MINIMALLY NON-DIATONIC PC-SETS

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1. Minimally non-diatonic sets

Throughout, we use the Forte notation, identifying the twelve pitch classes of the chromatic scale with elements of the set $\{0, 1, 2, \ldots, 9, \mathrm{T}, \mathrm{E}\}$. Call a pc-set *non-diatonic* if it is not contained in any diatonic major scale. For example, the entire chromatic scale is non-diatonic, as is the whole-tone scale 02468T. On the other hand, the pc-set 015 is *diatonic*, as it is contained in many diatonic major scales (C sharp major, for example).

Note that the non-diatonic property is, by definition monotonic; if X is non-diatonic and X is a subset of Y, then Y is non-diatonic as well. This suggests the following definition.

Definition 1.1. A pc-set X is *minimally non-diatonic* if it is non-diatonic, but the removal of any pitch class from X yields a diatonic set.

The chromatic scale is non-diatonic, but it is far from being minimally non-diatonic, as one must remove at least five pitch classes before the resulting set is contained in some major scale. The pc-set 012 *is* minimally non-diatonic, though, as 01, 02, and 12 are all diatonic sets. The following computation was performed using the computer algebra system Macaulay 2.

Observation 1.2. Up to prime form, there are only six minimally non-diatonic sets:

$$012, 014, 048, 0167, 0268, \text{ and } 0369.$$

Thus any non-diatonic pc-set must contain a transposition and / or inversion of at least one of the above sets. For example, the whole-tone scale contains the sets 048 and 0268.

The three minimally non-diatonic tetrachords can be thought of as starting with the tritone 06 and combining it with its transpositions 17, 28, and 39.

It also follows that any pc-set of eight or more pitch classes must contain one of the minimally non-diatonic sets.

2. Complexes

Implicit in the above construction is a simplicial complex whose faces correspond to diatonic sets. Formally, we define the complex Δ as follows.

Definition 2.1. The complex Δ has vertex set $\{0, 1, 2, \ldots, 9, \mathrm{T}, \mathrm{E}\}$ and faces corresponding to diatonic sets. Thus, Δ is pure of dimension six, and has twelve facets, one for each major scale.

The question of the topology of Δ is an interesting one; while many "naturally occurring" complexes in combinatorics and commutative algebra are homotopy equivalent to bouquets of spheres, Δ is not. In fact, Δ only has non-vanishing reduced homology in dimension two. Computations show that the rank of the second reduced homology group of Δ is 5.

The f-vector of Δ has been computed as:

$$1, 12, 66, 180, 240, 180, 72, 12,$$

and thus its reduced Euler characteristic is

$$\tilde{\chi}(\Delta) = 5.$$

In terms of Stanley-Reisner Theory, the complex Δ has minimal non-faces given by all transpositions and inversions of the minimally non-diatonic sets discussed in the previous section. In order, there are 12 minimal non-faces corresponding to 012, 24 corresponding to 014, 4 corresponding to 048, 6 corresponding to 0167, 6 corresponding to 0268, and 3 corresponding to 0369.

This suggests similar analyses for other scales. As modes have the same pc-set, the minimally non-diatonic sets are also, for example, the minimally non-lydian sets. As mentioned before, the minimally non-diatonic tetrachords all consist of two tritones; thus they stem from the fact that a major scale contains only one tritone. It is thus natural to ask if the scales somehow "closest" to the major scale are those whose minimal forbidden tetrachords consists of two copies of the same interval.

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