LECTURE 17

Green's Identities and Green's Functions

Let us recall The Divergence Theorem in n-dimensions.

THEOREM 17.1. *Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field over $\mathbb{R}^n$ that is of class C^1 on some closed, connected, simply connected n-dimensional region $D \subset \mathbb{R}^n$. Then*

$$\int_D \nabla \cdot \mathbf{F}\, dV = \int_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS$$

where ∂D is the boundary of D and $\mathbf{n}(\mathbf{r})$ is the unit vector that is (outward) normal to the surface ∂D at the point $\mathbf{r} \in \partial D$.

As a special case of Stokes' theorem, we may set

$$\mathbf{F} = \boldsymbol{\nabla}\phi \tag{1}$$

with ϕ a C^2 function on D. We then obtain

$$\int_D \boldsymbol{\nabla}^2 \phi \, dV = \int_{\partial D} \boldsymbol{\nabla}\phi \cdot \, dS \quad . \tag{2}$$

Another special case of Stokes' theorem comes from the choice

$$\mathbf{F} = \phi \boldsymbol{\nabla}\psi \quad . \tag{3}$$

For this case, Stokes' theorem says

$$\int_D \boldsymbol{\nabla} \cdot (\phi \boldsymbol{\nabla}\psi) \, dV = \int_{\partial D} \phi \boldsymbol{\nabla}\psi \cdot \mathbf{n} dS \quad . \tag{4}$$

Using the identity

$$\boldsymbol{\nabla} \cdot (\phi \mathbf{F}) = \boldsymbol{\nabla}\phi \cdot \mathbf{F} + \phi \boldsymbol{\nabla} \cdot \mathbf{F} \tag{5}$$

we find (4) is equivalent to

$$\int_D \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi \, dV + \int_D \phi \boldsymbol{\nabla}^2 \psi \, dV = \int_{\partial D} \phi \boldsymbol{\nabla}\psi \cdot \mathbf{n}\, dS \quad . \tag{6}$$

Equation (6) is known as **Green's first identity**.

Reversing the roles of ϕ and ψ in (6) we obtain

$$\int_D \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\phi \, dV + \int_D \psi \boldsymbol{\nabla}^2 \phi \, dV = \int_{\partial D} \psi \boldsymbol{\nabla}\phi \cdot \mathbf{n}\, dS \quad . \tag{7}$$

Finally, subtracting (7) from (6) we get

$$\int_D \left(\phi \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \phi\right) \, dV = \int_{\partial D} (\phi \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\phi) \cdot \mathbf{n}\, dS \quad . \tag{8}$$

Equation (8) is known as **Green's second identity**.

Now set

$$\psi(\mathbf{r}) = \frac{1}{|\mathbf{r} - \mathbf{r}_o| + \epsilon}$$

and insert this expression into (8). We then get

$$\begin{aligned}\int_D \phi \left(\boldsymbol{\nabla}^2 \frac{1}{|\mathbf{r}-\mathbf{r}_o|+\varepsilon} \right) dV &= \int_D \frac{1}{|\mathbf{r}-\mathbf{r}_o|+\varepsilon} \boldsymbol{\nabla}^2 \phi \, dV \\ &\quad + \int_{\partial D} \left(\frac{1}{|\mathbf{r}-\mathbf{r}_o|+\varepsilon} \boldsymbol{\nabla}\phi - \phi \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}_o|+\varepsilon} \right) \cdot \mathbf{n} dS \right) \quad .\end{aligned}$$

Taking the limit $\epsilon \to 0$ and using the identities

$$\begin{aligned}\lim_{\epsilon\to 0} \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}_o|+\varepsilon} &= -4\pi\delta^n (\mathbf{r}-\mathbf{r}_o) \\ \lim_{\epsilon\to 0} \frac{1}{|\mathbf{r}-\mathbf{r}_o|+\varepsilon} &= \frac{1}{|\mathbf{r}-\mathbf{r}_o|} \\ \lim_{\epsilon\to 0} \nabla \frac{1}{|\mathbf{r}-\mathbf{r}_o|+\varepsilon} &= \nabla \frac{1}{|\mathbf{r}-\mathbf{r}_o|}\end{aligned}$$

we obtain

$$\begin{aligned}-4\pi\phi(\mathbf{r}_o) &= \int_D \tfrac{1}{|\mathbf{r}-\mathbf{r}_o|} \nabla^2 \phi \, dV \\ &\quad + \int_{\partial D} \left(\tfrac{1}{|\mathbf{r}-\mathbf{r}_o|} \nabla\phi - \phi \left(\nabla \tfrac{1}{|\mathbf{r}-\mathbf{r}_o|} \right) \cdot \mathbf{n} dS \right) \quad .\end{aligned} \tag{9}$$

Equation (9) is known as **Green's third identity**.

Notice that if ϕ satisfies Laplace's equation the first term on the right hand side vanishes and so we have

$$\begin{aligned}\phi(\mathbf{r}_o) &= \tfrac{-1}{4\pi} \int_{\partial D} \left(\tfrac{1}{|\mathbf{r}-\mathbf{r}_o|} \nabla\phi - \phi \left(\nabla \tfrac{1}{|\mathbf{r}-\mathbf{r}_o|} \right) \cdot \mathbf{n} dS \right) \\ &= \tfrac{1}{4\pi} \int_{\partial D} \left(\phi \tfrac{\partial}{\partial n} \tfrac{1}{\mathbf{r}-\mathbf{r}_o} - \tfrac{1}{|\mathbf{r}-\mathbf{r}_o|} \tfrac{\partial \phi}{\partial n} \right) dS \quad .\end{aligned} \tag{10}$$

Here $\frac{\partial}{\partial n}$ is the directional derivative corresponding to the surface normal vector $\mathbf{n}$. Thus, if ϕ satisfies Laplace's equation in D then its value at any point $\mathbf{r}_o \in D$ is completely determined by the values of ϕ and $\frac{\partial \phi}{\partial n}$ on the boundary of D.

1. Green's Functions and Solutions of Laplace's Equation, II

Recall the fundamental solutions of Laplace's equation in n-dimensions

$$\Phi_n\left(r,\psi,\theta_1,\ldots,\theta_{n-2}\right)=\begin{cases}\log|r| & , \quad \text{if } n=2\\ \frac{1}{r^{n-2}} & , \quad \text{if } n>2\end{cases} \quad . \tag{11}$$

Each of these solutions really only makes sense in the region $\mathbb{R}^n-\mathbf{O}$; for each possesses a singularity at the origin.

We studied the case when $n=3$, a little more closely and found that we could actually write

$$\nabla^2\left(\frac{1}{r}\right)=-4\pi\delta^3\left(\mathbf{r}\right)=\begin{cases}0 & , \quad \text{if } \mathbf{r}\neq\mathbf{0}\\ \infty & , \quad \text{if } \mathbf{r}=\mathbf{0}\end{cases} \tag{12}$$

In fact, using similar arguments one can show that

$$\nabla^2\Phi(\mathbf{r})=-c_n\delta^n\left(\mathbf{r}\right) \tag{13}$$

where c_n is the surface area of the unit sphere in $\mathbb{R}^n$. Thus, the fundamental solutions can actually be regarded as solutions of an **inhomogeneous** Laplace equation where the driving function is concentrated at a single point.

Let us now set $n=3$ and consider the following PDE/BVP

$$\begin{aligned}\nabla^2\Phi(\mathbf{r}) &= f(\mathbf{r}) \quad , \quad r\in D\\ \Phi(\mathbf{r})|_{\partial D} &= h(\mathbf{r})|_{\partial D}\end{aligned} \tag{14}$$

where D is some closed, connected, simply connected region in $\mathbb{R}^3$. Let $\mathbf{r}_o$ be some fixed point in D and set

$$G\left(\mathbf{r},\mathbf{r}_o\right)=\frac{-1}{4\pi\left|\mathbf{r}-\mathbf{r}_o\right|}+\phi_o(\mathbf{r},\mathbf{r}_o) \tag{15}$$

where $\phi_o(\mathbf{r},\mathbf{r}_o)$ is some solution of the homogeneous Laplace equation

$$\nabla^2\phi_o(\mathbf{r},\mathbf{r}_o)=0 \quad . \tag{16}$$

Then

$$\nabla^2G\left(\mathbf{r},\mathbf{r}_o\right)=\delta^3\left(\mathbf{r}-\mathbf{r}_o\right) \quad . \tag{17}$$

Now recall Green's third identity

$$\int_D\left(\Phi\nabla^2\Psi-\Psi\nabla^2\Phi\right)\,dV=\int_{\partial D}\left(\Phi\nabla\Psi-\Psi\nabla\Phi\right)\cdot\mathbf{n}\,dS \quad . \tag{18}$$

If we replace ψ in (18) by $G\left(\mathbf{r},\mathbf{r}_o\right)$ we get

$$\begin{aligned}\Phi\left(\mathbf{r}_o\right) &= \int_D\Phi(\mathbf{r})\delta^3\left(\mathbf{r}-\mathbf{r}_o\right)dV\\ &= \int_D\Phi\nabla^2G dV\\ &= \int_D G\nabla^2\Phi\,dV+\int_{\partial D}\left(\Phi\nabla G-G\nabla\Phi\right)\cdot\mathbf{n}\,dS\\ &= \int_D Gf\,dV+\int_{\partial D}\left(h\frac{\partial G}{\partial n}-G\frac{\partial\Phi}{\partial n}\right)\,dS\\ &= \int_D Gf\,dV+\int_{\partial D}h\frac{\partial G}{\partial n}dS-\int_{\partial D}G\frac{\partial\Phi}{\partial n}dS \quad .\end{aligned} \tag{19}$$

Up to this point we have only required that the function ϕ_o satisfies Laplace's equation. We will now make our choice of ϕ_o more particular; we shall choose $\phi_o(\mathbf{r},\mathbf{r}_o)$ to be the unique solution of Laplace's equation in D satisfying the boundary condition

$$\left.\frac{1}{4\pi\left|\mathbf{r}-\mathbf{r}_o\right|}\right|_{\partial D}=\phi_o(\mathbf{r},\mathbf{r}_o)|_{\partial D} \tag{20}$$

so that

$$G\left(\mathbf{r},\mathbf{r}_o\right)|_{\partial D}=0 \quad .$$

Then the last integral on the right hand side of (19) vanishes and so we have

$$\Phi(\mathbf{r}_o) = \int_D G(\mathbf{r},\mathbf{r}_o) f(\mathbf{r})\, dV + \int_{\partial D} h(\mathbf{r}) \frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}_o)\, dS \quad . \tag{21}$$

Thus, once we find a solution $\phi_o(\mathbf{r},\mathbf{r}_o)$ to the homogenenous Laplace equation satisfying the boundary condition (21), we have a closed formula for the solution of the PDE/BVP (14) in terms of integrals of $G(\mathbf{r},\mathbf{r}_o)$ times the driving function $f(\mathbf{r})$, and of $\frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}_o)$ times the function $h(\mathbf{r})$ describing the boundary conditions on Φ. Note that the Green's function $G(\mathbf{r},\mathbf{r}_o)$ is fixed once we fix ϕ_o which in turn depends only on the nature of the boundary of the region D (through condition (20)).

Example

Let us find the Green's function corresponding to the interior of sphere of radius R centered about the origin. We seek to find a solution of ϕ_o of the homogenous Laplace's equation such that (20) is satisfied. This is accomplished by the following trick.

Suppose $\Phi(r,\psi,\theta)$ is a solution of the homogeneous Laplace equation inside the sphere of radius R centered at the origin. For $r > R$, we define a function

$$\tilde{\Phi}(r,\psi,\theta) = \frac{R}{r}\Phi\left(\frac{R^2}{r},\psi,\theta\right) \quad . \tag{22}$$

I claim that $\tilde{\Phi}(r,\psi,\theta)$ so defined also satisfies Laplace's equation in the region exterior to the sphere.

To prove this, it suffices to show that

$$0 = r^2\nabla\tilde{\Phi} \;=\; \frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{\Phi}}{\partial r}\right) + \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\tilde{\Phi}}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2\tilde{\Phi}}{\partial\psi^2} \tag{23}$$

or

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{\Phi}}{\partial r}\right) = -\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\tilde{\Phi}}{\partial\theta}\right) - \frac{1}{\sin^2(\theta)}\frac{\partial^2\tilde{\Phi}}{\partial\psi^2} \quad . \tag{24}$$

Set

$$u = \frac{R^2}{r} \quad . \tag{25}$$

so that

$$\begin{aligned} r &= \tfrac{R^2}{u} \\ \tilde{\Phi}(r,\psi,\theta) &= \tfrac{u}{R}\Phi(u,\psi,\theta) \\ \tfrac{\partial}{\partial r} &= -\tfrac{du}{dr}\tfrac{\partial}{\partial u} = -\tfrac{R^2}{r^2}\tfrac{\partial}{\partial u} = -\tfrac{u^2}{R^2}\tfrac{\partial}{\partial u} \end{aligned} \tag{26}$$

and so

$$\begin{aligned} \tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial\tilde{\Phi}}{\partial r}\right) &= \left(-\tfrac{u^2}{R^2}\tfrac{\partial}{\partial u}\right)\left(\tfrac{R^4}{u^2}\left(-\tfrac{u^2}{R^2}\tfrac{\partial}{\partial u}\right)\left(\tfrac{u}{R}\Phi\right)\right) \\ &= \tfrac{u^2}{R}\tfrac{\partial}{\partial u}\left(\tfrac{\partial}{\partial u}(u\Phi)\right) \\ &= \tfrac{u^2}{R}\left(u\tfrac{\partial^2\Phi}{\partial u^2} + 2\tfrac{\partial\Phi}{\partial u}\right) \\ &= \tfrac{u}{R}\left(\tfrac{\partial}{\partial u}\left(u^2\tfrac{\partial\Phi}{\partial u}\right)\right) \\ &= -\tfrac{u}{R}\left(\tfrac{1}{\sin(\theta)}\tfrac{\partial}{\partial\theta}\left(\sin(\theta)\tfrac{\partial\Phi}{\partial\theta}\right) + \tfrac{1}{\sin^2(\theta)}\tfrac{\partial^2\Phi}{\partial\psi^2}\right) \\ &= -\left(\tfrac{1}{\sin(\theta)}\tfrac{\partial}{\partial\theta}\left(\sin(\theta)\tfrac{\partial\tilde{\Phi}}{\partial\theta}\right) + \tfrac{1}{\sin^2(\theta)}\tfrac{\partial^2\tilde{\Phi}}{\partial\psi^2}\right) \end{aligned} \tag{27}$$

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Notice that

$$\lim_{r\to R}\tilde{\Phi}(r,\psi,\theta) = \Phi(r,\psi,\theta) \tag{28}$$

This transform is called *Kelvin inversion.*

Now let return to the problem of finding a Green's function for the interior of a sphere of radius. Let

$$\tilde{\mathbf{r}} = \mathbf{r}\left(\frac{R^2}{r}, \psi, \theta\right) = \frac{R^2}{r^2}\mathbf{r} \quad . \tag{29}$$

In view of the preceding remarks, we know that the functions

$$\begin{aligned} \Phi_1(\mathbf{r}) &= \tfrac{1}{|\mathbf{r}-\mathbf{r}_o|} \\ \Phi_2(\mathbf{r}) &= \tfrac{R}{r}\tfrac{1}{|\tilde{\mathbf{r}}-\mathbf{r}_o|} = \tilde{\Phi}_1(\mathbf{r}) \end{aligned} \tag{30}$$

will satisfy, respectively,

$$\begin{aligned} \nabla^2\Phi_1(\mathbf{r}) &= -4\pi\delta^3(\mathbf{r}-\mathbf{r}_o) \\ \nabla^2\Phi_2(\mathbf{r}) &= -\tfrac{4\pi R}{r}\delta^3\left(\tfrac{R^2\mathbf{r}}{r^2}-\mathbf{r}_o\right) \quad . \end{aligned} \tag{31}$$

However, notice that the support of $\nabla^2\Phi_2(\mathbf{r})$ lies completely outside the sphere. Therefore, in the interior of the sphere, Φ_2 is a solution of the homogenous Laplace equation. We also know that on the boundary of the sphere that we have

$$\Phi_1(\mathbf{r}) = \Phi_2(\mathbf{r}) \quad . \tag{32}$$

Thus, the function

$$\begin{aligned} G(\mathbf{r},\mathbf{r}_o) &= \tfrac{R}{r}\tfrac{1}{4\pi|\tilde{\mathbf{r}}-\mathbf{r}_o|} - \tfrac{1}{4\pi|\mathbf{r}-\mathbf{r}_o|} \\ &= \tfrac{1}{4\pi\left|\frac{R}{r}\mathbf{r}-\frac{r}{R}\mathbf{r}_o\right|} - \tfrac{1}{4\pi|\mathbf{r}-\mathbf{r}_o|} \end{aligned} \tag{33}$$

thus satisfies

$$\nabla^2_{\mathbf{r}}G(\mathbf{r},\mathbf{r}_o) = \delta^3(\mathbf{r}-\mathbf{r}_o) \tag{34}$$

for all $\mathbf{r}$ inside the sphere and

$$G(\mathbf{r},\mathbf{r}_o) = 0 \tag{35}$$

or all $\mathbf{r}$ on the boundary of the sphere. Thus, the function $G(\mathbf{r},\mathbf{r}_o)$ defined by (33) is the Green's function for Laplace's equation within the sphere.

Now consider the following PDE/BVP

$$\begin{aligned} \nabla^2\Phi(\mathbf{r}) &= f(\mathbf{r}) \quad , \quad r \in B \\ \Phi(R,\psi,\theta) &= 0 \quad . \end{aligned} \tag{36}$$

where B is a ball of radius R centered about the origin.

According to the formula (21) and (33), the solution of (36) is given by

$$\begin{aligned} \Phi(\mathbf{r}_o) &= \int_B G(\mathbf{r},\mathbf{r}_o)f(\mathbf{r})\,dV + \int_{\partial B} h(\psi,\theta)\frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}_o)\,dS \\ &= \int_B G(\mathbf{r},\mathbf{r}_o)f(\mathbf{r})\,dV \end{aligned}$$

To arrive at a more explicit expression, we set

$$\begin{aligned} \mathbf{r}_o &= (r\cos(\psi)\sin(\theta), r\sin(\psi)\sin(\theta), r\cos(\theta)) \\ \mathbf{r} &= (\rho\cos(\alpha)\sin(\beta), \rho\sin(\alpha)\sin(\beta), \rho\cos(\beta)) \quad . \end{aligned}$$

Then

$$\begin{aligned} dV &= \rho^2\sin^2(\theta)\,d\rho\,d\alpha\,d\beta \\ dS &= \rho^2\sin^2(\theta)\,d\alpha\,d\beta \end{aligned}$$

and after a little trigonometry one finds

$$\begin{aligned}
\frac{1}{4\pi\left|\mathbf{r}-\mathbf{r}_o\right|} &= \frac{1}{4\pi\sqrt{r^2+\rho^2-2r\rho\left(\cos(\psi-\alpha)\sin(\theta)\sin(\beta)+\cos(\theta)\cos(\beta)\right)}} \\
\frac{1}{4\pi\left|\frac{R}{r}\mathbf{r}_o-\frac{r}{R}\mathbf{r}_o\right|} &= \frac{R}{4\pi\sqrt{R^4+r^2\rho^2-2R^2r\rho\left(\cos(\psi-\alpha)\sin(\theta)\sin(\beta)+\cos(\theta)\cos(\beta)\right)}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Phi(r,\psi,\theta) &= \int_0^R\int_0^{2\pi}\int_0^{\pi}\frac{Rf(r,\psi,\theta)r^2\sin(\theta)drd\theta d\psi}{4\pi\sqrt{R^4+r^2\rho^2-2R^2r\rho\left(\cos(\psi-\alpha)\sin(\theta)\sin(\beta)+\cos(\theta)\cos(\beta)\right)}} \\
&\quad -\int_0^R\int_0^{2\pi}\int_0^{\pi}\frac{f(r,\psi,\theta)r^2\sin(\theta)drd\theta d\psi}{4\pi\sqrt{r^2+\rho^2-2r\rho\left(\cos(\psi-\alpha)\sin(\theta)\sin(\beta)+\cos(\theta)\cos(\beta)\right)}}
\end{aligned}$$