Lagrange interpolation

1. **An example with** \( n = 2 \). It is easy to check that quadratic polynomials

\[
L_{2,0}(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1 \quad \text{goes through 3 points} \ (0, 1), \ (1, 0), \ (2, 0) , \\
L_{2,1}(x) = -x^2 + 2x \quad \text{goes through 3 points} \ (0, 0), \ (1, 1), \ (2, 0), \\
L_{2,2}(x) = \frac{1}{2}x^2 - \frac{1}{2}x \quad \text{goes through 3 points} \ (0, 0), \ (1, 0), \ (2, 1).
\]

Their graphs look like the following:

![Graphs of L2,0, L2,1, and L2,2](image)

Now, if we would like to find a quadratic polynomial that goes through points

\((0, 5), \ (1, -3), \ (2, 4)\),

all we need to do is to compute the linear combination

\[
5L_{2,0}(x) - 3L_{2,1}(x) + 4L_{2,2}(x) = 5 \left( \frac{1}{2}x^2 - \frac{3}{2}x + 1 \right) - 3 \left( -x^2 + 2x \right) + 4 \left( \frac{1}{2}x^2 - \frac{1}{2}x \right)
\]

It is not hard to check the above quadratic polynomial goes through \((0, 5), \ (1, -3), \ (2, 4)\). Its graph is given below:

![Graph of the linear combination](image)

In general,

\[
f_0L_{2,0}(x) + f_1L_{2,1}(x) + f_2L_{2,2}(x) \quad \text{goes through 3 points} \ (0, f_0), \ (1, f_1), \ (2, f_2).
\]
2. **An example with \( n = 3 \).** Now let’s look at the case of cubic polynomials. Clearly,

\[
L_{3,0}(x) = \frac{-1}{6}x^3 + x^2 - \frac{11}{6}x + 1 \quad \text{goes through 4 points } (0,1), (1,0), (2,0), (3,0),
\]

\[
L_{3,1}(x) = \frac{1}{2}x^3 - \frac{5}{2}x^2 + 3x \quad \text{goes through 4 points } (0,0), (1,1), (2,0), (3,0),
\]

\[
L_{3,2}(x) = \frac{-1}{2}x^3 + 2x^2 - \frac{3}{2}x \quad \text{goes through 4 points } (0,0), (1,0), (2,1), (3,0),
\]

\[
L_{3,3}(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x \quad \text{goes through 4 points } (0,0), (1,0), (2,0), (3,1),
\]

Their graphs look like the following:

![Graphs of cubic polynomials](image)

and the linear combination

\[
f_0L_{3,0}(x) + f_1L_{3,1}(x) + f_2L_{3,2}(x) + f_3L_{3,3} \quad \text{goes through 4 points } (0, f_0), (1, f_1), (2, f_2), (3, f_3).\]