A study on the global regularity for a model of the 3D axisymmetric Navier–Stokes equations

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ABSTRACT

This paper investigates the global regularity issue concerning a model equation proposed by Hou and Lei (2008) [9] to understand the stabilizing effects of the nonlinear terms in the 3D axisymmetric Navier–Stokes and Euler equations. We establish the global regularity of a generalized version of their model with a fractional Laplacian when the fractional power satisfies an explicit condition. This condition is exactly the same as in the case of the 3D generalized Navier–Stokes equations and is due to the balance between a more regular nonlinearity and a less effective (five-dimensional) Laplacian.

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1. Introduction

The global regularity issue concerning the 3D axisymmetric Navier–Stokes and Euler equations has recently attracted a lot of attention and much progress has been made (see e.g. [1–8]). The results presented here were motivated by recent work of Hou and his collaborators on two models for the axisymmetric Navier–Stokes and Euler equations [4,6,7].

In several recent papers [3–5,9], Hou et al. proposed two systems of equations for study in order to understand the stabilizing effects of the nonlinear terms in the 3D axisymmetric Navier–Stokes and Euler equations. We shall briefly summarize their derivation of these model equations. The incompressible 3D axisymmetric Navier–Stokes equations can be written as

$$
\begin{cases}
\dfrac{\widetilde{D}}{Dt} u^r - \dfrac{(u^\theta)^2}{r} = -p_r + \nu \left(\partial_{rr} + \dfrac{1}{r}\partial_r + \partial_{zz} - \dfrac{1}{r^2} \right) u^r, \\
\dfrac{\widetilde{D}}{Dt} u^\theta + \dfrac{u^r u^\theta}{r} = \nu \left(\partial_{rr} + \dfrac{1}{r}\partial_r + \partial_{zz} - \dfrac{1}{r^2} \right) u^\theta, \\
\dfrac{\widetilde{D}}{Dt} u^z = -p_z + \nu \left(\partial_{rr} + \dfrac{1}{r}\partial_r + \partial_{zz} \right) u^z, \\
\partial_r u^r + \dfrac{1}{r} u^r + \partial_z u^z = 0,
\end{cases}
\tag{1.1}
$$

where u^r, u^θ and u^z are the cylindrical coordinates of the velocity field **u**, and

$$
\frac{\widetilde{D}}{Dt} = \partial_t + u^r \partial_r + u^z \partial_z.
$$

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When $\nu = 0$, these equations reduce to the axisymmetric Euler equations. The corresponding vorticity $\omega = \nabla \times \mathbf{u}$ obey

$$
\begin{cases}
\dfrac{\widetilde{D}}{Dt}\omega^r = \nu \left(\partial_{rr} + \dfrac{1}{r}\partial_r + \partial_{zz} - \dfrac{1}{r^2} \right) \omega^r + (\omega^r \partial_r + \omega^z \partial_z)u^r, \\[2mm]
\dfrac{\widetilde{D}}{Dt}\omega^\theta + \dfrac{u^\theta \omega^r}{r} = \nu \left(\partial_{rr} + \dfrac{1}{r}\partial_r + \partial_{zz} - \dfrac{1}{r^2} \right) \omega^\theta + (\omega^r \partial_r + \omega^z \partial_z)u^\theta + \dfrac{u^r \omega^\theta}{r}, \\[2mm]
\dfrac{\widetilde{D}}{Dt}\omega^z = \nu \left(\partial_{rr} + \dfrac{1}{r}\partial_r + \partial_{zz} \right) \omega^z + (\omega^r \partial_r + \omega^z \partial_z)u^z.
\end{cases}
\tag{1.2}
$$

Noticing that u^r and u^z can be represented by ψ^θ, ω^r and ω^z by u^θ and ω^θ and ψ^θ are related by

$$
-\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2} \right) \psi^\theta = \omega^\theta,
\tag{1.3}
$$

the axisymmetric Navier–Stokes equations reduce to a system of equations for the swirl components ψ^θ, u^θ and ω^θ. By substituting the expansions of ψ^θ, u^θ and ω^θ near $r = 0$ and keeping the leading order terms, Hou and Li derived an one-dimensional model that approximates the Navier–Stokes equations along the symmetric axis [9]. This model has some interesting properties. In particular, the nonlinear terms have a very special structure and appear to have depletion mechanism that prevents a finite-time singularity.

By substituting the new variables

$$
u_1 = \frac{u^\theta}{r}, \qquad \omega_1 = \frac{\omega^\theta}{r}, \qquad \psi_1 = \frac{\psi^\theta}{r}
$$

in the swirl component equations of (1.1), (1.2) and in (1.3), and dropping the convection terms, Hou and Lei [4] obtained the following system of model equations

$$
\begin{cases}
\partial_t u_1 = \nu \left(\partial_{rr} + \dfrac{3}{r}\partial_r + \partial_{zz} \right) u_1 + 2\partial_z \psi_1 \, u_1, \\[2mm]
\partial_t \omega_1 = \nu \left(\partial_{rr} + \dfrac{3}{r}\partial_r + \partial_{zz} \right) u_1 + \partial_z(u_1^2), \\[2mm]
-\left(\partial_{rr} + \dfrac{3}{r}\partial_r + \partial_{zz} \right) \psi_1 = \omega_1.
\end{cases}
\tag{1.4}
$$

Clearly this system of equations is self-contained. When the convection terms are added back to this system of equations, the 3D axisymmetric Navier–Stokes equations can be recovered. Even without the convection terms, these equations possess many similarities as the 3D axisymmetric Navier–Stokes equations. As demonstrated in [4,5], regularity criteria of the Prodi–Serrin type and of the Beal–Kato–Majda type still hold for this system of equations.

Our attention is focused on the open problem of whether classical solutions of (1.4) are global in time. This is an extremely difficult problem and the intention here is to examine it from a more general point of view. We generalize this model to include dissipation given by a general fractional Laplacian. For this purpose, we need to interpret these equations as a system of equations in five-dimensional space. To be more precise, we set $y = (y_1, y_2, y_3, y_4, z) \in \mathbf{R}^5$ and write Δ_y for the 5D Laplacian, namely

$$
\Delta_y = \sum_{j=1}^{4} \partial_{y_j y_j} + \partial_{zz}.
$$

If a function $f = f(y)$ is axisymmetric about the z-axis, then

$$
\Delta_y f = \left(\partial_{rr} + \frac{3}{r}\partial_r + \partial_{zz} \right) f.
$$

Identifying u_1, ω_1 and ψ_1 as 5D axisymmetric functions, we can write the equations in (1.4) as

$$
\begin{cases}
\partial_t u_1 = \nu \Delta_y u_1 + 2\partial_z \psi_1 \, u_1, \\
\partial_t \omega_1 = \nu \Delta_y \omega_1 + \partial_z(u_1^2), \\
-\Delta_y \psi_1 = \omega_1.
\end{cases}
$$

Replacing Δ_y by the fractional Laplacian $-(-\Delta_y)^\alpha$ for a parameter $\alpha > 0$ in the first two equations, we obtain the generalized Hou–Lei model

$$
\begin{cases}
\partial_t u_1 = -\nu(-\Delta_y)^\alpha u_1 + 2\partial_z \psi_1 \, u_1, \\
\partial_t \omega_1 = -\nu(-\Delta_y)^\alpha \omega_1 + \partial_z(u_1^2), \\
(-\Delta_y)\psi_1 = \omega_1.
\end{cases}
\tag{1.5}
$$

More generally, for any integer $n \geq 3$, we can consider the following equations of $n+2$-dimensional axisymmetric functions u_1, ω_1 and ψ_1,

$$\begin{cases} \partial_t u_1 = -\nu(-\Delta_{n+2})^{\alpha} u_1 + 2\partial_z \psi_1 \, u_1 \\ \partial_t \omega_1 = -\nu(-\Delta_{n+2})^{\alpha} \omega_1 + \partial_z(u_1^2), \\ (-\Delta_{n+2})\psi_1 = \omega_1, \end{cases} \tag{1.6}$$

where Δ_{n+2} denotes the Laplacian operator in $\mathbf{R}^{n+2}$. We study the initial-value problems of these generalized Hou–Lei equations with the initial data

$$u_1(x, 0) = u_{10}(x), \qquad \omega_1(x, 0) = \omega_{10}(x), \qquad \psi_1(x, 0) = \psi_{10}(x). \tag{1.7}$$

This paper establishes the global regularity of (1.5) for $\alpha \geq \frac{5}{4}$ and that of (1.6) for $\alpha \geq \frac{1}{2} + \frac{n}{4}$. We remark that the condition on α is exactly the same as the condition for the generalized Navier–Stokes equations (see e.g. [10]). Through the construction of the model (1.5), the convection terms in the Navier–Stokes equations have been removed and the nonlinear terms in (1.5) are significantly weakened. The dissipation in (1.5) grows out of the original 3D Laplacian in the Navier–Stokes equations and is represented by the axisymmetric 5D Laplacian. When $\alpha \geq \frac{1}{2} + \frac{n}{4}$, we are able to control the nonlinear part through the dissipation.

The global regularity results can be stated as the following theorems. In the these theorems and in the rest of this paper, $\|f\|_q$ with $1 \leq q \leq \infty$ denotes the norm in the Lebesgue space $L^q(\mathbf{R}^5)$, $\|f\|_{H^k}$ denotes the norm in the space $H^k(\mathbf{R}^5)$ and $\|f\|_{k,q}$ the norm in the Sobolev space $W^{k,q}(\mathbf{R}^5)$.

Theorem 1.1. *Consider the generalized 3D model* (1.5). *Assume that the initial data* $(u_{10}, \omega_{10}, \psi_{10})$ *in* (1.7) *satisfies*

$$u_{10} \in H^1(\mathbf{R}^5), \qquad \psi_{10} \in H^2(\mathbf{R}^5) \quad and \quad \omega_{10} = -\Delta_y \psi_{10}.$$

When $\alpha \geq \frac{5}{4}$, *the solution* (u_1, ω_1, ψ_1) *emanating from* $(u_{10}, \omega_{10}, \psi_{10})$ *remains bounded in* $H^1(\mathbf{R}^5) \times L^2(\mathbf{R}^5) \times H^2(\mathbf{R}^5)$ *for all time. More precisely, we have, for any* $0 \leq t < \infty$,

$$\left(\|u_1\|_{H^1(\mathbf{R}^5)} + 2\|\omega_1\|_2^2 \right) + \nu \int_0^t \left(\|\Lambda_y^{\alpha} u_1\|_2^2 + \|\Lambda_y^{1+\alpha}(u_1, \psi_1)\|_2^2 + 2\|\Lambda_y^{\alpha} \omega_1\|_2^2 \right) dt \leq C,$$

where $\Lambda_y = (-\Delta_y)^{1/2}$ *and* C *is a constant depending on* $\|u_{10}\|_{H^1}$, $\|\omega_1\|_2$ *and* $\|\psi_{10}\|_{H^2}$ *only.*

A similar global result holds for the general system of equations given by (1.6).

Theorem 1.2. *Consider the generalized model* (1.6) *with the initial data given by* (1.7). *Assume that* $(u_{10}, \omega_{10}, \psi_{10})$ *satisfies*

$$u_{10} \in H^1(\mathbf{R}^{n+2}), \qquad \psi_{10} \in H^2(\mathbf{R}^{n+2}) \quad and \quad \omega_{10} = -\Delta_{n+2}\psi_{10}.$$

If

$$\alpha \geq \frac{1}{2} + \frac{n}{4},$$

then any solution of (1.6) *emanating from* $(u_{10}, \omega_{10}, \psi_{10})$ *remains bounded in* $H^1(\mathbf{R}^{n+2}) \times L^2(\mathbf{R}^{n+2}) \times H^2(\mathbf{R}^{n+2})$ *for all time.*

The next section presents the proofs of these two theorems.

2. Proofs

This section proves Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Multiplying the first equation in (1.5) by u_1, the second by $2\psi_1$, integrating over $y \in \mathbf{R}^5$ and performing several integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbf{R}^5} (u_1^2 + 2|\nabla_y \psi_1|^2) \, dy + \nu \int_{\mathbf{R}^5} (|\Lambda_y^{\alpha} u_1|^2 + 2|\Lambda_y^{1+\alpha} \psi_1|^2) \, dy = 0,$$

where $\Lambda_y = (-\Delta_y)^{\frac{1}{2}}$. Integrating in time yields

$$\int_{\mathbf{R}^5} (u_1^2 + 2|\nabla_y \psi_1|^2) \, dy + 2\nu \int_0^t \int_{\mathbf{R}^5} (|\Lambda_y^{\alpha} u_1|^2 + 2|\Lambda_y^{1+\alpha} \psi_1|^2) \, dy \, dt = \int_{\mathbf{R}^5} (u_{10}^2 + 2|\nabla_y \psi_{10}|^2) \, dy. \tag{2.1}$$

To obtain further bounds, we multiply the first equation in (1.5) by $\Delta_y u_1$, the second by $2\omega_1$, integrate over $y \in \mathbf{R}^5$ to obtain

$$\frac{1}{2}\frac{d}{dt} \int \left(|\nabla_y u_1|^2 + 2|\omega_1|^2 \right) dy + \nu \int \left(|\Lambda_y^{1+\alpha} u_1|^2 + 2|\Lambda_y^{\alpha} \omega_1|^2 \right) dy = J_1 + J_2, \tag{2.2}$$

where

$$J_1 = \int 2\partial_z \psi_1 u_1 \Delta_y u_1 \, dy, \qquad J_2 = \int 2\omega_1 \partial_z u_1^2 \, dy.$$

We estimate J_1 and J_2. By Hölder's inequality,

$$|J_1| \le C \, \|\Delta_y u_1\|_2 \, \|\partial_z \psi_1\|_4 \, \|u_1\|_4. \tag{2.3}$$

By the Gagliardo–Nirenberg type inequality, for $\alpha \ge 1$,

$$\|\Delta_y u_1\|_2 \le C \, \|u_1\|_2^{\frac{\alpha-1}{\alpha+1}} \, \|\Lambda_y^{1+\alpha} u_1\|_2^{\frac{2}{1+\alpha}}, \tag{2.4}$$

we obtain

$$\|u_1\|_4 \le C \, \|u_1\|_2^a \, \|\nabla_y u_1\|_2^b \, \|\Lambda^\alpha u_1\|_2^c \, \|\Lambda_y^{1+\alpha} u_1\|_2^d, \tag{2.5}$$

where the indices $a, b, c, d \in [0, 1]$ and satisfy

$$a + b + c + d = 1, \qquad \frac{1}{4} = \frac{a}{2} + b\left(\frac{1}{2} - \frac{1}{5}\right) + c\left(\frac{1}{2} - \frac{\alpha}{5}\right) + d\left(\frac{1}{2} - \frac{1+\alpha}{5}\right). \tag{2.6}$$

Writing a and b in terms of c and d, we have

$$a = -\frac{1}{4} + (\alpha - 1)c + \alpha d, \qquad b = \frac{5}{4} - \alpha c - (1 + \alpha)d. \tag{2.7}$$

Similarly,

$$\begin{aligned}\|\partial_z \psi_1\|_4 &\le C \|\partial_z \psi_1\|_2^e \, \|\nabla_y \partial_z \psi_1\|_2^f \, \|\Lambda_y^\alpha \partial_z \psi_1\|_2^g \, \|\Lambda_y^{1+\alpha} \partial_z \psi_1\|_2^h \\ &\le C \, \|\nabla_y \psi_1\|_2^e \, \|\omega_1\|_2^f \, \|\Lambda_y^{1+\alpha} \psi_1\|_2^g \, \|\Lambda_y^\alpha \omega_1\|_2^h, \end{aligned} \tag{2.8}$$

where the indices $e, f, g, h \in [0, 1]$ and satisfy

$$e + f + g + h = 1, \qquad \frac{1}{4} = \frac{e}{2} + f\left(\frac{1}{2} - \frac{1}{5}\right) + g\left(\frac{1}{2} - \frac{\alpha}{5}\right) + h\left(\frac{1}{2} - \frac{1+\alpha}{5}\right). \tag{2.9}$$

Or

$$e = (\alpha - 1)g + \alpha h - \frac{1}{4}, \qquad f = \frac{5}{4} - \alpha g - (1 + \alpha)h. \tag{2.10}$$

Inserting (2.4), (2.5) and (2.8) in (2.3), we obtain

$$|J_1| \le C \, \|u_1\|_2^{\frac{\alpha-1}{\alpha+1}+a} \, \|\nabla_y \psi_1\|_2^e \, \|\nabla_y u_1\|_2^b \, \|\omega_1\|_2^f \, \|\Lambda^\alpha u_1\|_2^c \, \|\Lambda_y^{1+\alpha} \psi_1\|_2^g \|\Lambda_y^{1+\alpha} u_1\|_2^{\frac{2}{1+\alpha}+d} \, \|\Lambda_y^\alpha \omega_1\|_2^h. \tag{2.11}$$

When

$$\frac{2}{1+\alpha} + d + h \le 2,$$

we apply Young's inequality with

$$\frac{h}{2} + \frac{1}{1+\alpha} + \frac{d}{2} + \frac{1}{p} = 1 \quad \text{or} \quad p = \frac{2(\alpha+1)}{2\alpha - (\alpha+1)(h+d)} \tag{2.12}$$

to obtain

$$|J_1| \le \frac{\nu}{2} \|\Lambda_y^\alpha \omega_1\|_2^2 + \frac{\nu}{2} \, \|\Lambda_y^{1+\alpha} u_1\|_2^2 + C(\nu) \, \|u_1\|_2^{\gamma_1} \, \|\nabla_y \psi_1\|_2^{\gamma_2} \|\nabla_y u_1\|_2^{\gamma_3} \, \|\omega_1\|_2^{\gamma_4} \, \|\Lambda^\alpha u_1\|_2^{\gamma_5} \, \|\Lambda_y^{1+\alpha} \psi_1\|_2^{\gamma_6},$$

where

$$\gamma_1 = p\left(\frac{\alpha-1}{\alpha+1} + a\right), \qquad \gamma_2 = p\,e, \qquad \gamma_3 = p\,b, \qquad \gamma_4 = p\,f, \qquad \gamma_5 = p\,c, \qquad \gamma_6 = p\,g.$$

When $\gamma_3 + \gamma_4 \le 2$ and $\gamma_5 + \gamma_6 \le 2$, namely

$$p(b+f) \le 2 \quad \text{and} \quad p(c+g) \le 2, \tag{2.13}$$

we can apply Young's inequality again to further bound J_1 by

$$|J_1| \leq \frac{\nu}{2} \|\Lambda_y^\alpha \omega_1\|_2^2 + \frac{\nu}{2} \|\Lambda_y^{1+\alpha} u_1\|_2^2 + C(\nu) \|u_1\|_2^{\gamma_1} \|\nabla_y \psi_1\|_2^{\gamma_2} \left(\|\nabla_y u_1\|_2^2 + \|\omega_1\|_2^2 \right) \left(\|\Lambda_y^\alpha u_1\|^2 + \|\Lambda_y^{1+\alpha} \psi_1\|_2^2 \right). \tag{2.14}$$

Invoking (2.7), (2.10) and (2.12), the conditions in (2.13) can be rewritten as

$$\frac{2(\alpha+1)}{2\alpha - (\alpha+1)(d+h)} \cdot \left(\frac{5}{2} - \alpha(c+g) - (1+\alpha)(d+h) \right) \leq 2, \tag{2.15}$$

$$\frac{2(\alpha+1)}{2\alpha - (\alpha+1)(d+h)} (c+g) \leq 2. \tag{2.16}$$

Equivalently,

$$\frac{\alpha+5}{2\alpha(\alpha+1)} \leq (c+g) + (d+h) \leq \frac{2\alpha}{\alpha+1}. \tag{2.17}$$

When $\alpha \geq \frac{5}{4}$,

$$\frac{\alpha+5}{2\alpha(\alpha+1)} \leq \frac{2\alpha}{\alpha+1}$$

and we can select suitable c, g, d and h so that (2.17) holds and thus (2.13) holds. Some special choices of the indices a, b, c, d and e, f, g, h are

$$a = 0, \qquad b = \frac{4}{9}, \qquad c = \frac{4}{9}, \qquad d = \frac{1}{9}, \qquad e = 0, \qquad f = \frac{4}{9}, \qquad g = \frac{4}{9}, \qquad h = \frac{1}{9}$$

in the case $\alpha = \frac{5}{4}$, and

$$a = e = 0, \qquad b = f = \frac{4\alpha^2 + 3\alpha - 5}{4\alpha(\alpha+1)}, \qquad c = g = \frac{1}{\alpha+1}, \qquad d = h = \frac{5 - 3\alpha}{4\alpha(\alpha+1)}$$

in the case of $\alpha \geq \frac{5}{4}$.

We now bound J_2. By the third equation in (1.5), J_2 can be written as

$$J_2 = -4 \int u_1 \, \partial_z u_1 \, \Delta_y \psi_1 \, dy.$$

For any $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, we have, by Hölder's inequality,

$$|J_2| \leq \|u_1\|_p \|\partial_z u_1\|_q \|\omega_1\|_2. \tag{2.18}$$

Furthermore, by the Gagliardo–Nirenberg type inequalities

$$\|u_1\|_p \leq C \|u_1\|_2^{a_1} \|\nabla_y u_1\|_2^{b_1} \|\Lambda^\alpha u_1\|_2^{c_1} \|\Lambda^{1+\alpha} u_1\|_2^{d_1},$$

$$\|\partial_z u_1\|_q \leq C \|\nabla_y u_1\|_2^{b_2} \|\Lambda^\alpha u_1\|_2^{c_2} \|\Lambda^{1+\alpha} u_1\|_2^{d_2} \tag{2.19}$$

with the indices satisfying

$$a_1 + b_1 + c_1 + d_1 = 1, \qquad b_2 + c_2 + d_2 = 1,$$

$$\frac{1}{p} = \frac{a_1}{2} + b_1 \left(\frac{1}{2} - \frac{1}{5} \right) + c_1 \left(\frac{1}{2} - \frac{\alpha}{5} \right) + d_1 \left(\frac{1}{2} - \frac{1+\alpha}{5} \right),$$

$$\frac{1}{q} - \frac{1}{5} = b_2 \left(\frac{1}{2} - \frac{1}{5} \right) + c_2 \left(\frac{1}{2} - \frac{\alpha}{5} \right) + d_2 \left(\frac{1}{2} - \frac{1+\alpha}{5} \right),$$

we obtain

$$\|u_1\|_p \|\partial_z u_1\|_q \leq C \|u_1\|_2^{a_1} \|\nabla_y u_1\|_2^{b_3} \|\Lambda^\alpha u_1\|_2^{c_3} \|\Lambda^{1+\alpha} u_1\|_2^{d_3}, \tag{2.20}$$

where $b_3 = b_1 + b_2$, $c_3 = c_1 + c_2$ and $d_3 = d_1 + d_2$. Clearly

$$a_1 + b_3 + c_3 + d_3 = 2, \tag{2.21}$$

$$\frac{a_1}{2} + b_3 \frac{3}{10} + c_3 \frac{5 - 2\alpha}{10} + d_3 \frac{3 - 2\alpha}{10} = \frac{3}{10}. \tag{2.22}$$

Inserting (2.20) in (2.18) and applying Young's inequality, we obtain

$$|J_2| \le \frac{\nu}{2} \|\Lambda^{1+\alpha} u_1\|_2^2 + C(\nu) \|u_1\|_2^{\frac{2a_1}{2-d_3}} \|\nabla_y u_1\|_2^{\frac{2b_3}{2-d_3}} \|\Lambda^\alpha u_1\|_2^{\frac{2c_3}{2-d_3}} \|\omega_1\|_2^{\frac{2}{2-d_3}}.$$

If

$$\frac{2c_3}{2-d_3} \le 2, \qquad \frac{2b_3}{2-d_3} + \frac{2}{2-d_3} \le 2, \tag{2.23}$$

a further application of Young's inequality implies

$$|J_2| \le \frac{\nu}{2} \|\Lambda^{1+\alpha} u_1\|_2^2 + C(\nu) \|u_1\|_2^{\frac{2a_1}{2-d_3}} \|\Lambda^\alpha u_1\|_2^2 \left(\|\nabla_y u_1\|_2^2 + \|\omega_1\|_2^2 \right). \tag{2.24}$$

When $\alpha \ge \frac{5}{4}$, we can choose suitable a_1, b_2, c_3 and d_3 so that they satisfy (2.21)–(2.23). In fact, these conditions are equivalent to

$$a_1 + c_3 = 2 - (b_3 + d_3),$$
$$(b_3 + d_3) + \alpha(c_3 + d_3) = \frac{7}{2},$$
$$c_3 + d_3 \le 2, \qquad b_3 + d_3 \le 1$$

and all of them are obviously satisfied if we set

$$a_1 = 0, \qquad b_3 = 2 - \frac{5}{2\alpha}, \qquad c_3 = 1 \quad \text{and} \quad d_3 = \frac{5}{2\alpha} - 1.$$

Combining (2.2), (2.14) and (2.24), we find that

$$\begin{aligned} \frac{d}{dt} &\left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2 \right) + \nu \left(\|\Lambda_y^{1+\alpha} u_1\|_2^2 + 2\|\Lambda_y^\alpha \omega_1\|_2^2 \right) \\ &\le C(\nu) \|u_1\|_2^{\gamma_1} \|\nabla_y \psi_1\|_2^{\gamma_2} \left(\|\Lambda_y^\alpha u_1\|^2 + \|\Lambda_y^{1+\alpha} \psi_1\|_2^2 \right) \left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2 \right) \\ &\quad + C(\nu) \|u_1\|_2^{\frac{2a_1}{2-d_3}} \|\Lambda^\alpha u_1\|_2^2 \left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2 \right). \end{aligned}$$

It then follows from Gronwall's inequality and (2.1) that

$$\left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2 \right) + \nu \int_0^t \left(\|\Lambda_y^{1+\alpha} u_1\|_2^2 + 2\|\Lambda_y^\alpha \omega_1\|_2^2 \right) dt \le C,$$

where C is a constant depending on the norms of the initial data, namely $\|u_{10}\|_2 + \|\nabla_y u_{10}\|_2$, $\|\nabla_y \psi_{10}\|_2$ and $\|\omega_{10}\|_2$. When the initial data are more regular, the solution of (1.5) can be shown to be more regular. In particular, smooth data yield smooth solutions. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. The proof is similar to the proof of Theorem 1.1. The estimates in the proof of Theorem 1.1 remain valid although the associated indices should be suitably modified. For example, (2.6), (2.7), (2.9), (2.10) and (2.15)–(2.17) should be changed to (2.25)–(2.31), respectively, where the new equations are given by

$$a + b + c + d = 1, \tag{2.25}$$
$$\frac{1}{4} = \frac{a}{2} + b \left(\frac{1}{2} - \frac{1}{n+2} \right) + c \left(\frac{1}{2} - \frac{\alpha}{n+2} \right) + d \left(\frac{1}{2} - \frac{1+\alpha}{n+2} \right),$$
$$a = -\frac{n-2}{4} + (\alpha - 1)c + \alpha d, \qquad b = \frac{n+2}{4} - \alpha c - (1+\alpha)d, \tag{2.26}$$
$$e + f + g + h = 1, \tag{2.27}$$
$$\frac{1}{4} = \frac{e}{2} + f \left(\frac{1}{2} - \frac{1}{n+2} \right) + g \left(\frac{1}{2} - \frac{\alpha}{n+2} \right) + h \left(\frac{1}{2} - \frac{1+\alpha}{n+2} \right),$$
$$e = -\frac{n-2}{4} + (\alpha - 1)g + \alpha h, \qquad f = \frac{n+2}{4} - \alpha g - (1+\alpha)h, \tag{2.28}$$
$$\frac{2(\alpha+1)}{2\alpha - (\alpha+1)(d+h)} \cdot \left(\frac{n+2}{2} - \alpha(c+g) - (1+\alpha)(d+h) \right) \le 2, \tag{2.29}$$
$$\frac{2(\alpha+1)}{2\alpha - (\alpha+1)(d+h)} (c+g) \le 2, \tag{2.30}$$

$$\frac{n(\alpha + 1) + 2 - 2\alpha}{2\alpha(\alpha + 1)} \le (c + g) + (d + h) \le \frac{2\alpha}{\alpha + 1}. \tag{2.31}$$

When $\alpha \ge \frac{1}{2} + \frac{n}{4}$,

$$\frac{n(\alpha + 1) + 2 - 2\alpha}{2\alpha(\alpha + 1)} \le \frac{2\alpha}{\alpha + 1}$$

and the indices a, b, c, d, e, f, g and h can be selected so that all the estimates in the proof of Theorem 1.1 remain valid. A special set of indices is

$$a = e = 0, \qquad b = f = \frac{4\alpha^2 + (6 - n)\alpha - n - 2}{4\alpha(\alpha + 1)}, \qquad c = g = \frac{1}{\alpha + 1}, \qquad d = h = \frac{(n - 6)\alpha + n + 2}{4\alpha(\alpha + 1)}.$$

We omit further details and this completes the proof of Theorem 1.2. □

Acknowledgments

References

[1] D. Chae, On the Lagrangian dynamics of the axisymmetric 3D Euler equations, J. Differential Equations 249 (2010) 571–577.
[2] A. Hammadi, H. Taoufik, K. Sahbi, On the global well-posedness for the axisymmetric Euler equations, Math. Ann. 347 (2010) 15–41.
[3] T.Y. Hou, Z. Lei, C. Li, Global regularity of the 3D axi-symmetric Navier–Stokes equations with anisotropic data, Comm. Partial Differentail Equations 33 (2008) 1622–1637.
[4] T.Y. Hou, Z. Lei, On the stabilizing effect of convection in 3D incompressible flow, Commun. Pure Appl. Math. 62 (2009) 501–564.
[5] T.Y. Hou, Z. Lei, On the partial regularity of a 3D model of the Navier–Stokes equations, Comm. Math. Phys. 287 (2009) 589–612.
[6] T.Y. Hou, C. Li, Z. Shi, S. Wang, X. Yu, On singularity formation of a nonlinear nonlocal system, Arch. Ration. Mech. Anal. 199 (2011) 117–144.
[7] T.Y. Hou, Z. Shi, S. Wang, On singularity formation of a 3D model for incompressible Navier–Stokes equations, arXiv:0912.1316v1 [math.AP] (2009).
[8] Q. Chen, C. Miao, Z. Zhang, Global well-posedness for the 3D rotating Navier–Stokes equations with highly oscillation initial data, arXiv:0910.3064v2 [math.AP] (2009).
[9] T.Y. Hou, C. Li, Dynamic stability of the 3D axi-symmetric Navier–Stokes equations with swirl, Commun. Pure Appl. Math. 61 (2008) 661–697.
[10] J. Wu, Generalized MHD equations, J. Differential Equations 195 (2003) 284–312.