Math 4513, Solution to Homework 1

1. (a) Recall that the Taylor expansions for $\sin x$ and $\cos x$ at 0 are

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \\
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.
\]

We have

\[
\frac{\sin h - h \cos h}{h} = \frac{\left( h - \frac{h^3}{3!} + \frac{h^5}{5!} + \cdots \right) - h \left( 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \cdots \right)}{h}
\]

\[
= \frac{\frac{h^3}{3!} - h^5}{h} + \cdots \\
= \frac{h^2}{3} + O(h^4).
\]

Hence the convergence rate is $O(h^2)$.

(b) The following matlab commands define the vectors

\[
h = 2.^{-(1:20)}; \\
f = (\sin(h) - h.* \cos(h))./h;
\]

Now plot the graph of $f$ versus $h$:

\[
figure(1); \\
plot(h,f,'*', 'MarkerSize',12);
\]

The is shown below:

(c) Since we know that the convergence rate is $O(h^2)$, from part (a). It is fine to use polynomials of degree greater than or equal to 2 in polyfit. The command

\[
p = \text{polyfit}(h,f,2)
\]
gives
p =

0.3207  0.0023  -0.0000

These are coefficients of the fitting polynomial. In other words, the polynomial is

\[ p(x) = 0.3207x^2 + 0.0023x - 0.0000 \]

(d) Now lets add the polynomial to the graph:

```matlab
figure(2);
plot(h,f,'*', 'MarkerSize',12);
hold on;
ezplot('0.3207*x^2 + 0.0023*x', [0, 0.5]);
hold off
```

And the graph is shown below:

2. (a) The matlab problem and its output is given below:

```matlab
TOL = 1e-8;
N=100;
for r=2.6, 3.2, 3.5, 3.8, 4.2
  a = 0.5;
b = 1;
i = 1;
while ( i<=N & b-a>2*TOL )
i = i+1;
p = (a+b)/2;
if  -r*p^2+(r-1)*p>0
    a = p;
else
    b = p;
end
end
fprintf('r = %4.2f, Exact sol = %12.10f
', r, (r-1)/r);
if ( b-a<2*TOL )
```

2
fprintf(' Found the root %12.10f after %d iterations.

' , p, i-1);
else
    fprintf(' Method failed after %d iterations

', i-1);
end
end

and the outputs:

r = 2.60, Exact sol = 0.6153846154
    Found the root 0.6153846234 after 25 iterations.

r = 3.20, Exact sol = 0.6875000000
    Found the root 0.6874999851 after 25 iterations.

r = 3.50, Exact sol = 0.7142857143
    Found the root 0.7142857164 after 25 iterations.

r = 3.80, Exact sol = 0.7368421053
    Found the root 0.7368421108 after 25 iterations.

r = 4.20, Exact sol = 0.7619047619
    Found the root 0.7619047612 after 25 iterations.

(b) The matlab program is

TOL = 1e-8;
N = 100;
figInd = 3;
for r = [0.5, 1.5, 2.6, 3.2, 3.5, 3.8, 4.2]
    x = zeros(100,1);
    x(1) = 0.5;
    i = 1;
    err = 1;
    while ( i<=N & err>TOL )
        i = i+1;
        x(i) = r*x(i-1)-r*x(i-1)^2;
        err = abs( x(i-1)-x(i) );
    end
    fprintf('r = %4.2f, Exact sol = %12.10f
', r, max(0,(r-1)/r));
    if ( err<TOL )
        fprintf(' Found the root %12.10f after %d iterations

', x(i), i-1);
    else
        fprintf(' Method failed after %d iterations

', i-1);
    end
figure(figInd);
plot(x(1:i));
title(['r = ', num2str(r)],'FontSize',18);
figInd = figInd + 1;
end

The out puts are

3
\[ r = 0.50, \text{ Exact sol } = 0.0000000000 \]
Found the root \(0.0000000059\) after 25 iterations.

\[ r = 1.50, \text{ Exact sol } = 0.3333333333 \]
Found the root \(0.333333411\) after 23 iterations.

\[ r = 2.60, \text{ Exact sol } = 0.6153846154 \]
Found the root \(0.6153846183\) after 33 iterations.

\[ r = 3.20, \text{ Exact sol } = 0.6875000000 \]
Method failed after 100 iterations

\[ r = 3.50, \text{ Exact sol } = 0.7142857143 \]
Method failed after 100 iterations

\[ r = 3.80, \text{ Exact sol } = 0.7368421053 \]
Method failed after 100 iterations

\[ r = 4.20, \text{ Exact sol } = 0.7619047619 \]
Method failed after 13 iterations

The graphs are given below: