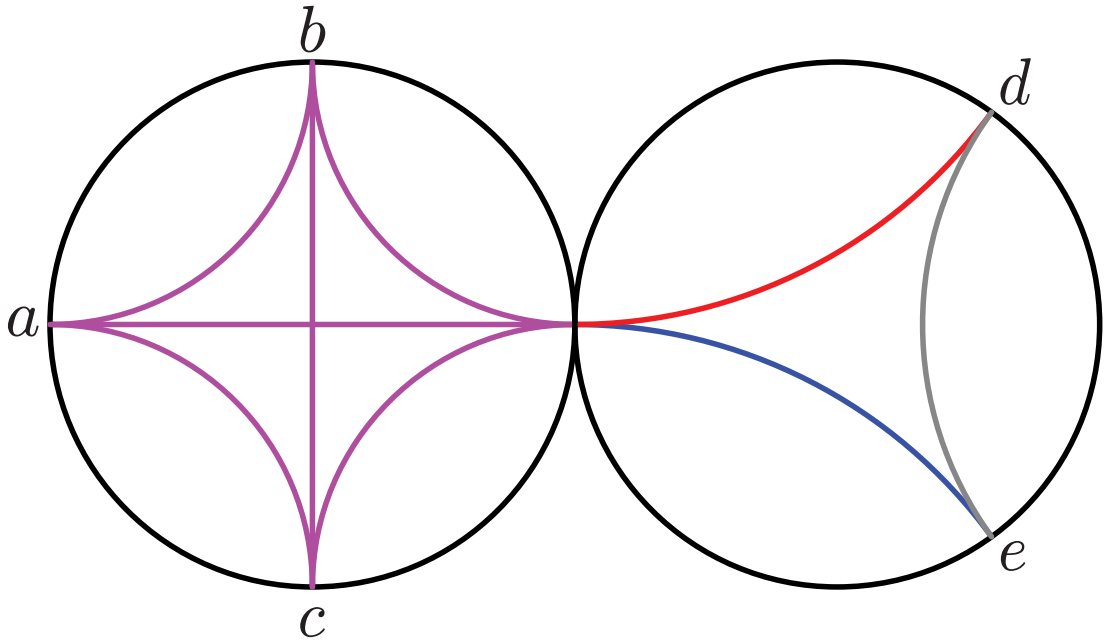

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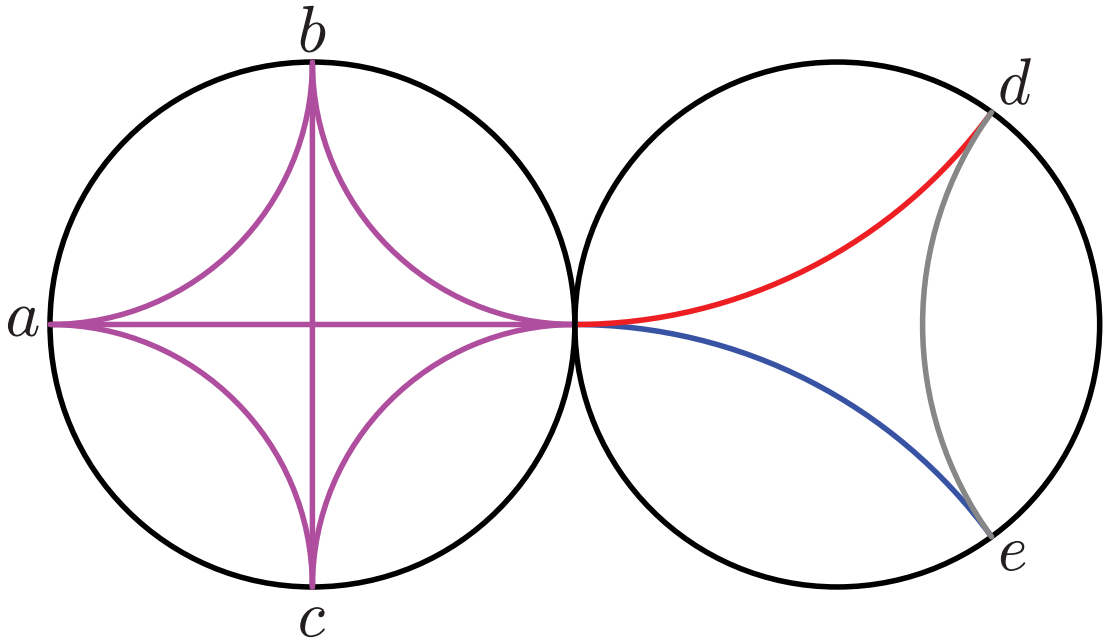

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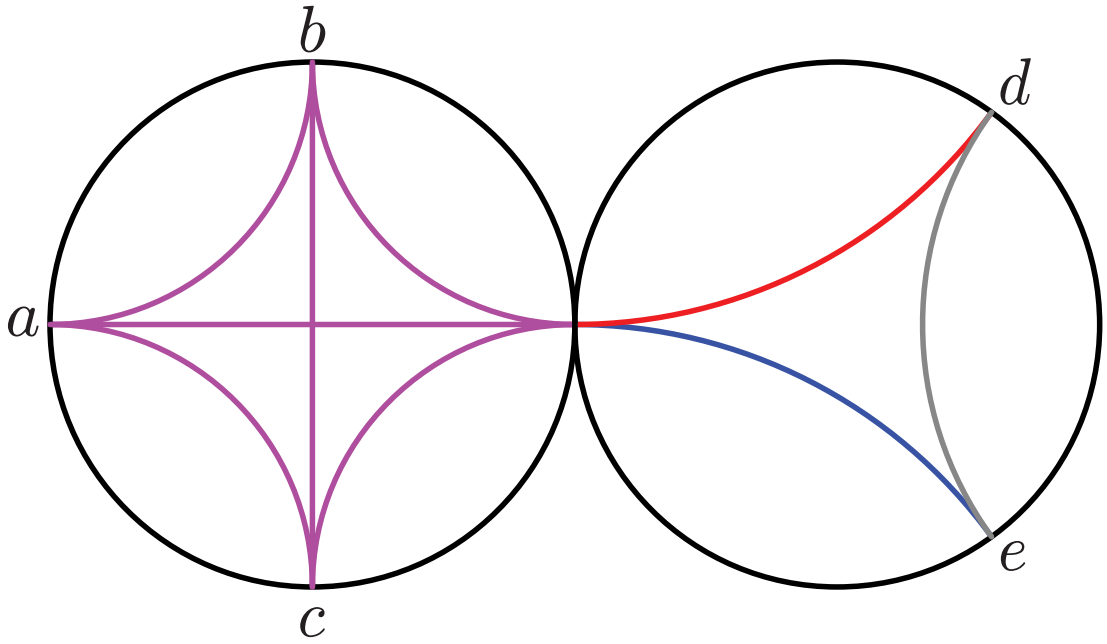

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Developing into ∂(tree of $\mathbb{H}^3$'s)

Previously, we developed ideal tetrahedra into $\mathbb{H}^3$ by tracking positions of vertices on $\partial\mathbb{H}^3 = \mathbb{CP}^1$, now we need to track positions of vertices on ∂(tree of $\mathbb{H}^3$'s)

Instead of $\mathbb{CP}^1$ we use Serre's tree (so called by Ohtsuki), T_ζ, tree with ends $\mathbb{C}((\zeta)) \cup \{\infty\}$. (Laurent series, plus ∞)

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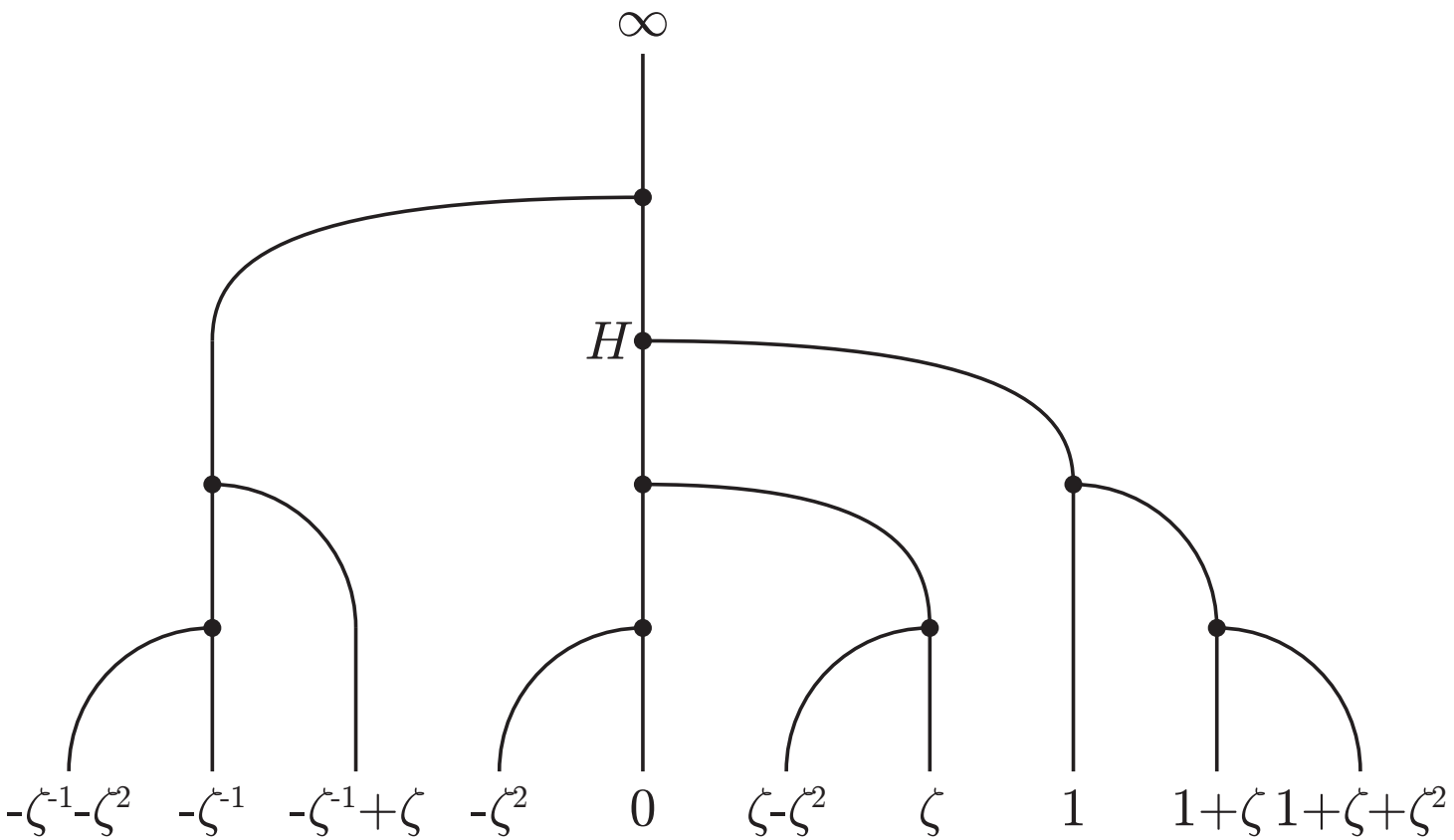

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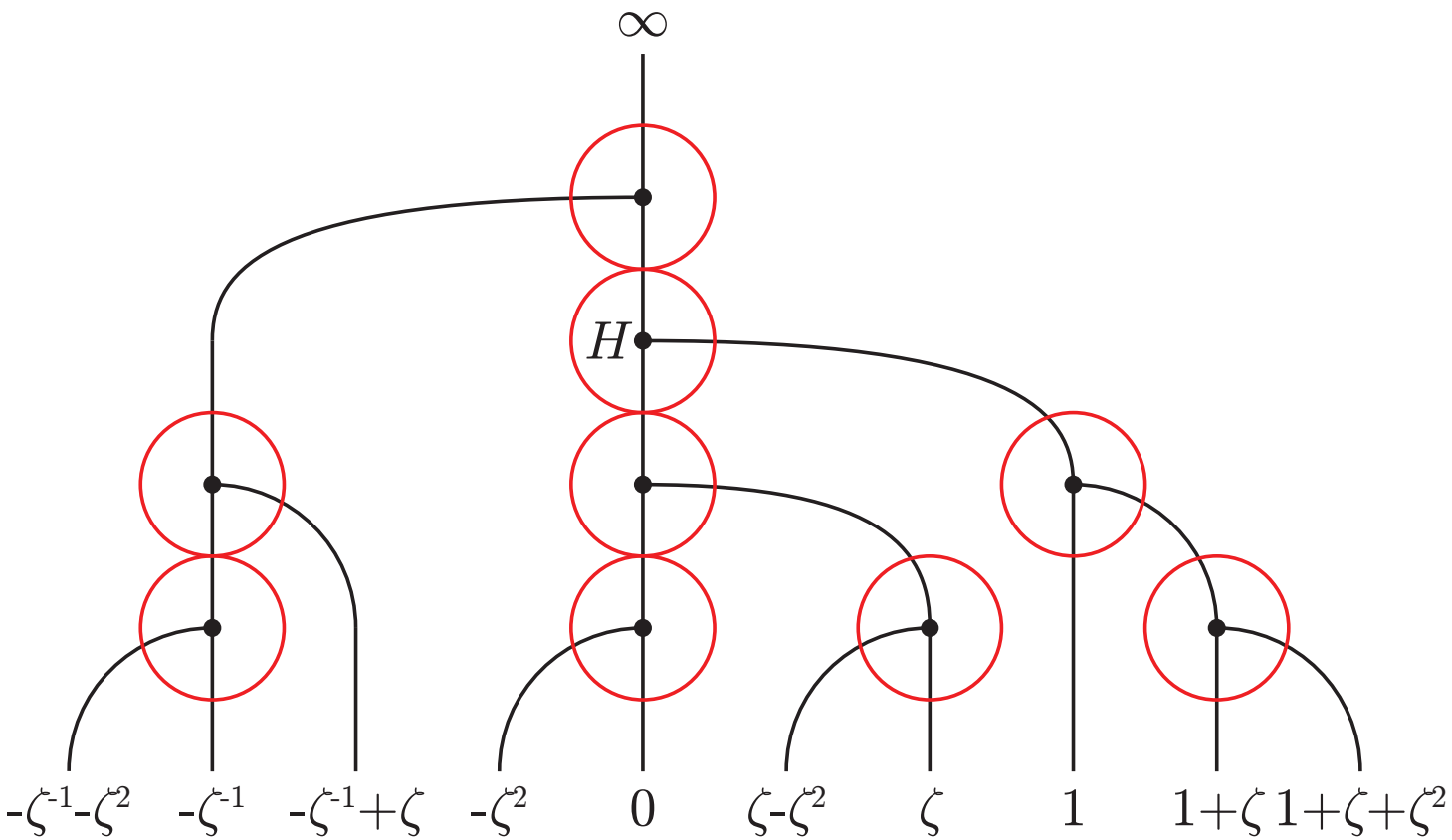

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We develop in the same way: given $a, b, c \in \mathbb{C}((\zeta)) \cup \{\infty\}$ and knowing that the cross ratio (aka dihedral angle) z of (a, b, c, d) is given for this tetrahedron and

$$z = \frac{(a-c)(b-d)}{(a-d)(b-c)} \in \mathbb{C}((\zeta)) \setminus \{0, 1\}$$

We can solve for d:

$$d = \frac{(b-c)za - (a-c)b}{(b-c)z - (a-c)}$$

We also need consistency of the developing map through different paths of tetrahedra.

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Example: developing an ideal point into T_ζ

$\widetilde{p}$ is an ideal point of $\mathfrak{D}(M;\mathcal{T})$ (which we assume is a 1 (complex) dimensional variety)

Cross ratios in $\mathbb{C}((\zeta))$: By some algebraic geometry magic, we can parametrise a neighbourhood of $\widetilde{p}$ by a complex variable ζ (with $\zeta = 0$ corresponding to $\widetilde{p}$). For each tetrahedron, the cross ratio $z_i = z_i(\zeta) \in \mathbb{C}((\zeta))$.

Develop consistently: For all $\zeta \neq 0$ the $z_i(\zeta)$ develop cusp positions consistently, because they correspond to shapes of tetrahedra that satisfy the gluing equations, so the $z_i(\zeta)$ develop consistently as elements of $\mathbb{C}((\zeta))$.

For an isolated ideal point of $\mathfrak{D}(M;\mathcal{T}_3)$ corresponding to "$x = 1/y$" in $\mathfrak{D}(M;\mathcal{T}_2)$ this gives us the right developing map into one "central" $\mathbb{H}^3$ of the tree.

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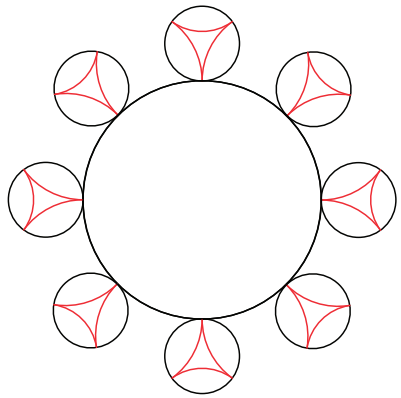

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- This deals with an isolated ideal point of $\mathfrak{D}(M;\mathcal{T}_3)$ corresponding to "$x = 1/y$" for $\mathfrak{D}(M;\mathcal{T}_2)$, but doesn't help for the other case, of an entire "ideal component". There is no way to approach the needed degenerate shapes of tetrahedra.
- An element of $\mathbb{C}((\zeta))$ for each cross ratio is an enormous amount of data in comparison to an element of $\mathbb{C}$: finding sets of cross ratios that develop consistently looks hard (unless we can apply magic through being able to approach from non-degenerate shapes).
- Similarly, we don't obviously have a finite dimensional algebraic variety of solutions, assuming that solutions for "ideal components" even exist.

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Developing with less information

Idea: Low order information determines low order behaviour.

Definition: If a is an end of T_ζ, so $a \in \mathbb{C}((\zeta)) \cup \{\infty\}$, let

$$a_H := \begin{cases} a_0 & \text{if} \quad a \in \mathbb{C}[[\zeta]] \\ \infty & \text{if} \quad a \in (\mathbb{C}((\zeta)) \cup \infty) \setminus \mathbb{C}[[\zeta]] \end{cases}$$

Example: Suppose p is an interior point (i.e. not an ideal point) of $\mathcal{D}(M;\mathcal{T})$, so the data of p is a tuple of cross ratios $(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in (\mathbb{C} \setminus \{0,1\})^n$.

Arbitrarily extend $z^{(i)}$ to $Z^{(i)}(\zeta) = \sum_{k=0}^{\infty} Z_k^{(i)} \zeta^k \in \mathbb{C}[[\zeta]]$ such that $Z_0^{(i)} = z^{(i)}$, but $Z_k^{(i)}, k > 0$ is chosen arbitrarily.

Now develop the $Z^{(i)}(\zeta)$ into T_ζ. If a is the position of some developed vertex then a_H depends only on the $Z_0^{(i)} = z^{(i)}$.

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Example: Suppose p is an interior point (i.e. not an ideal point) of $\mathcal{D}(M;\mathcal{T})$, so the data of p is a tuple of cross ratios $(z^{(1)}, z^{(2)}, \ldots, z^{(n)}) \in (\mathbb{C} \setminus \{0,1\})^n$.

Arbitrarily extend $z^{(i)}$ to $Z^{(i)}(\zeta) = \sum_{k=0}^{\infty} Z_k^{(i)} \zeta^k \in \mathbb{C}[[\zeta]]$ such that $Z_0^{(i)} = z^{(i)}$, but $Z_k^{(i)}, k > 0$ is chosen arbitrarily.

Now develop the $Z^{(i)}(\zeta)$ into T_ζ. If a is the position of some developed vertex then a_H depends only on the $Z_0^{(i)} = z^{(i)}$.

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Idea: Low order information determines low order behaviour.

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So if we only care about one $\mathbb{H}^3$ of the tree, we can develop without knowing higher order information.

Question: Is something like this true when some tetrahedra are degenerate?

Answer: Yes, but we have to be careful about which tetrahedra we develop into.

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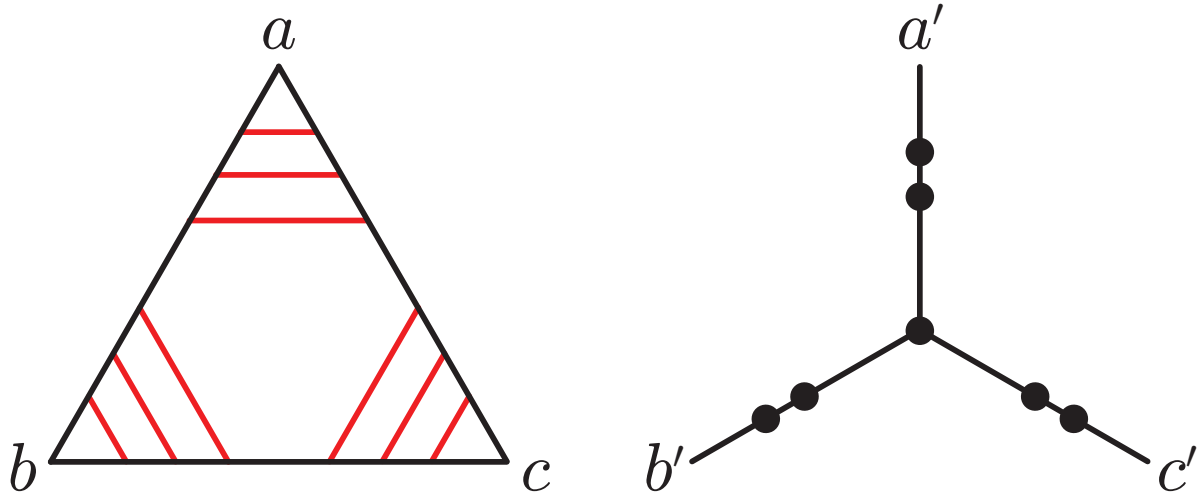
First, some needed machinery:



A triangle of $\widetilde{M}$ maps to a tripod in T_ζ. Normal curves are inverse images of midpoints of the edges of the tripod.

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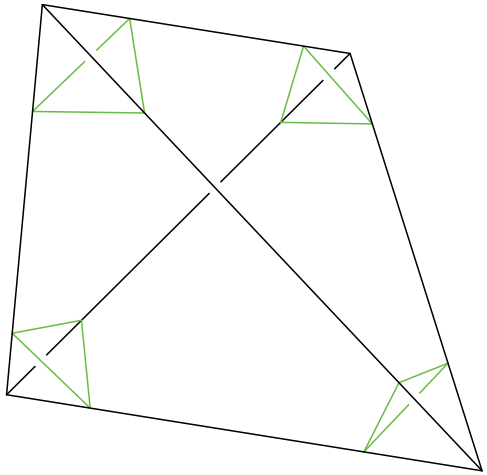

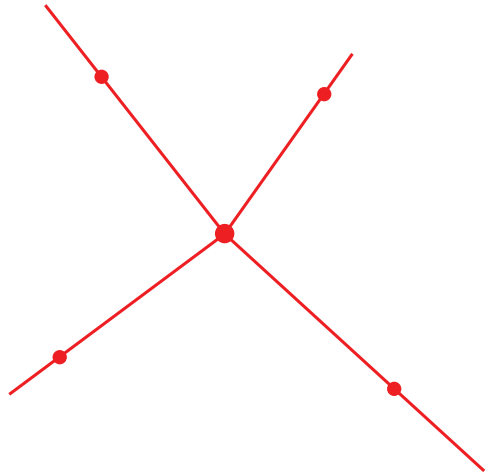

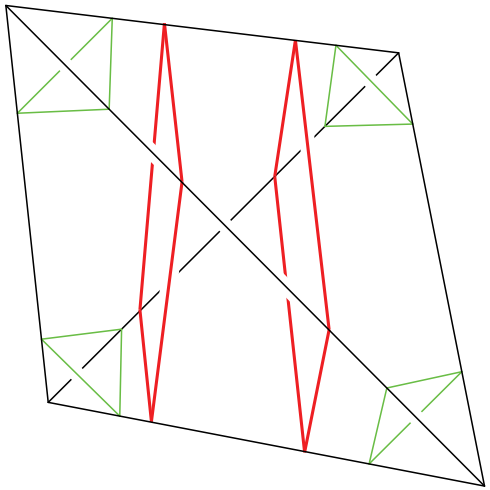

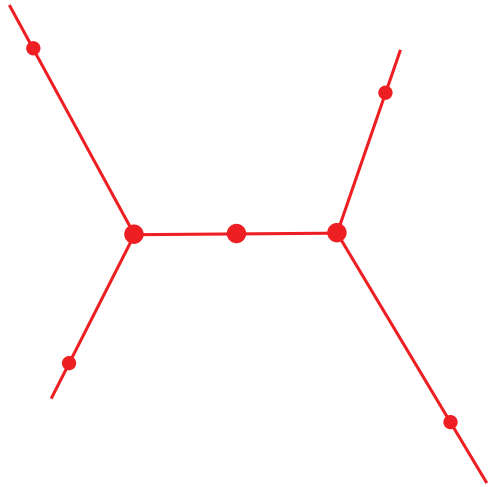

A tetrahedron of $\widetilde{M}$ maps to a spine in T_ζ. Normal triangles and quadrilaterals are inverse images of midpoints of the edges of the spine.

If the lowest non-zero term of a cross ratio $z(\zeta) \in \mathbb{C}[[\zeta]]$ is $z_k\zeta^k$ then $\text{order}(z(\zeta)) = k$, which is also the length of the spine.

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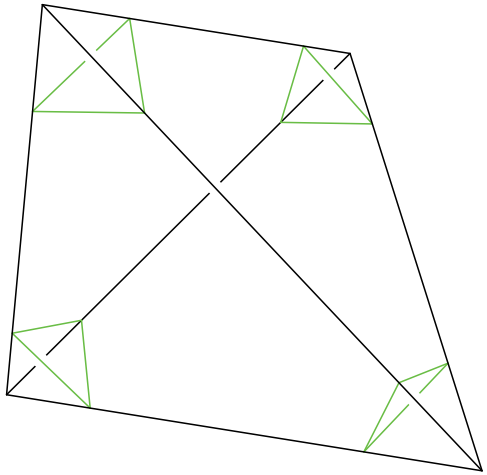

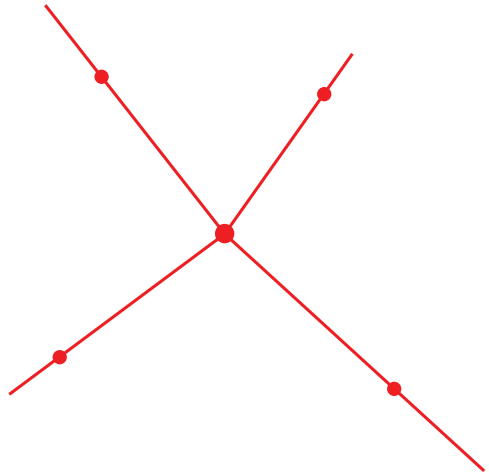

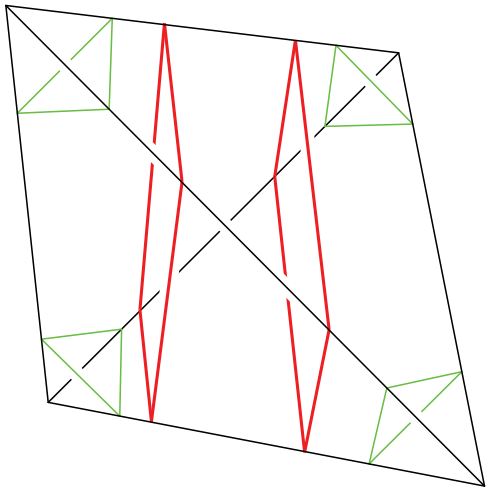

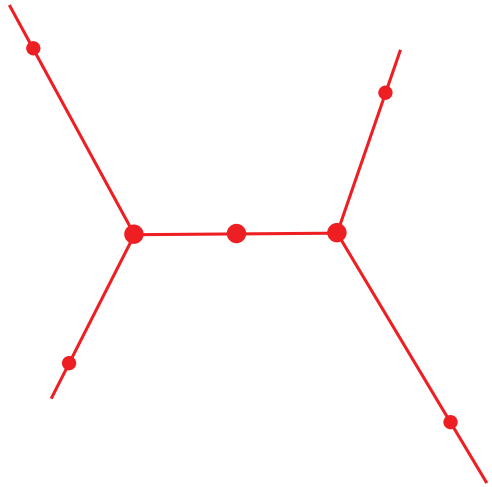

A tetrahedron of $\widetilde{M}$ maps to a spine in T_ζ. Normal triangles and quadrilaterals are inverse images of midpoints of the edges of the spine.

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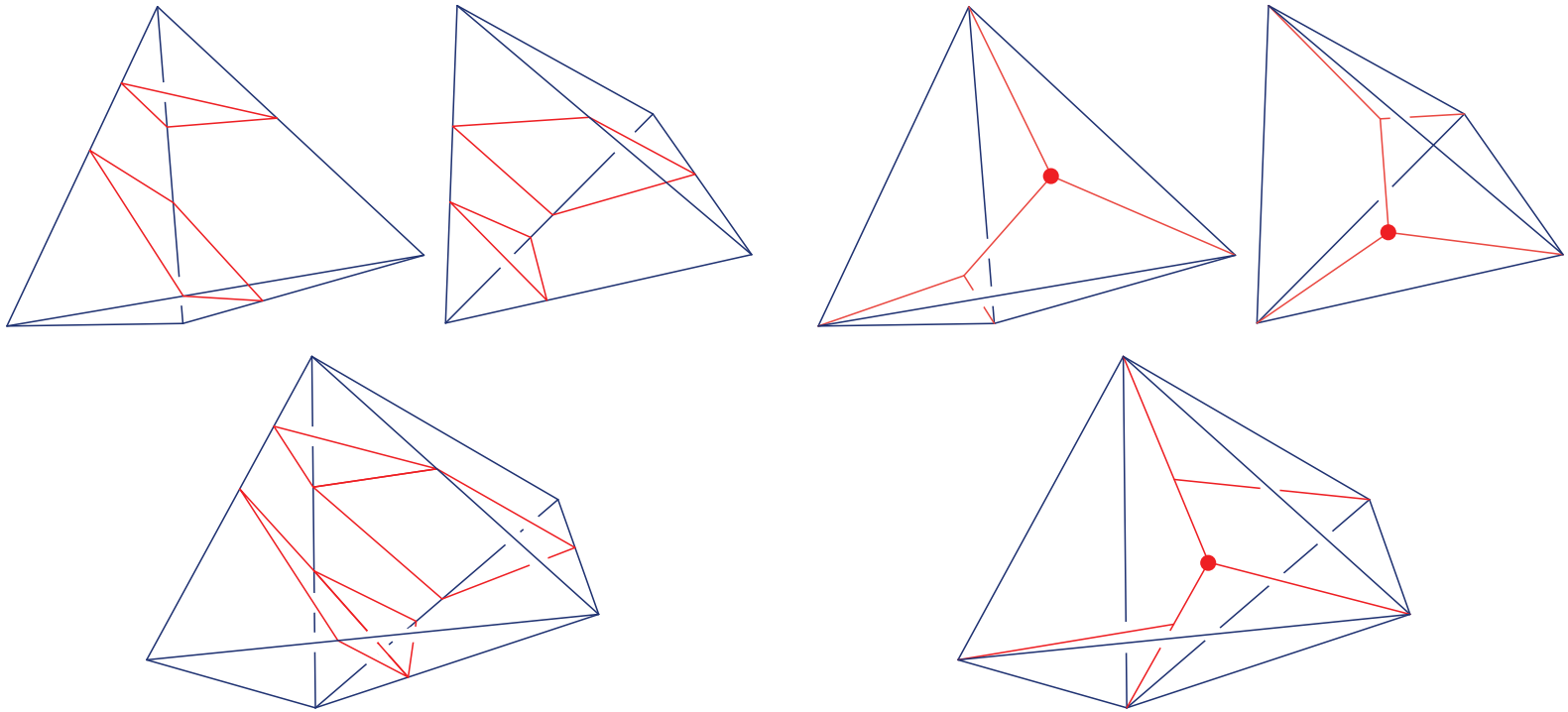

The combinatorial data of spines and how they glue together allows us to develop a purely combinatorial tree from the degeneration data. We require that this construction be independent of the path of tetrahedra we develop along.

This consistency implies that the normal triangles and quadrilaterals link up to form a spun-normal surface $S \subset \widetilde{M}$ (related to work of Tillmann).

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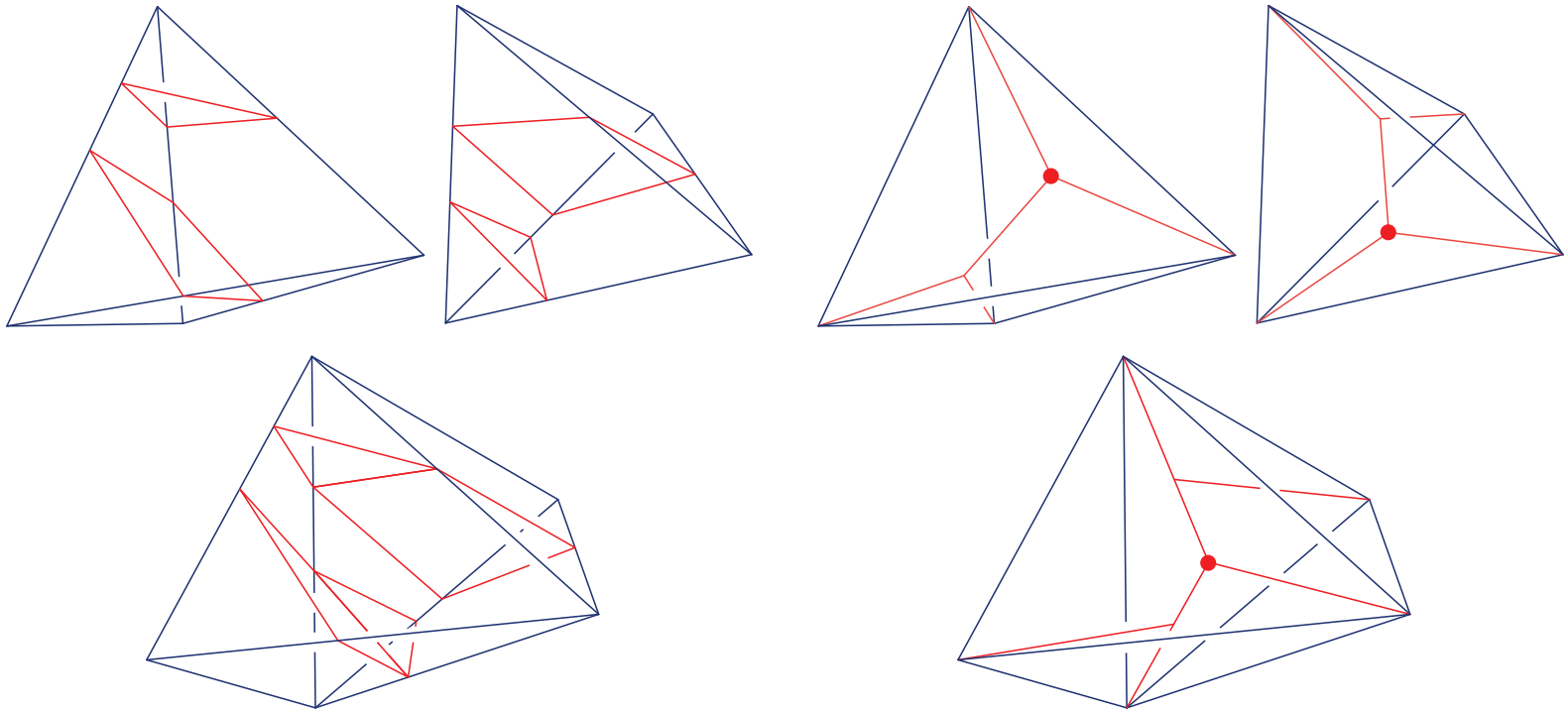

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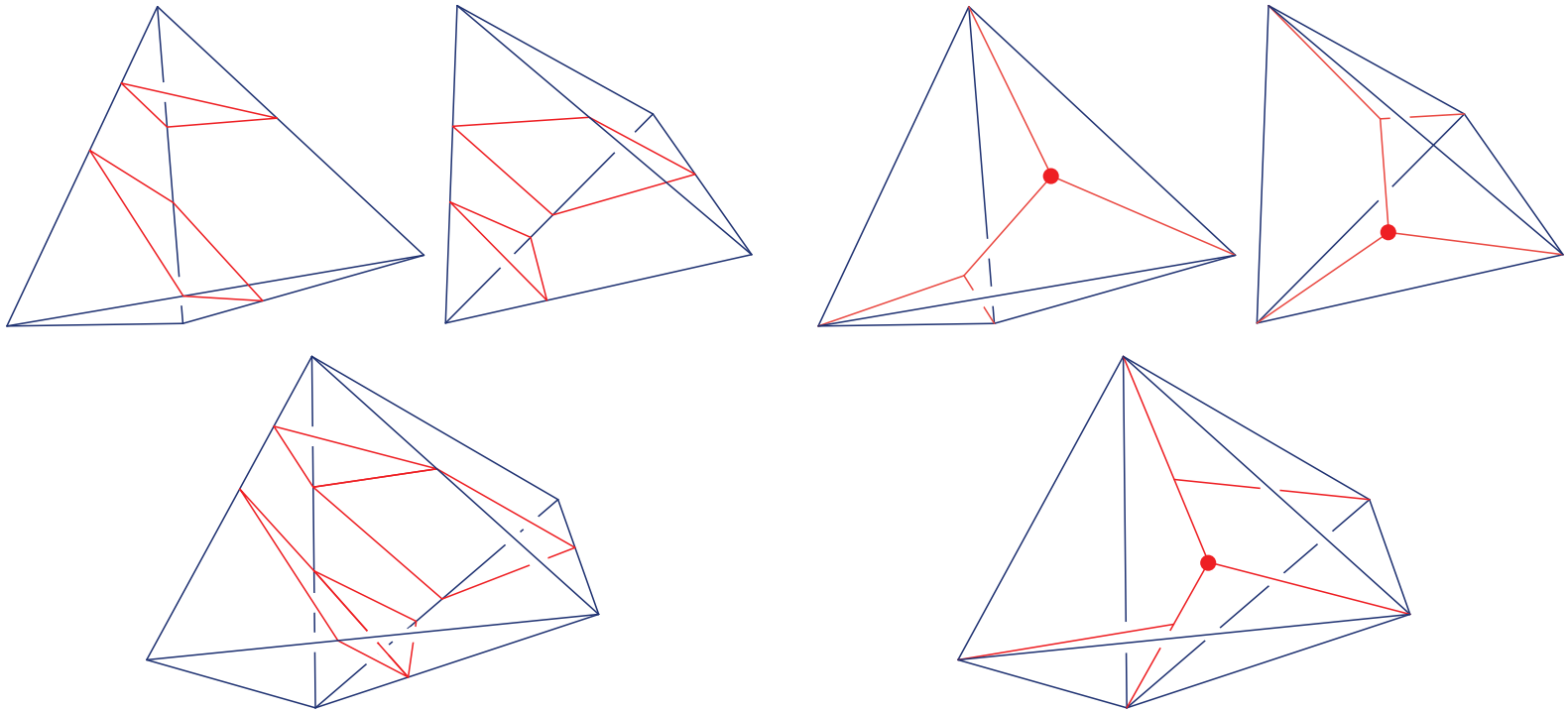

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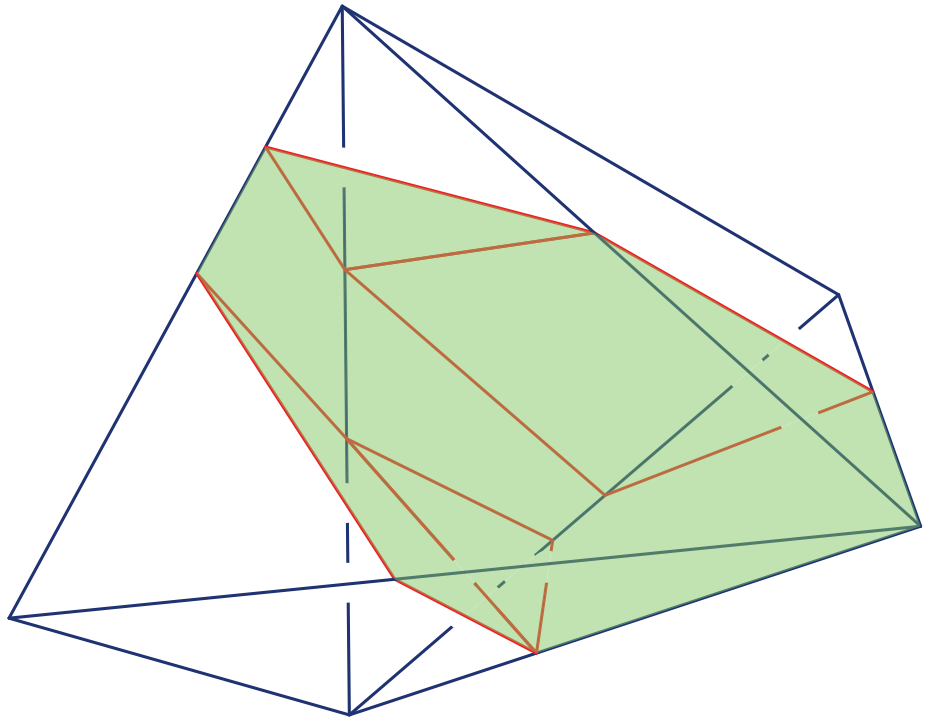
We can then define a tree T_S with vertices the components of $\widetilde{M} \setminus S$ and edges corresponding to the components of S, and T_S is the same as the tree we get by gluing spines.



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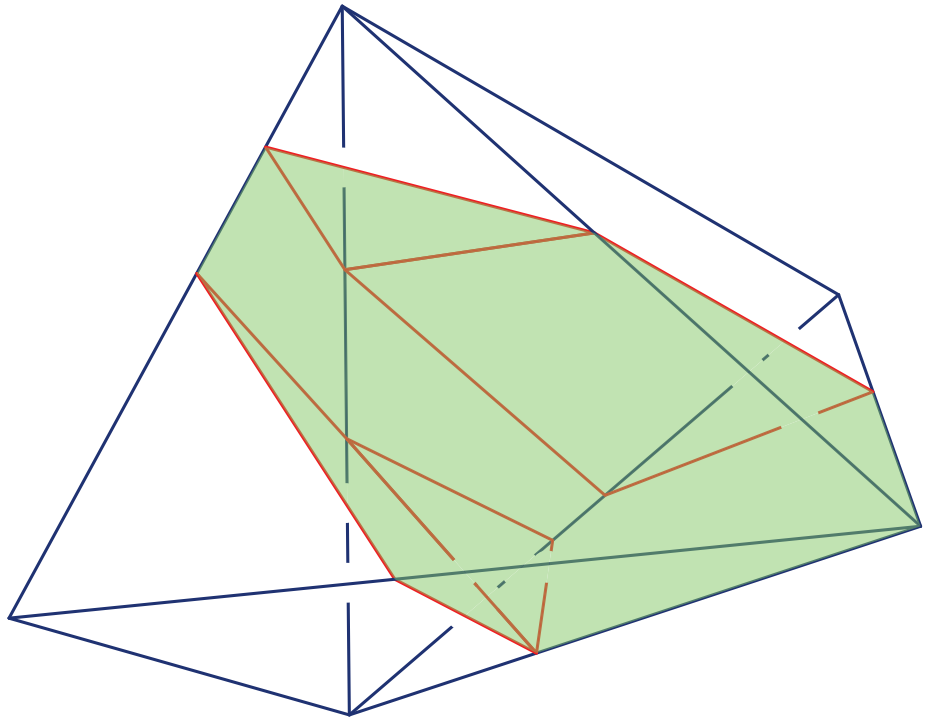

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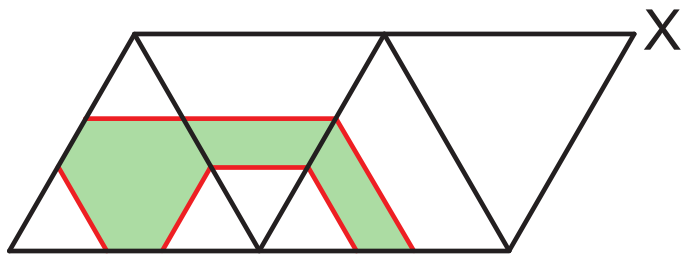
Theorem Suppose we have a set of $z^{(i)}(\zeta)$, one for each tetrahedron of $\mathcal{T}$ such that the tree generated by the spines of lengths given by the orders of the $z^{(i)}$ is consistent. Let $\widetilde{M}_0$ be a component of $\widetilde{M} \setminus S$ and suppose a is the developed position of a vertex obtained from a path of tetrahedra that has a contiguous non-empty intersection with $\widetilde{M}_0$. Then a_H depends only on the lowest order terms of the $z^{(i)}(\zeta)$.



What this means is that we can work out the developing map for one vertex (one $\mathbb{H}^3$) of the T_ζ knowing only the orders and the lowest order coefficients of the cross ratios.

We still need some conditions on these coefficients to get that the developing map is independent of the path of tetrahedra we take to get to a vertex.

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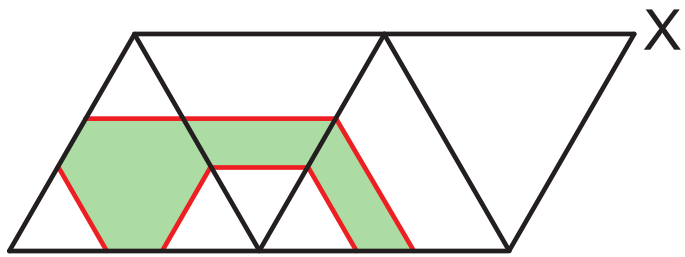

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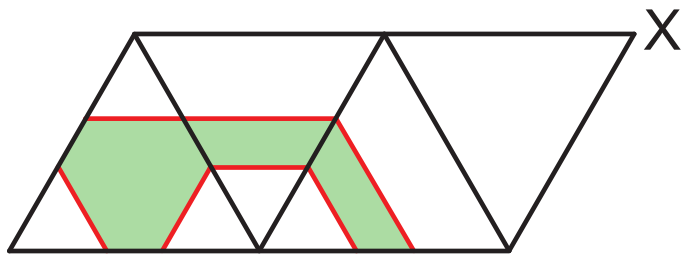

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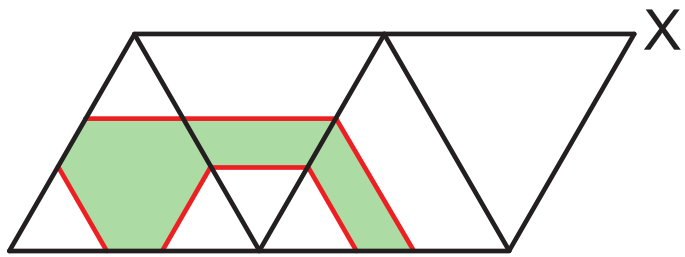

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The extension to the deformation variety, $\widetilde{\mathfrak{D}}(M, \mathcal{T})$ then should consist of points containing the following data:

- Order and choice of degeneration of cross ratio $z^{(i)}$ for each tetrahedron.
- Lowest order coefficient for each $z^{(i)}$.

subject to the conditions that:

- The spines corresponding to the combinatorial degeneration information fit together into a consistent tree.
- The positions of vertices a_H developed from paths following each component $\widetilde{M}_0$ are independent of the path.

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- Is this structure actually a variety?
- Does this work nicely with the 2-3 and 3-2 moves?
- Given triangulations $\mathcal{T}, \mathcal{T}'$ and a point $x \in \widetilde{\mathfrak{D}}(M, \mathcal{T})$, is there a corresponding $x' \in \widetilde{\mathfrak{D}}(M, \mathcal{T}')$? If so, we effectively have a triangulation independent structure.
- We should then, for instance, be able to take limits of points of the extended deformation variety(?) to (even more ideal) ideal points.
- This could then give us a way to look for essential surfaces from ideal points, and know that we would get all possible such surfaces, no matter which triangulation we used.