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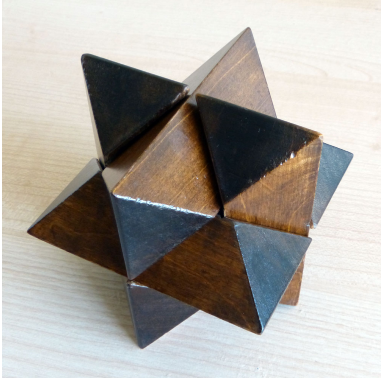
Oklahoma State University

Saul Schleimer
University of Warwick

Puzzling the 120–cell

Burr puzzles

The goal of a burr puzzle is to assemble a number of "notched sticks" into a single object.



In this talk, I will describe Quintessence, a family of burr puzzles based on the 120–cell.

The 120-cell

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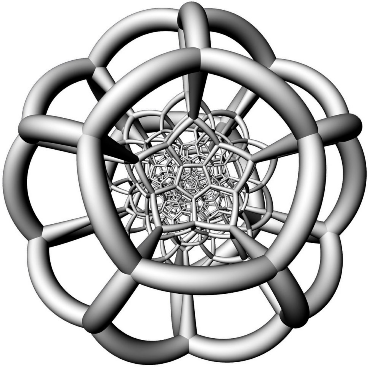

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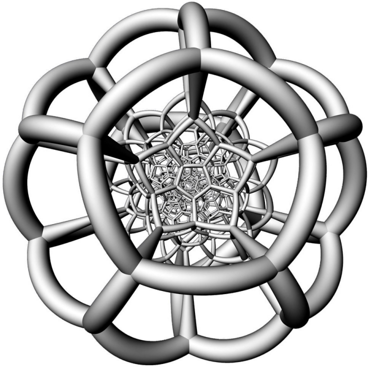
It has

- 120 dodecahedral cells,
- 720 pentagonal faces,
- 1200 edges, and
- 600 vertices.



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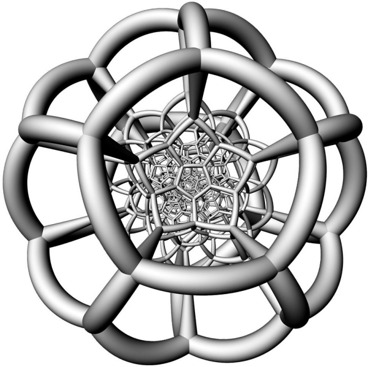

We use radial projection followed by stereographic projection to help us visualise the 120–cell.

$$\mathbb{R}^4 \setminus \{0\} \to S^3 \subset \mathbb{R}^4 \qquad S^3 \setminus \{N\} \to \mathbb{R}^3$$

$$(w,x,y,z) \mapsto \frac{(w,x,y,z)}{|(w,x,y,z)|} \qquad (w,x,y,z) \mapsto \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}\right)$$

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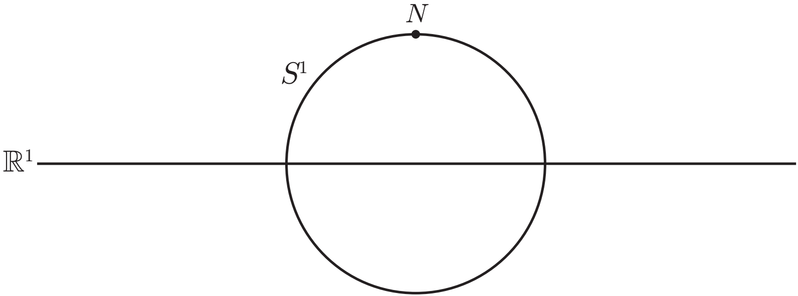

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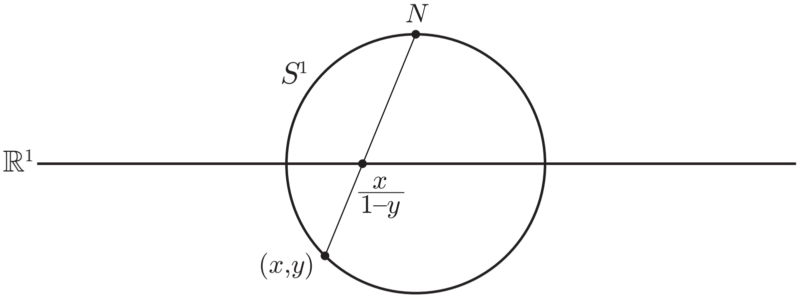

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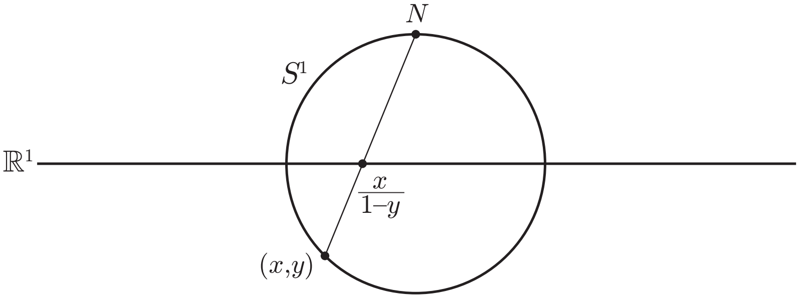
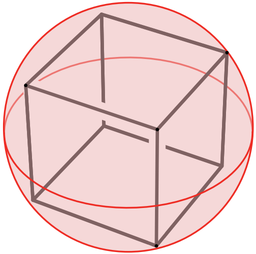


This is a cross-section of stereographic projection for $n > 1$.

Example: projecting a cube into $\mathbb{R}^2$



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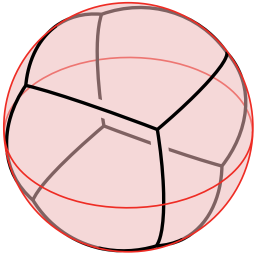

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$$\mathbb{R}^3 \setminus \{0\} \to S^2$$

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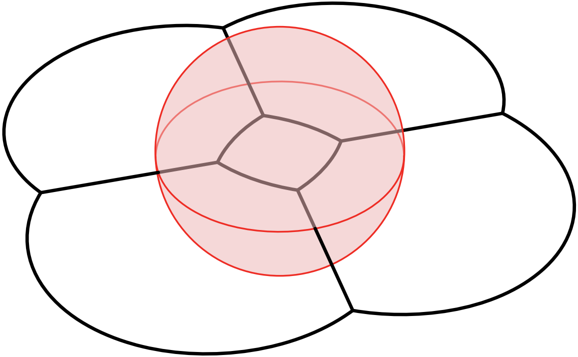

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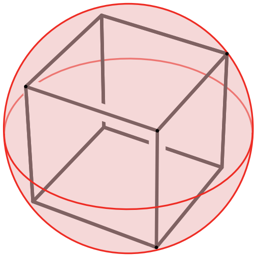
Vertex-centered versus cell-centered projection



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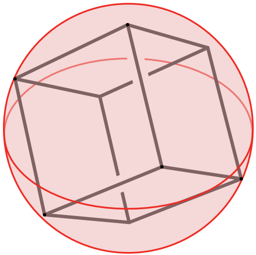

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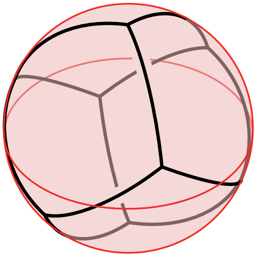

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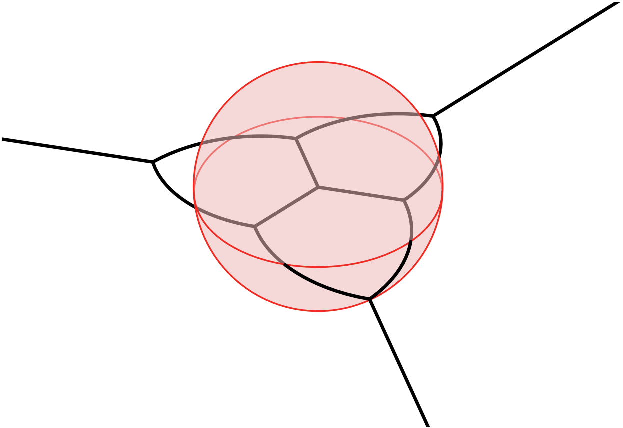

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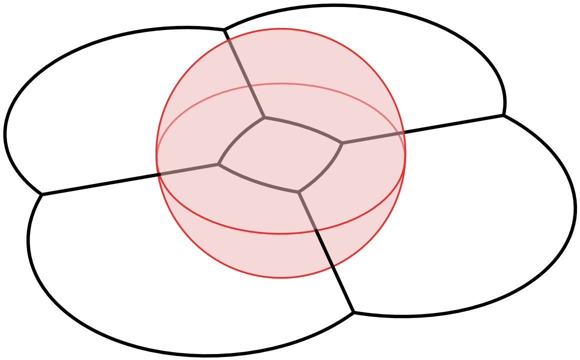
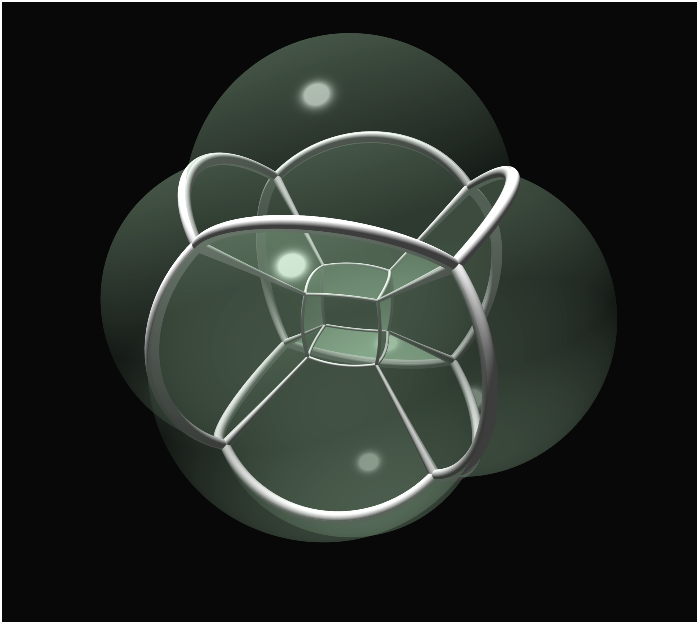
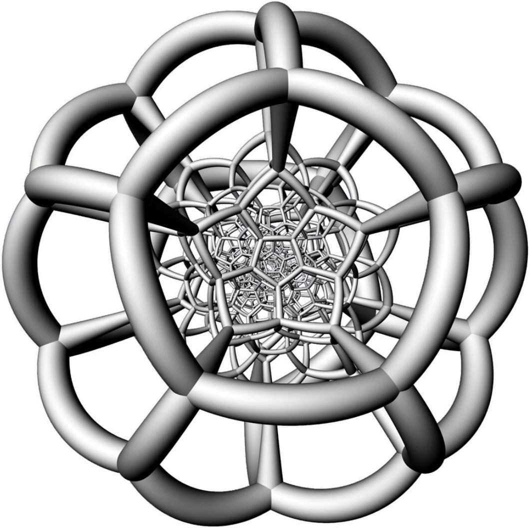


Do the same one dimension up to see a hypercube

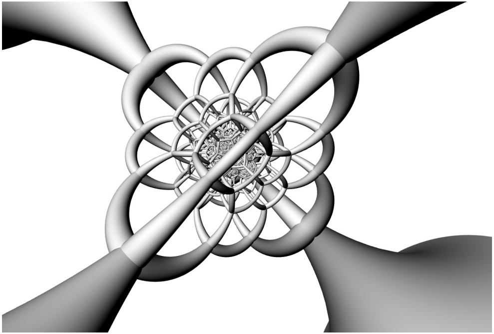


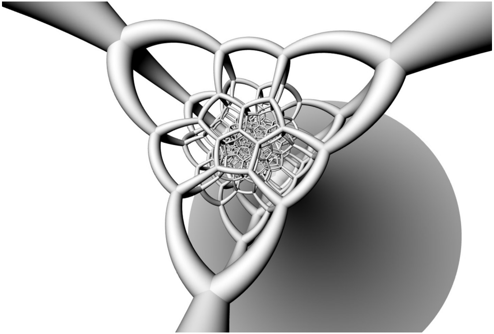
This is the
cell-centered projection
of the 120-cell; it has
dodecahedral symmetry
in $\mathbb{R}^3$.



The vertex-centered projection has tetrahedral symmetry in $\mathbb{R}^3$ and so has fewer possibilities for puzzle making.

Other choices have even less symmetry, and so have even fewer interesting ways to combine pieces.





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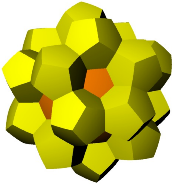

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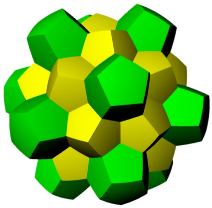

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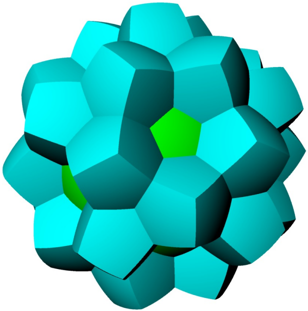

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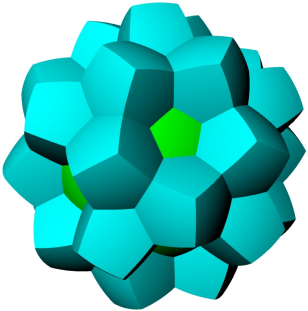

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The pattern is mirrored in the last four layers.

$$1 + 12 + 20 + 12 + 30 + 12 + 20 + 12 + 1 = 120$$

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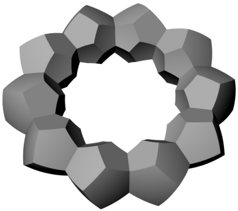

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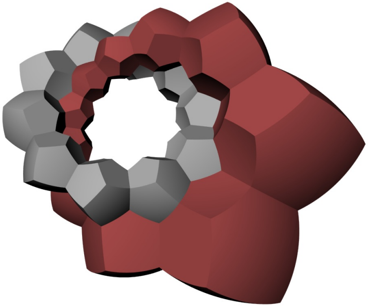

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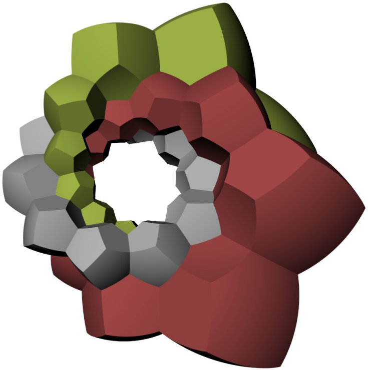

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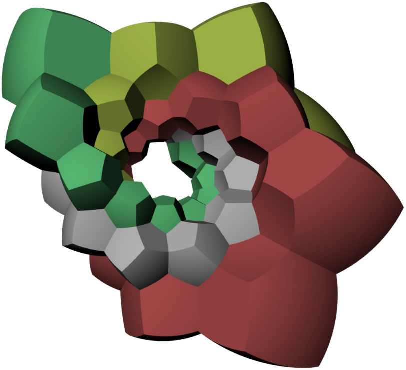

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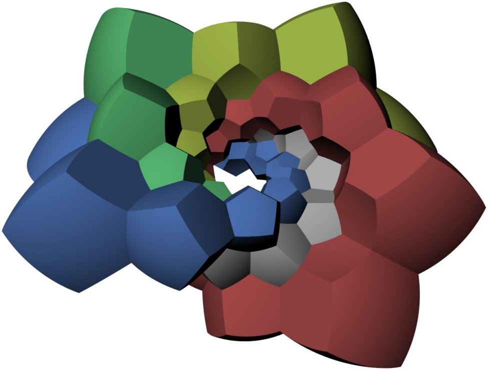

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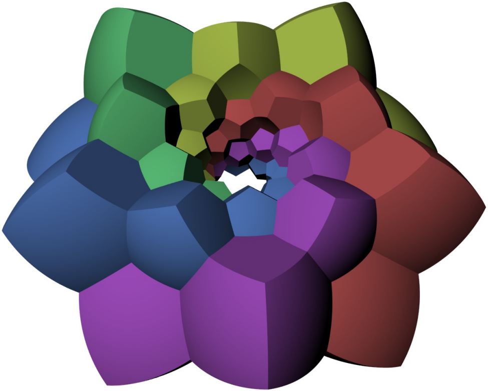

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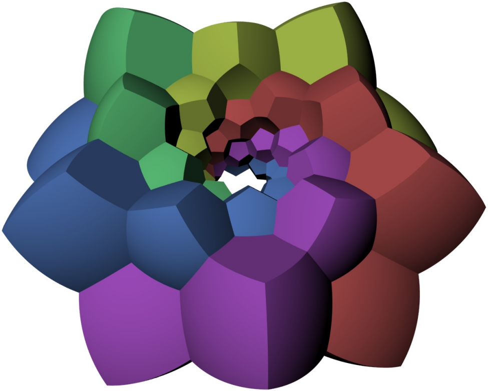

These six rings make up half of the 120–cell. The other half consists of five more rings that wrap around these, and one more ring "dual" to the original grey one.

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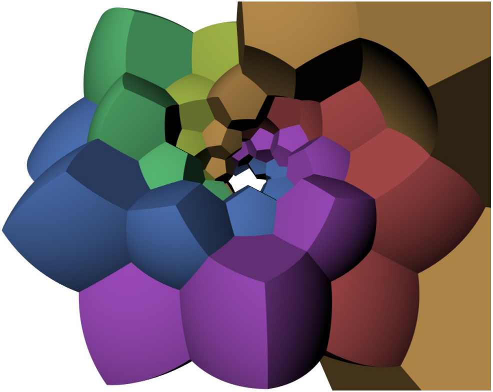

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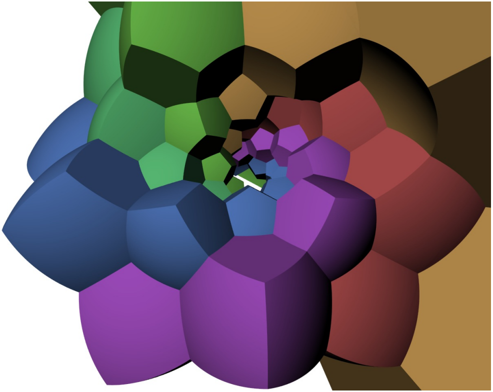

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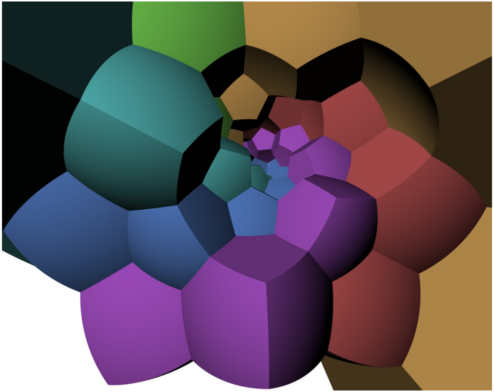

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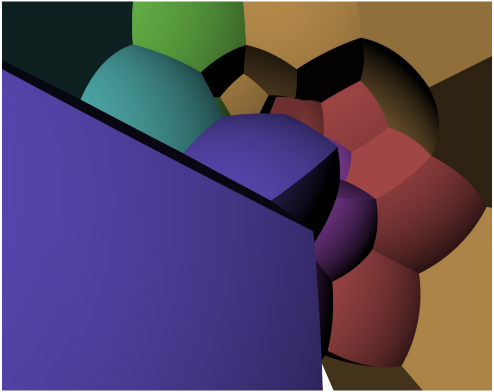

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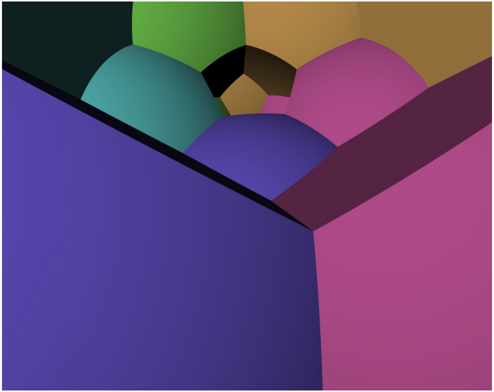

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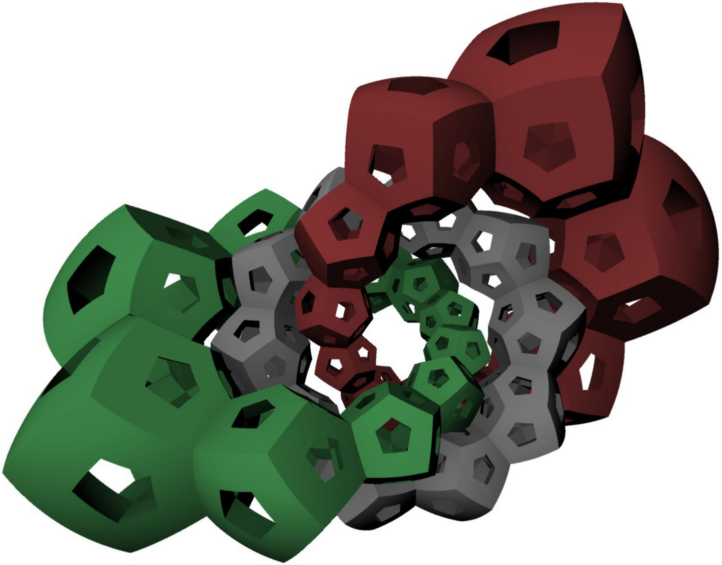
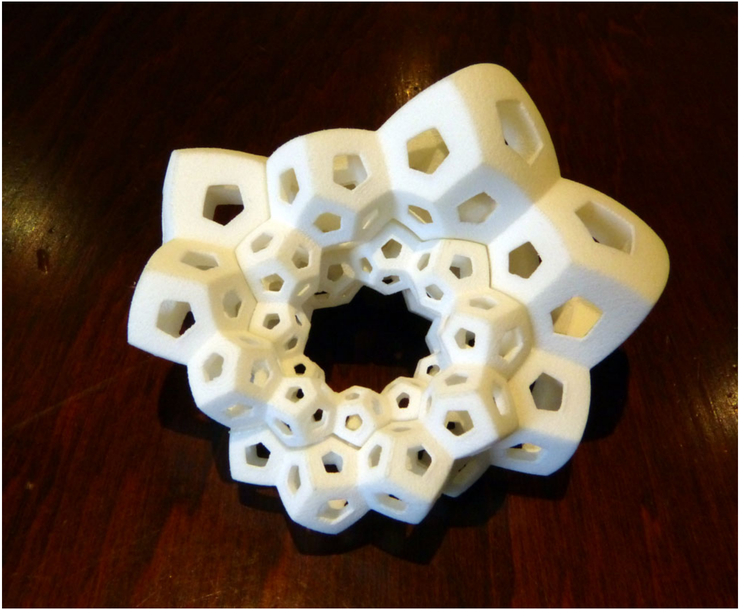
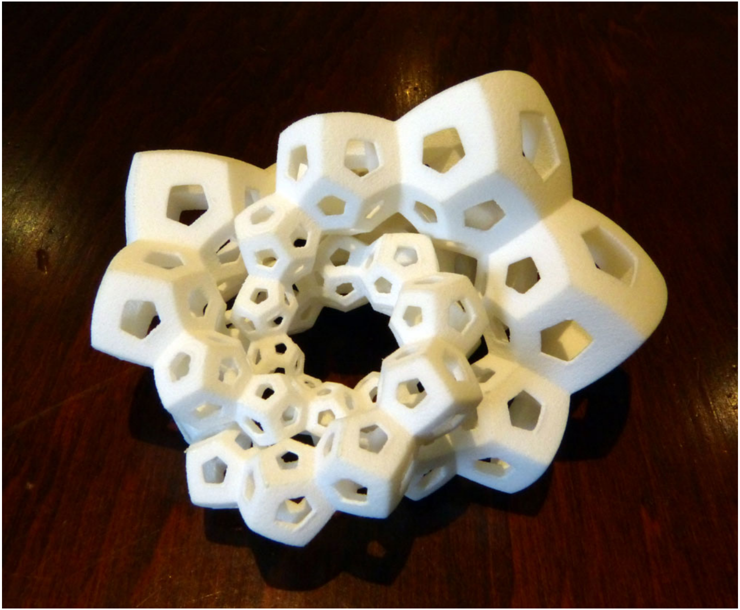
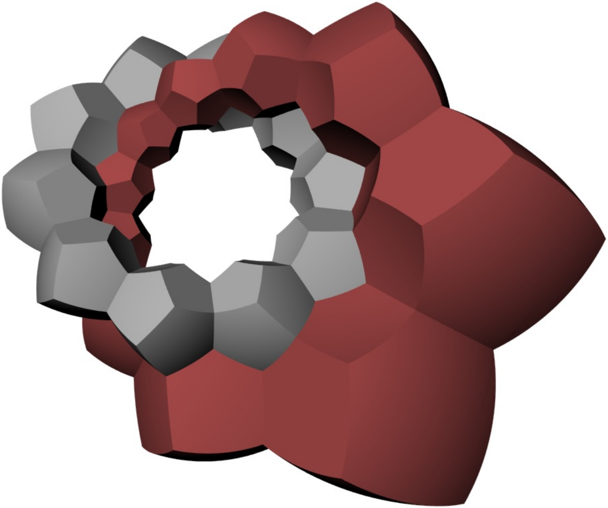
We wanted to 3D print all six of the inner rings together; it seems this cannot be done without them touching each other. (Parts intended to move must not touch during the printing process.)



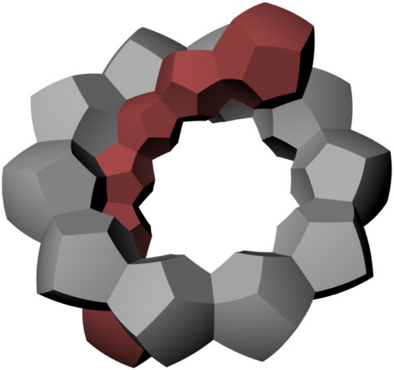




To print all five we use a trick...



To print all five we use a trick... don't print the whole ring. We call part of a ring a rib.



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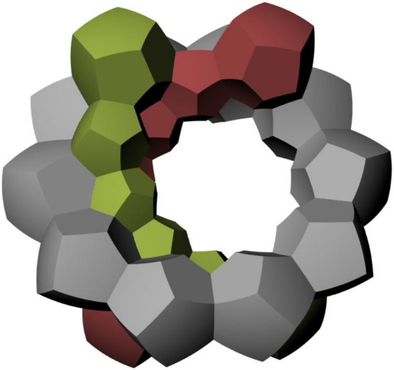

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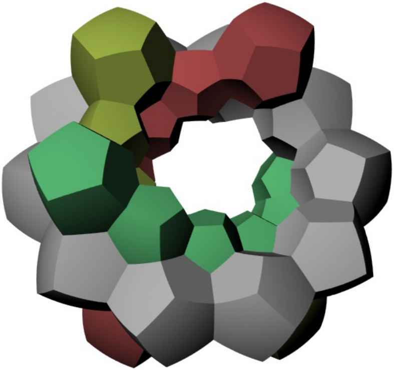

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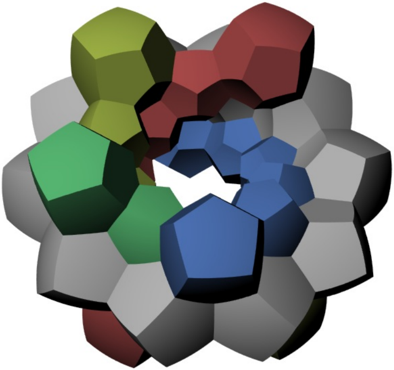

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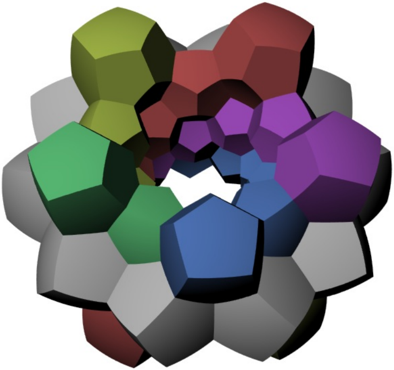

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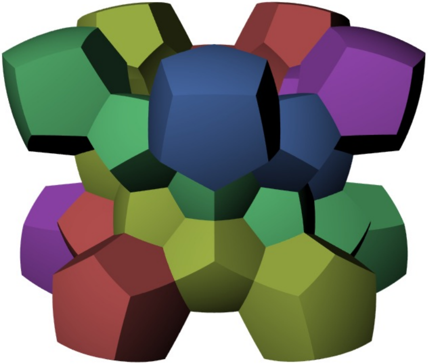
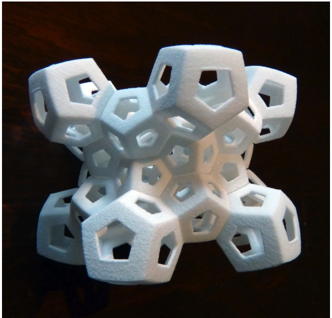
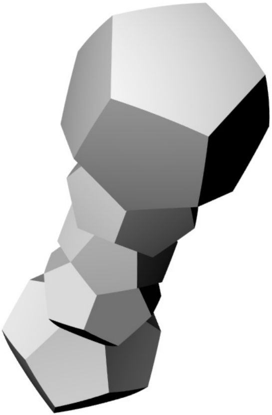


Dc30 Ring puzzle



Another decomposition, with even shorter ribs.



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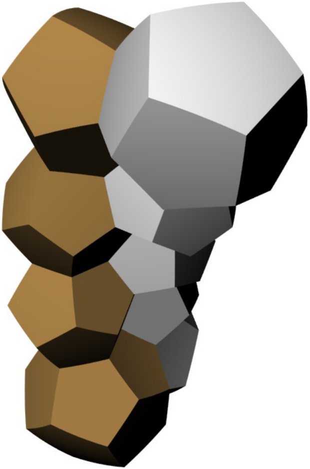

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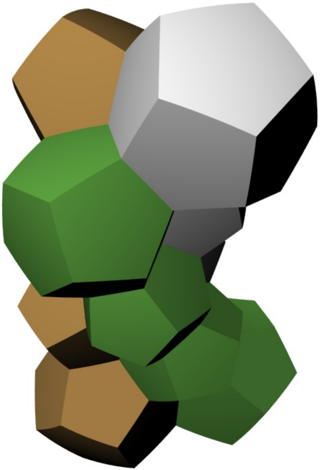

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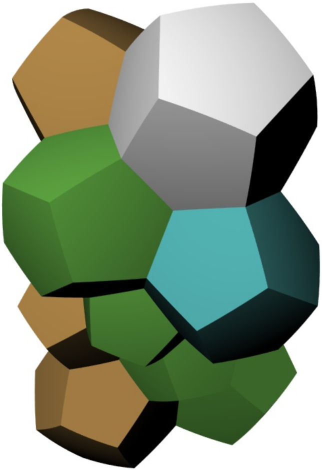

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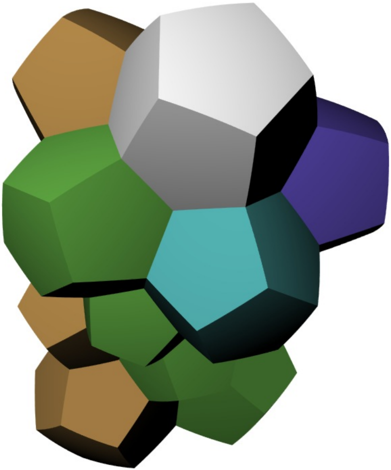

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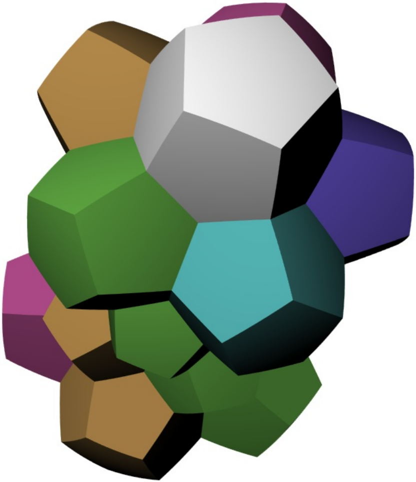

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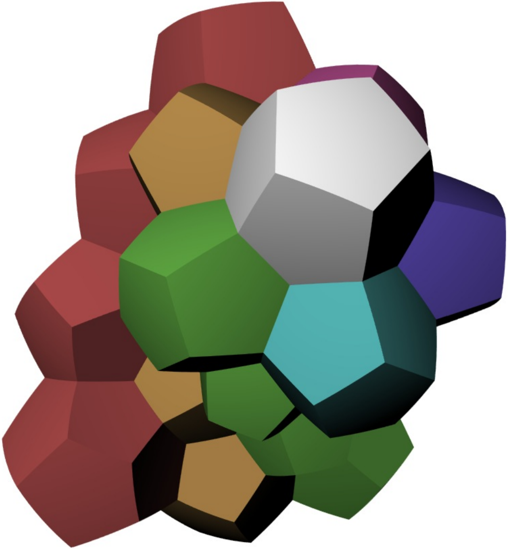
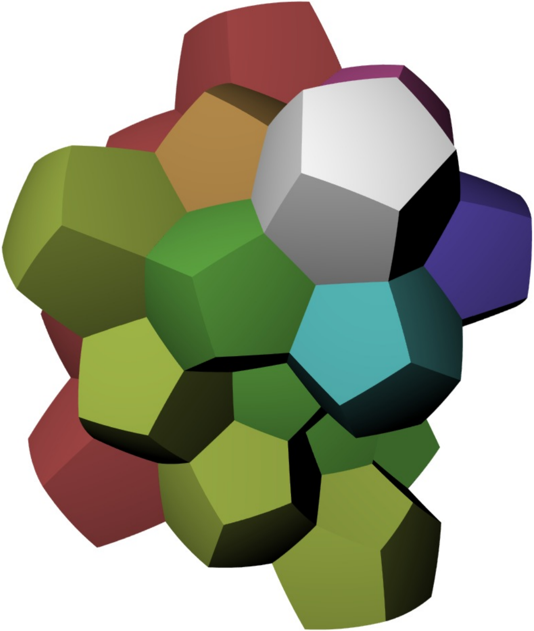


Another decomposition, with even shorter ribs.



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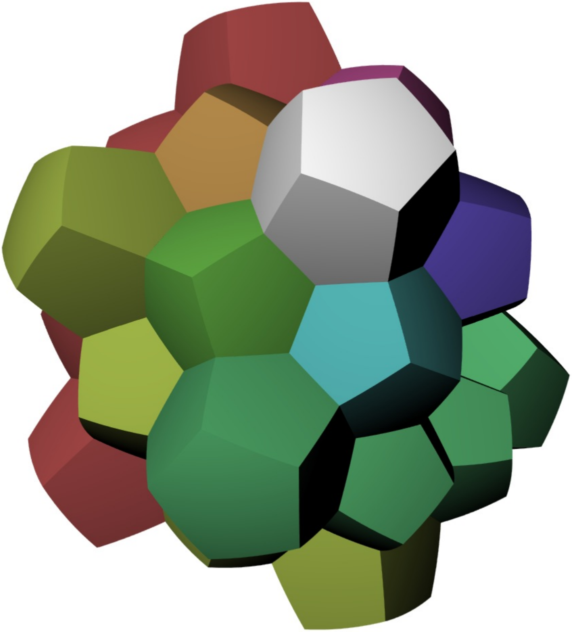

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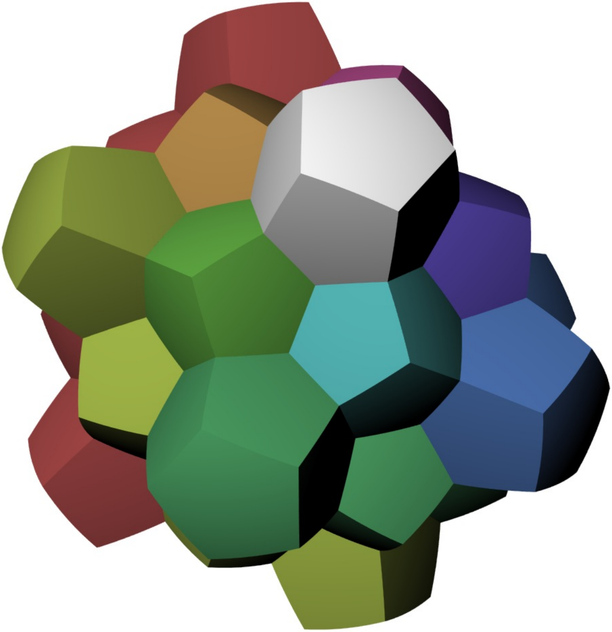

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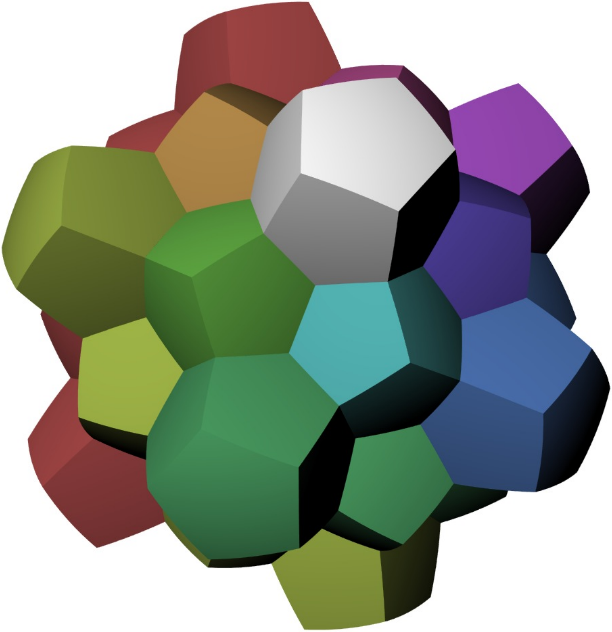
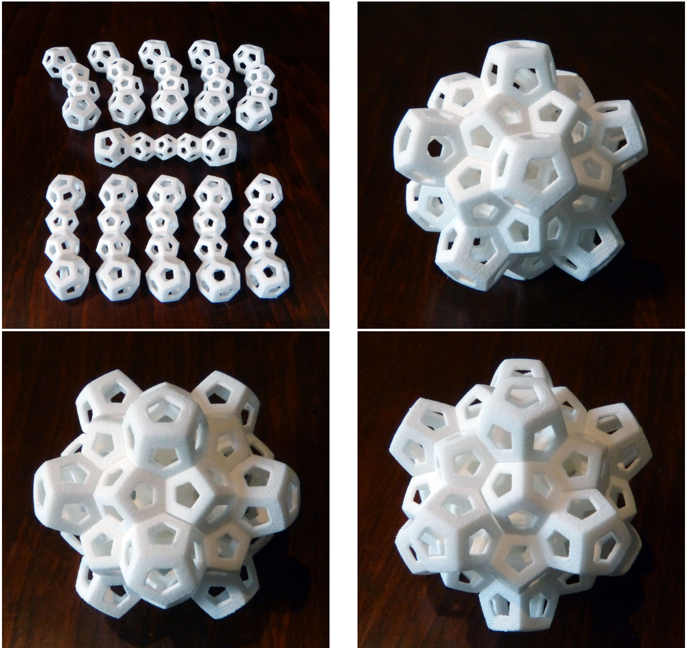
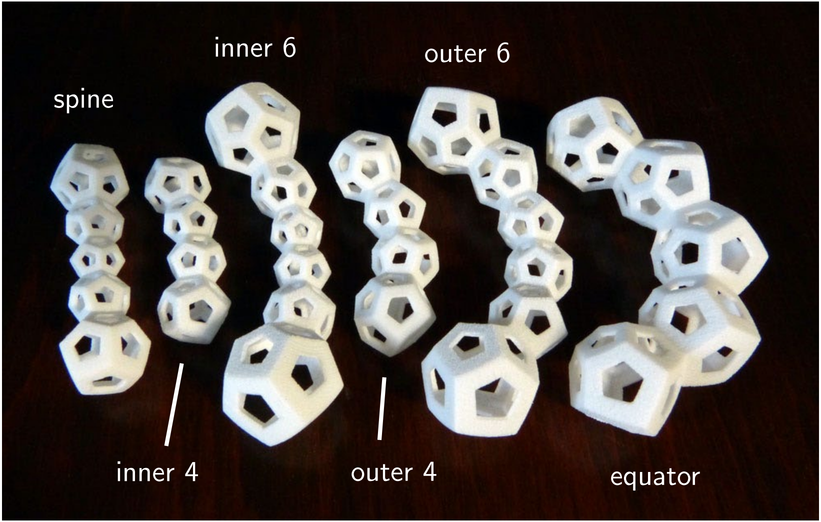


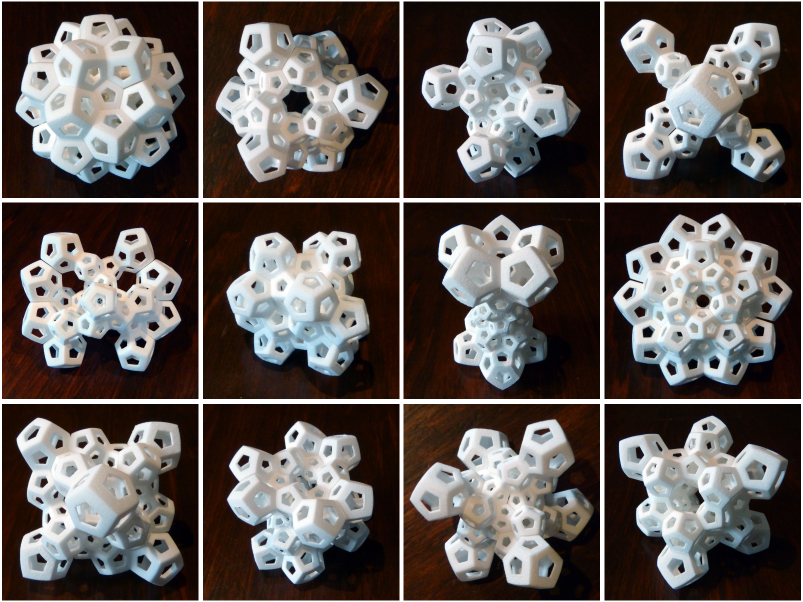
Dc45 Meteor puzzle



Six kinds of ribs



These make many puzzles, which we collectively call Quintessence.



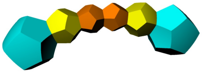
Theorem

- *At most six inner ribs are used in any puzzle.*
- *At most six outer ribs are used in any puzzle.*
- *At most ten inner and outer ribs are used in any puzzle.*

Theorem

- *At most six inner ribs are used in any puzzle.*
- *At most six outer ribs are used in any puzzle.*
- *At most ten inner and outer ribs are used in any puzzle.*

Proof.



□

Theorem

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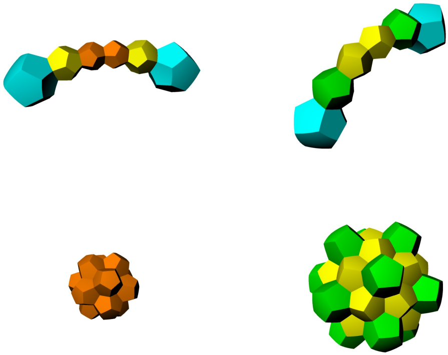
- *At most ten inner and outer ribs are used in any puzzle.*

Proof.



□

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- *At most six inner ribs are used in any puzzle.*
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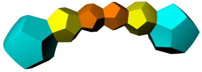
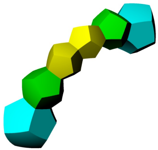
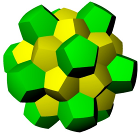
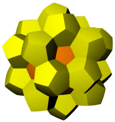
- *At most ten inner and outer ribs are used in any puzzle.*

Proof.









$\square$

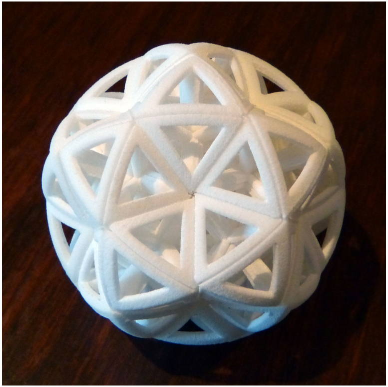
Further possibilities: vertex centered projection

Dv30 Asteroid puzzle

Further possibilities: other polytopes

The 600-cell works, although the ribs now have handedness.

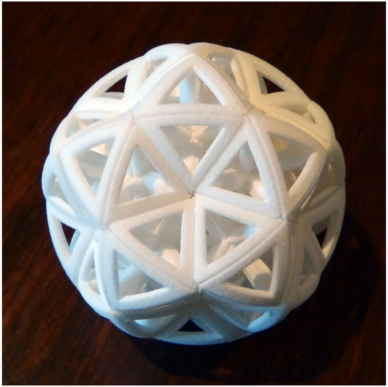
Tv270 Meteor puzzle



Further possibilities: other polytopes

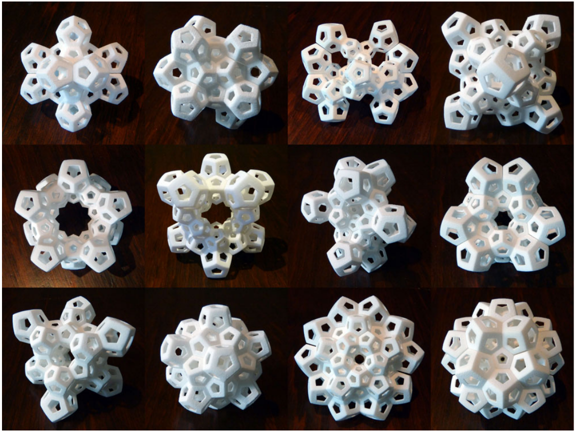
The 600-cell works, although the ribs now have handedness.

Tv270 Meteor puzzle



The other regular polytopes seem to have too few cells to make interesting puzzles.

Thanks!



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http://ms.unimelb.edu.au/~segerman/
http://youtube.com/user/henryseg
http://www.shapeways.com/shops/henryseg?section=Quintessence