The Minimal Surface Problem

The minimal surface problem  Given \( u(x, y) = g \) on the boundary of a domain \( \Omega \), find the surface \( u \) which has the minimal surface area.

Mathematical model  Let \( \Omega = (0, 1) \times (0, 1) \), find \( u(x, y) \) satisfying

\[
\begin{aligned}
\begin{cases}
\nabla \cdot \left( \frac{\nabla u}{\sqrt{1 + u_x^2 + u_y^2}} \right) = \frac{\partial}{\partial x} \left( \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}} \right) + \frac{\partial}{\partial y} \left( \frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}} \right) = 0 & \text{in } \Omega, \\
\quad u = g & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]

Finite difference scheme  We discretize the above model problem on an \((M+1) \times (M+1)\) uniform grid. The step size is \( h = 1/M \) and \( x_m = mh, \ y_l = lh \). Find the grid function \( v_{ml} = v(x_m, y_l) \) satisfying the following scheme

\[
\begin{aligned}
\begin{cases}
\delta^+_x \left( \frac{\delta^-_x v}{\sqrt{1 + (\delta^-_x v)^2 + (\delta^-_y v)^2}} \right) + \delta^+_y \left( \frac{\delta^-_y v}{\sqrt{1 + (\delta^-_x v)^2 + (\delta^-_y v)^2}} \right) = 0, \\
\end{cases}
\end{aligned}
\]

where

\[
\begin{aligned}
\delta^+_x v_{ml} &= \frac{v_{m+1,l} - v_{ml}}{h}, & \delta^-_x v_{ml} &= \frac{v_{ml} - v_{m-1,l}}{h}, \\
\delta^+_y v_{ml} &= \frac{v_{m+1,l} - v_{ml}}{h}, & \delta^-_y v_{ml} &= \frac{v_{ml} - v_{m,l-1}}{h}.
\end{aligned}
\]

Let us simplify the above scheme as follows. Define

\[
DS_{ml} = \sqrt{1 + (\delta^-_x v_{ml})^2 + (\delta^-_y v_{ml})^2} = \sqrt{1 + \left(\frac{v_{m+1,l} - v_{ml}}{h} \right)^2 + \left(\frac{v_{m,l} - v_{m,l-1}}{h} \right)^2}.
\]

Then (1) can be rewritten as

\[
F_{ml}(\vec{v}) = \frac{1}{h} \left( \frac{v_{m+1,l} - v_{ml}}{h \cdot DS_{m+1,l}} - \frac{v_{ml} - v_{m-1,l}}{h \cdot DS_{ml}} \right) + \frac{1}{h} \left( \frac{v_{m,l+1} - v_{ml}}{h \cdot DS_{m,l+1}} - \frac{v_{m,l} - v_{m,l-1}}{h \cdot DS_{ml}} \right) = 0 \tag{2}
\]

Newton’s method for solving nonlinear system of equations  System (2) is nonlinear. We will use the Newton’s method to solve it. In general, given a nonlinear system of equations

\[
\mathbf{F}(\mathbf{V}) = 0
\]

where \( \mathbf{V} \) is an \( n \)-dimensional vector and \( F_i(\mathbf{V}), i = 1, \ldots, n \) are \( n \) nonlinear equations. We consider the following algorithm for the Newton’s method:

1. set initial guess \( \mathbf{V}_0 \)
2. for \( k \) from 0 to maximum iteration do:
   (a) compute \( \mathbf{V}(\mathbf{V}_k) \) and the Jacobian \( \frac{\partial \mathbf{F}}{\partial \mathbf{V}}(\mathbf{V}_k) = \left( \frac{\partial F_i}{\partial V_j}(\mathbf{V}_k) \right)_{1 \leq i, j \leq n} \)
   (b) compute \( \mathbf{V}_{k+1} = \mathbf{V}_k - 2^{-b} \left[ \frac{\partial \mathbf{F}}{\partial \mathbf{V}}(\mathbf{V}_k) \right]^{-1} \mathbf{F}(\mathbf{V}_k) \), where \( b \geq 0 \) is the smallest integer which guarantees \( \| \mathbf{F}(\mathbf{V}_{k+1}) \| < (1 - \alpha 2^{-b}) \| \mathbf{F}(\mathbf{V}_k) \| \) for an \( \alpha \in (0, 1) \) (\( \alpha = 10^{-4} \) is typical)
(c) stop the iteration when $\|V_{k+1} - V_k\|$ is small enough, for example, less than $10^{-6}$

**Compute the Jacobian** We need to compute the Jacobian for the scheme (2), whose entries are $\frac{\partial F_{ml}(\vec{v})}{\partial v_{ij}}$, for $1 \leq m, l \leq M + 1$ and $1 \leq i, j \leq M + 1$. Indeed, notice that $F_{ml}(\vec{v})$ only depends on 7 nodes (see the graph of stencil), it is not hard to see that

\[
\frac{\partial F_{m,l}}{\partial v_{m,l+1}} = \frac{\delta^-_x v_{m,l+1} \delta^-_y v_{m,l+1}}{h^2(DS_{m,l+1})^3}
\]

\[
\frac{\partial F_{m,l}}{\partial v_{m-1,l}} = \frac{1 + (\delta^-_y v_{m,l})^2 - (\delta^-_x v_{m,l})^2}{h^2(DS_{m,l})^3}
\]

\[
\frac{\partial F_{m,l}}{\partial v_{m,l}} = \frac{2 + (\delta^-_x v_{m,l})^2 + (\delta^-_y v_{m,l})^2 - 2 \delta^-_x v_{m,l} \delta^-_y v_{m,l}}{h^2(DS_{m,l})^3}
\]

\[
\frac{\partial F_{m,l}}{\partial v_{m+1,l}} = \frac{\delta^-_x v_{m+1,l} \delta^-_y v_{m+1,l}}{h^2(DS_{m+1,l})^3}
\]

\[
\frac{\partial F_{m,l}}{\partial v_{m-1,l+1}} = \frac{\delta^-_x v_{m-1,l} \delta^-_y v_{m-1,l+1}}{h^2(DS_{m,l+1})^3}
\]

\[
\frac{\partial F_{m,l}}{\partial v_{m+1,l+1}} = \frac{\delta^-_x v_{m+1,l+1} \delta^-_y v_{m+1,l+1}}{h^2(DS_{m+1,l+1})^3}
\]

**Boundary condition** On the boundary, we set

$$F_{ml}(\vec{v}) = g_{ml} - v_{ml} = 0 \quad \text{on } \partial \Omega.$$  

The corresponding entry in the Jacobian is

$$\frac{\partial F_{m,l}}{\partial v_{m,l}} = -1 \quad \text{on } \partial \Omega.$$

**Matlab code** Combining all the above, the Matlab code is given in `minimalsurface.m` file.