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流体—固体耦合问题的全离散有限元估计

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摘 要：本文讨论了一类流体—固体耦合问题的全离散有限元方法。时间方向的离散，采用向后差分公式和复化左矩形公式来分别近似连续弱形式中关于时间 t 的一阶导数和积分. 文章还给出了全离散有限元解的存在唯一性，并推导了基于全离散解的误差估计.

关键词：全离散；流体—固体耦合；有限元；误差估计
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Fully discrete finite element approximation of a fluid-structure interaction problem

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Abstract: In this paper, we consider fully discrete finite element methods for the Fluid-Structure Interaction system. For the time discretization, backward difference algorithm and composite left rectangular methods are adopted to approximate the continuous derivative and integration with respect to t, respectively. Existence and uniqueness of finite element solutions are proved, fully discrete error estimates are obtained.

Key words: fully discrete, fluid-structure interaction, finite element, error estimate
(2000 MSC 45K05, 65M60, 65Q05, 76M10)

1 Introduction

The analysis of fluid-structure interaction (FSI) problems has attracted growing attention during recent years. The FSI modeling describes the dynamics of fluids in contact with the elastic structures with natural transmission conditions, coupled the solid unknown with the velocity of the fluid solution, at a common interface. There are many numerical studies of the FSI modeling in recent years. C. Hirt introduced the Arbitrary Lagrangian-Eulerian (ALE) method as a suitable procedure for the analysis of FSI modeling[1]. In the ALE method, the grid nodes may be moved with the fluid in normal Lagrangian way, or be held fixed in Eulerian manner. A series of detailed research[2,3] of ALE finite clement methods for FSI modeling are given. T. Tezduyar and S. Sathe[4] developed space-time FSI techniques that have been applied to a wide range of 3D computation of FSI problems. These techniques enhanced the scope, accuracy, robustness and efficiency of traditional space-time methods.

In Ref. [5], Q. Du, M. Gunzburger, L.

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Hou and J. Lee explained the physical validity of the FSI modeling. In addition, they considerd weak fomulations for FSI modeling and established the existence of weak solutions. In Ref. [6], They further discussed a divergence-free formulation of FSI modeling which does not involve the fluid pressure field. Based on this formulation, they presented a semi-discrete finite element formulation of FSI modeling and derived error estimates.

The object of this paper is to discuss the fully discrete finite element approximation, prove the existence and uniqueness of finite element solutions, and derive the error estimates for the fully discrete finte element approximations. The rest of paper is arranged as follows. In section 2, we recall some relevant results of Ref. [5,6]. In section 3, we discuss the fully discrete finite element approximation and establish the existence and uniqueness of the finite element solutions and derive the error estimates.

2 Results for semi-discrete finite element approximation

2.1 Weak formulation

Assume that the fluid and solid occupy two neighboring open Lipschitz domains, $\Omega_1\subset\mathbf{R}^d$ and $\Omega_2\subset\mathbf{R}^d$ where $d=2,3$. Ω is the interior of $\bar{\Omega}_1\cup\bar{\Omega}_2$, i. e. Ω is the entire fluid-solid region. Moreover, we let $\gamma=\partial\Omega_1\cap\partial\Omega_2$ denote the interface between fluid and solid, and let $\Gamma_1=\partial\Omega_1\backslash\gamma$, $\Gamma_2=\partial\Omega_2\backslash\gamma$, denote the parts of the fluid and solid boundaries respectively, excluding the interface γ. Let $n_i(i=1,2)$ denotes the unit outward normal vector to $\Omega_i(i=1,2)$.

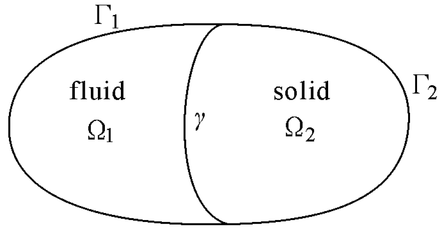

Fig. 1 Geometric description of the fluid —structure interaction model

We consider the following FSI system, in the fluid region Ω_1, we apply the Stokes system

$$\begin{cases}\rho_1\boldsymbol{v}_t+\nabla p-\mu_1\triangle\boldsymbol{v}=\rho_1\boldsymbol{f}_1 & \text{in}\quad \Omega_1,\\ \mathrm{div}\boldsymbol{v}=0 & \text{in}\ \Omega_1,\\ \boldsymbol{v}=0 & \text{on}\quad \Gamma_1,\boldsymbol{v}|_{t=0}=\boldsymbol{v}_0\end{cases}\tag{2,1}$$

where $\boldsymbol{v}=(v_1,\cdots,v_d)^T$ denotes the fluid velocity, p represents the fluid pressure, $\boldsymbol{f}$ denotes the given body force per unit mass, ρ_1 and μ_1 represent the constant fluid density and viscosity, and $\boldsymbol{v}_0$ denotes the given initial velocity.

In the solid Ω_2, we consider the equations of linear elasticity

$$\begin{cases}\rho_2\boldsymbol{u}_{tt}-2\mu_2\mathrm{div}\varepsilon(\boldsymbol{u})-\lambda_2\mathrm{div}\,[(\mathrm{div}\boldsymbol{u})I]=\rho_2\boldsymbol{f}_2 & \text{in}\quad\Omega_2,\\ \boldsymbol{u}=0 & \text{on}\quad\Gamma_2,\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0\ \text{and}\ \boldsymbol{u}_t|_{t=0}=\boldsymbol{u}_1 & \text{in}\quad\Omega_2,\end{cases}\tag{2.2}$$

where $\boldsymbol{u}=(u_1,\cdots,u_d)^T$ and $\boldsymbol{u}_t$ denote the displacement and velocity of the solid, respectively. $\varepsilon(\boldsymbol{u})=\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_j}\right)\right]_{ij}$ represents the strain tensor. $\boldsymbol{f}_2$ represents the given loading force per unit mass , μ_2 and λ_2 denote the Lame constants, ρ_2 represents the constant solid density , $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ denote the given initial data. I denotes the unit matrix.

Across the fixed interface γ between the fluid and solid, the velocity and stress vector are both continuous, i. e.

$$\boldsymbol{u}_t=\boldsymbol{v}\quad\text{on}\ \gamma,\tag{2.3}$$

$$2\mu_2\varepsilon(\boldsymbol{u})\cdot\boldsymbol{n}_2+\lambda_2(\mathrm{div}\,\boldsymbol{u})\boldsymbol{n}_2=p\boldsymbol{n}_1-\mu_1\nabla\boldsymbol{v}\cdot\boldsymbol{n}_1\quad\text{on}\ \gamma,\tag{2.4}$$

where $\boldsymbol{n}_i$ denotes the outward-pointing unit nornal vector along the boundary $\partial\Omega_i,i=1,2$.

Through out this paper, $H^m(K), m\in\mathbf{R}$, denotes the standard Sobolev spaces with order m defined on the region K equipped with the standard norm $\|\cdot\|_{m,K}$. Vector-valued Sobolev spaces are denoted by $\boldsymbol{H}^m(K)$. We use the following L^2 inner product notations on scalar and vector-valued L^2 spaces:

$$[p,q]_K=\int_K pq\,\mathrm{d}K\ \ \forall\, p,q\in L^2(K),$$

$$[\boldsymbol{u},\boldsymbol{v}]_K=\int_K \boldsymbol{u}\cdot\boldsymbol{v}\mathrm{d}K\quad \forall\,\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{L}^2(K),$$

where the spatial set K is Ω or γ or Ω_i, for $i=1,2$.

We define function spaces

$\boldsymbol{H}_i[\boldsymbol{H}_0^1(\Omega)]\mid_{\Omega i}$ with the norm $\|\cdot\|_{\boldsymbol{H}i}=\|\cdot\|_{1,\Omega_i}$, $\mathrm{i}=1,2$ (2.5)

We define the bilinear forms:

$$\begin{cases} a_1[\boldsymbol{u},\boldsymbol{v}]=\int_{\Omega_2}\mu_1\sum_{i,j=1}^{d}\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\mathrm{d}\Omega & \forall\,\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}_1,\\ a_2[\boldsymbol{u},\boldsymbol{v}]=\int_{\Omega_2}\{\lambda_2(\mathrm{div}\boldsymbol{u})(\mathrm{div}\boldsymbol{v})+2\mu_2\sum_{i,j=1}^{d}\varepsilon_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\}\mathrm{d}\Omega & \forall\,\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}_2,\\ b[\boldsymbol{v},q]=-\int_{\Omega_1}q\ \mathrm{div}\boldsymbol{v}\ \mathrm{d}\Omega & \forall\,\boldsymbol{v}\in\boldsymbol{H}_1,\forall\,q\in L^2(\Omega_1)\end{cases}\tag{2.6}$$

By using Korn's inequalities it can be verfied the bilinear forms $a_i[\cdot,\cdot]$ are coercive, i. e.

$$\begin{cases} a_i[\boldsymbol{u},\boldsymbol{u}]\ge k_i\ \|\boldsymbol{u}\|_{1,\Omega_i}^2\\ \forall\,\boldsymbol{u}\in\boldsymbol{H}_i,\ \text{if meas }(\Gamma_i)\neq 0\end{cases}\tag{2.7}$$

$$\begin{cases} [\boldsymbol{u},\boldsymbol{u}]_{\Omega_i}+a_i[\boldsymbol{u},\boldsymbol{u}]\ge k_i\ \|\boldsymbol{u}\|_{1,\Omega_i}^2\\ \forall\,\boldsymbol{u}\in\boldsymbol{H}_i,\quad \text{if meas }(\Gamma_i)=0\end{cases}\tag{2.8}$$

In this paper, h and τ, defined in the latter section, denote discretization parameter in space and time direction, respectively. The letter C denotes generic constant that may not be the same at different occurrences. For simplification, by $x\lesssim y$ ($y\gtrsim x$) we mean that there exists a constant C such that $x\le Cy$ ($y\ge Cx$), where C is independent with h and τ.

2.2 Semi-discrete approximation results

In this section, we will recall the weak formulations, and error estimates for the semi-discrete finite element approximation.

$$\boldsymbol{\xi}\overset{\text{def}}{=}\begin{cases}\boldsymbol{v} & \text{in }\Omega_1,\\ \boldsymbol{u}_t & \text{in }\Omega_2,\end{cases}\quad \boldsymbol{\xi}_0\overset{def}{=}\begin{cases}\boldsymbol{v}_0 & \text{in }\Omega_1,\\ \boldsymbol{u}_1 & \text{in }\Omega_2,\end{cases}\text{and}\quad \boldsymbol{f}\overset{\text{def}}{=}\begin{cases}\boldsymbol{f}_t & \text{in }\Omega_1,\\ \boldsymbol{f}_t & \text{in }\Omega_2,\end{cases}\tag{2.9}$$

and the weighted L^2 inner product $[[\cdot,\cdot]]$:

$$[[\xi,\eta]]=[\rho_1\xi,\eta]_{\Omega_1}+[\rho_2\xi,\eta]_{\Omega_2}\qquad \forall\,\xi,\eta\in\boldsymbol{L}^2(\Omega).\tag{2.10}$$

Thus the divergence—free weak formulation for (2.1)—(2.4) can be put conveniently into the following form: seek a ξ such that

$$\xi\in L^2(0,T;\boldsymbol{L}^2(\Omega)),\qquad \partial_t\xi\in L^2(0,T;\boldsymbol{\Psi}^*),\tag{2.11}$$

$$\xi\mid_{\Omega_1}\in L^2(0,T,\boldsymbol{H}_1),\quad \mathrm{div}\xi\mid_{\Omega_1}=0,\quad \int_0^t\xi(s)\mid_{\Omega_2}\mathrm{d}s\in L^2(0,T;H_2),\tag{2.12}$$

$$[[\xi_t,\eta]]+a_1[\xi,\eta]+a_2\left[\int_0^t\xi(s)\mathrm{d}s,\eta\right]=[[f,\eta]]-a_2[\boldsymbol{u}_0,\eta]\qquad\forall_\eta\in\boldsymbol{\Psi},\tag{2.13}$$

$$\xi(0)=\xi_0,\tag{2.14}$$

$$\int_0^t(\xi(s)\mid_{\Omega_1})\mid_\gamma\mathrm{d}s=\int_0^t(\xi(s)\mid_{\Omega_2})\mid_\gamma\mathrm{d}s\quad a.e.\ t,\tag{2.15}$$

The existence and uniqueness of the solution for the auxiliary prloblem (2.11)—(2.15) was proved in Ref. [6].

In what follows we assume that $\Omega_1\subset\mathbf{R}^2$ and

$\Omega_2 \subset \mathbf{R}^2$ are both polyhedral domains. Let h denote a discretization parameter associated with the triangulation $\mathcal{T}^h$ of Ω, i. e. $h = \max_{K_i \in \mathcal{T}^h} \{\mathrm{diam} K_i\}$. We also assume that K_i do not cross the interface γ. For each $0 < h < 1$, we choose that $\boldsymbol{H}^h \subset \boldsymbol{H}_0^1(\Omega) \cap C(\overline{\Omega})$ and $Q_1^h \subset L^2(\Omega_1)$ as finite element spaces that contain linear functions. We set $\boldsymbol{H}_i^h \overset{\text{def}}{=} \boldsymbol{H}^h|_{\Omega_i}$, $i = 1, 2$. and $\boldsymbol{\Psi}^h = \{\eta_h \in \boldsymbol{H}^h : b[\eta_h, q_h] = 0, \forall q_h \in Q_1^h\}$

Assume that $\boldsymbol{H}_1^h, \boldsymbol{H}_2^h$, and Q_1^h satisfy the standard approximation properties[7], i. e., there exists an integer $k > 0$, such that

$$\inf_{u_h \in H_i^h} \| \boldsymbol{u} - \boldsymbol{u}_h \|_{0,\Omega_i} \lesssim h^{r+1} \| \boldsymbol{u} \|_{r+1,\Omega_i} \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^{r+1}(\Omega_i) \cap H_i, \quad r \in [0, k]. \tag{2.16}$$

$$\inf_{u_h \in H_i^h} \| \boldsymbol{u} - \boldsymbol{u}_h \|_{1,\Omega_i} \lesssim h^{r} \| \boldsymbol{u} \|_{r+1,\Omega_i} \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^{r+1}(\Omega_i) \cap H_i, \quad r \in [0, k]. \tag{2.17}$$

$$\inf_{q_h \in Q_i^h} \| q - q_h \|_{0,\Omega_1} \lesssim h^{r} \| q \|_{r,\Omega_1} \qquad \forall q \in H^{r}(\Omega_1) \cap H_i, \quad r \in [0, k]. \tag{2.18}$$

Under the definition

$$\xi_h \overset{\text{def}}{=} \begin{cases} \boldsymbol{v}_h & \text{in } \Omega_1, \\ \partial_t \boldsymbol{u}_h & \text{in } \Omega_2, \end{cases} \tag{2.19}$$

the semi-discrete finite element formulation can be put as follows

$$[[\partial_t \xi_h, \eta_h]] + a_1[\xi_h, \eta_h] + a_2\left[\int_0^t \xi_h(s)\mathrm{d}s, \eta_h\right] = [[\boldsymbol{f}, \eta_h]] - a_2[\boldsymbol{u}_{0,h}, \eta_h] \quad \forall \eta_h \in \boldsymbol{H}^h, \tag{2.20}$$

$$\xi_{0,h} \overset{\text{def}}{=} \xi_h(0) = \begin{cases} \boldsymbol{v}_{0,h} & \text{in } \Omega_1, \\ \boldsymbol{u}_{1,h} & \text{in } \Omega_2, \end{cases} \tag{2.21}$$

***Lemma* 2.1** Assume that f_1, f_2, u_0, u_1 and $\boldsymbol{v}_0$ satisfy (2.11)—(2.12). Then, there , there exists a unique solution $\xi_h \in C^1(0, T; \boldsymbol{\Psi}^h)$ which satisfies (2.13)—(2.15).

The error estimate between the continuous solution defined by (2.11)—(2.15) and the semi-discrete finite element solution defined by (2.20)—(2.21) are derived in Ref. [6], i. e.

***Lemma* 2.2** Assume that $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{u}_0, \boldsymbol{u}_1$ and $\boldsymbol{v}_0$ satisfy (2.11)—(2.12). Let $(\boldsymbol{v}, \boldsymbol{u})$ be the solution of (2.13)—(2.15), and $(\boldsymbol{v}_h, \boldsymbol{u}_h)$ be the solution of (2.20)—(2.21). Assume that for some $r \in [1, k]$, $\boldsymbol{v} \in L^2(0, T; \boldsymbol{H}^{r+1}(\Omega_1))$, $\partial_t \boldsymbol{v} \in L^2(0, T; H^{r-1}(\Omega_1))$, $p \in L^2(0, T; H^r(\Omega_1))$, $\partial_t \boldsymbol{u} \in L^2(0, T; \boldsymbol{H}^{r-1}(\Omega_2))$, $\boldsymbol{v}_0 \in \boldsymbol{H}^{r+1}(\Omega_1)$, $\boldsymbol{u}_1 \in \boldsymbol{H}^{r+1}(\Omega_2)$, $\boldsymbol{u}_0 \in \boldsymbol{H}^{r+1}(\Omega_2)$, and $p_0 \in H^r(\Omega_1)$. *Then*,

$$\begin{aligned} &\| \boldsymbol{v}(t) - \boldsymbol{v}_h(t) \|_{0,\Omega_1} + \| \boldsymbol{v} - \boldsymbol{v}_h \|_{L^2(0,T;H_1)} + \\ &\| \partial_t \boldsymbol{u}_h(t) - \partial_t \boldsymbol{u}_h(t) \|_{0,\Omega_2} + \| \boldsymbol{u}(t) - \boldsymbol{u}_h(t) \|_{1,\Omega_2} \lesssim \\ &h^r(\| \boldsymbol{v}_0 \|_{r+1,\Omega_1} + \| \boldsymbol{u}_1 \|_{r+1,\Omega_2} + \| \boldsymbol{u}_0 \|_{r+1,\Omega_2} + \| p_0 \|_{r,\Omega_1} + \| p \|_{L^2(O,T;H^r(\Omega_1))}) + \\ &h^{r-\varepsilon}(\| \boldsymbol{v} \|_{L^2(0,T;H^{r+1}(\Omega_1))} + \| \boldsymbol{u}_t \|_{L^2(0,T;H^{r+1}(\Omega_2))} + \| \partial_t \boldsymbol{v} \|_{L^2(0,T;H^{r-1}(\Omega_1))} + \| \partial_{tt} \boldsymbol{u} \|_{L^2(0,T;H^{r-1}(\Omega_2))}) \end{aligned} \tag{2.22}$$

***Remark* 2.1** ε depends on the region regularity assumption[8]. In particular, if both Ω_1 and Ω_2 are convex (i. e. γ is necessarily a straight line), then ε can be chosen arbitrarily small.

3 Fully discrete finite element formulation and error estimate

We now consider a fully discrete scheme for (2.11)—(2.15). Suppose $[0, T]$ is partitioned into equal subintervals with time step $\tau = T/M$, where M is a positive integer. we denote $t_n = n\tau$ $(0 \le n \le M)$ in the following discussion. The fully discrete form of (2.11)—(2.15) is as follows

Find $\{\xi_h^n\}_{n=0}^M \subset \boldsymbol{H}^h$ such that:

$$[[\bar{\partial}_t \xi_h^n, \eta_h]] + a_1[\xi_h^n, \eta_h] + a_2[L_n(\xi_h), \eta_h] = [[f, \eta_h]] - a_2[u_{0,h}, \eta_h] \quad \forall \eta_h \in \boldsymbol{H}^h \tag{3.1}$$

$$\xi_h^0 = \xi_{0,h} = \begin{cases} \boldsymbol{v}_{0,h} & \text{in } \Omega_1, \\ \boldsymbol{u}_{1,h} & \text{in } \Omega_2, \end{cases} \tag{3.2}$$

where, the operator $\overline{\partial}_t$ denotes the backward difference operator, i. e.

$$\overline{\partial}_t \xi^n = (\xi^n - \xi^{n-1})/\tau, \qquad (3.3)$$

and L_n denotes the composite left rectangular operator, i. e.

$$L_n(\xi_h) = \tau\xi_h^0 + \tau\xi_h^1 + \cdots + \tau\xi_h^{n-1}. \qquad (3.4)$$

According to the fully-discrete form (3.1)-(3.2), $\boldsymbol{u}_h^n$ and $\boldsymbol{v}_h^n$ can be put as the following form:

$$\begin{cases} \boldsymbol{v}_h^n = \xi_h^n \mid_{\Omega_1}, \\ \boldsymbol{u}_h^n = L_n(\xi_h \mid_{\Omega_2}) + \boldsymbol{u}_{0,h}. \end{cases} \qquad (3.5)$$

***Theorem* 3.1** Assume that $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{u}_0, \boldsymbol{u}_1$ and $\boldsymbol{v}_0$ satisfy (2.11)-(2.12). Then, there exists a unique series of $\{\xi_h\}_{n=0}^M \subset H^h$ satisfy (3.1)-(3.2).

Proof According to the operator definitions of $\overline{\partial}_t$ and L_n, the fully-discrete formulation (3.1)-(3.2) is equivalent to the following

$$\frac{1}{\tau}[[\xi_h^n, \eta_h]] + a_1[\xi_h^n, \eta_h] = \frac{1}{\tau}[[\xi_h^{n-1}, \eta_h]] - a_2[\xi_h^{n-1} + \cdots + \xi_h^1 + \xi_h^0, \eta_h] + [[\boldsymbol{f}, \eta_h]] - a_2[u_{0,h}, \eta_h], \quad \forall \eta_h \in \boldsymbol{H}^h, \qquad (3.6)$$

$$\xi_h^0 = \xi_{0,h}. \qquad (3.7)$$

Recall that $\{\varphi_j\}_j^N = 1$ are a basis of H^h, therefore $\xi_h^n = \sum_{j=1}^M \varphi_j(x) g_j(t_n)$. Let $g = (g_1 \cdots, g_M)^T$, $h = (h_1, \cdots, h_M)^T$, $A_1 = \{a_1(\varphi_j, \varphi_i)\}_{M\times M}$, and $B = \{[[\varphi_j, \varphi_i]]\}_{M\times M}$, then we can use matrix format to rewrite (3.6)-(3.7) as follows

Find $g = (g_1, \cdots, g_N)^T$ such that:

$$\left(\frac{1}{\tau}B + A_1\right) g = F, \qquad (3.8)$$

where F is the corresponding term of the right side of (3.6). Since matrices A_1, B are all positive definite, the coefficient matrix $\frac{1}{\tau}B + A_1$ is positive definite, therefore, invertible. Thus there exists a unique $g = (g_1, \cdots, g_N)^T$ satisfying (3.8).

Noting the relations (3.5), we immediately obtain the existence of a series of $(\boldsymbol{v}_h^n, \boldsymbol{u}_h^n)$ satisfying the fully disctete formulation (3.1)-(3.2), as follows

***Theorem* 3.2** Assume that $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{u}_0, \boldsymbol{u}_1$ and $\boldsymbol{v}_0$ satisfy (2.11)-(2.12). Then, there exists a unique series of $\{(\boldsymbol{v}_h^n, \boldsymbol{u}_h^n)\}_{n=0}^M \subset \boldsymbol{H}_1^h \times \boldsymbol{H}_2^h$ satisfying (3.1)-(3.2).

Based on the fully discrete formulation (3.1)-(3.2), we derive the corresponding error estimates.

***Theorem* 3.3** Assume that $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{u}_0, \boldsymbol{u}_1$ and $\boldsymbol{v}_0$ satisfy (2.5). Let $(\boldsymbol{v}, \boldsymbol{u})$ be the solution of the continuous weak problem (2.6)-(2.8); and $(v_h^n, \boldsymbol{u}_h^n)$ be the solution of fully discrete problem (3.1)-(3.2). Assume for some $r \in [1,k]$, $\boldsymbol{v} \in L^2(0,T;H^{r+1}(\Omega_1))$, $\partial_t \boldsymbol{v} \in L^2(0,T;\boldsymbol{H}^{r-1}(\Omega_1))$, $\partial_t \boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^{r+1}(\Omega_2))$, $\partial_{tt}\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^{r-1}(\Omega_2))$, $\boldsymbol{v}_0 \in \boldsymbol{H}^{r+1}(\Omega_1)$, $\boldsymbol{u}_1 \in \boldsymbol{H}^{r+1}(\Omega_2)$ and $\boldsymbol{u}_0 \in H^{r+1}(\Omega_2)$. Then, the following error estimates hold: for $0 \le n \le M$

$$\begin{aligned} &\| v^n - v_h^n \|_{0,\Omega_1} + \sum_{n=1}^M \| v^n - v_h^n \|_{1,\Omega_1}^2 + \\ &\| u^n - u_h^n \|_{0,\Omega_2} + \| u^n - u_h^n \|_{1,\Omega_2} \lesssim \\ &(h^r + \tau)(\| \boldsymbol{v}_0 \|_{r+1,\boldsymbol{\Omega}_1} + \| \boldsymbol{u}_1 \|_{r+1,\boldsymbol{\Omega}_2} + \\ &\| \boldsymbol{u}_0 \|_{r+1,\Omega_2} + \\ &\| p_0 \|_{t,\Omega_1} + \| p \|_{L^2(0,T;H^r(\Omega_1))}) + \\ &(h^{r-\epsilon} + \tau)(\| \boldsymbol{v} \|_{L^2(0,T;\boldsymbol{H}^{r+1}(\boldsymbol{\Omega}_1))} + \\ &\| \boldsymbol{u}_t \|_{L^2(0,T;\boldsymbol{H}^{r+1}(\boldsymbol{\Omega}_2))} + \\ &\| \partial_t \boldsymbol{\eta v} \|_{L^2(0,T;\boldsymbol{H}^{r-1}(\boldsymbol{\Omega}_1))} + \\ &\| \partial_{tt} \boldsymbol{u} \|_{L^2(0,T;\boldsymbol{H}^{r-1}(\boldsymbol{\Omega}_2))}) \end{aligned} \qquad (3.9)$$

where τ is the time step, and $\boldsymbol{v}^n \overset{\text{def}}{=} \boldsymbol{v}(t_n)$, $\boldsymbol{u}^n \overset{\text{def}}{=} \boldsymbol{u}(t_n)$

Proof Subtract (3.1)-(3.2) from (2.20)-(2.21), and let $t = t_n$, then we have

$$\begin{aligned} 0 = &[[\partial_t \xi_h(t_n) - \overline{\partial}_t \xi_h^n, \eta_h]] + \\ &a_1[\xi_h(t_n) - \xi_h^n, \eta_h] + \\ &a_2\left[\int_0^{t_n} \xi_h^n(s)\,ds - L_n(\xi_h), \eta_h\right] = \\ &[[\partial_t \xi_h(t_n) - \overline{\partial}_t \xi_h(t_n), \eta_h]] + \\ &[[\overline{\partial}_t \xi_h(t_n) - \overline{\partial}_t \xi_h^n, \eta_h]] + \\ &a_1[\xi_h(t_n) - \xi_h^n, \eta_h] + \\ &a_2\left[\int_0^{t_n} \xi_h(s)\,ds - L_{t_n}(\xi_h), \eta_h\right] + \\ &a_2[L_{t_n}(\xi_h) - L_n(\xi_h), \eta_h] \quad \forall \eta_h \in H^h, \end{aligned} \qquad (3.10)$$

where

$$L_{t_n}(\xi_h) \overset{\text{def}}{=} \tau(\xi_h(t_0) + \xi_h(t_1) + \cdots + \xi_h(t_{n-1})).$$

Set $e^n = \xi_h(t_n) - \xi_h^n$ and choose $\eta_h = e^n \in H^h$ in (3.10), then we have

$$\left[\left[\frac{e^n-e^{n-1}}{\tau},e^n\right]\right]_\Omega+a_1[e^n,e^n]+ a_2[\tau(e^0+e^1+\cdots+e^{n-1}),e^n]= -[[\partial_t\xi_h(t_n)-\overline{\partial}_t\xi_h(t_n),e^n]]- a_2\left[\int_0^{t_n}\xi_h(s)\mathrm{d}s-L_{t_n}(\xi_h),e^n\right]$$

Moreover, applying the interpolation error theory, the following operator error estimates holds:

$$\left\|\int_0^{t_n}\xi_h(s)\mathrm{d}s-L_{t_n}(\xi_h)\right\|_{1,\Omega_2}\lesssim \tau\|\partial_t\xi\|_{L^2(0,T;H^1(\Omega_2))},$$

$$\|\partial_t\xi_h(t_n)-\overline{\partial}_t\xi_h(t_n)\|_{0,\Omega}\lesssim \tau\|\partial_t\xi\|_{L^2(0,T;L^2(\Omega))}.$$

Thus, the following inequality holds

$$\frac{1}{\tau}\|e^n\|^2_{0,\Omega}+\|e^n\|^2_{1,\Omega_1}\leq \frac{1}{2\tau}\|e^{n-1}\|^2_{0,\Omega}+\frac{1}{2\tau}\|e^n\|^2_{0,\Omega}+ \tau\sum_{j=0}^{n-1}\|e^j\|^2_{1,\Omega_2}+\frac{1}{2}\|e^n\|^2_{1,\Omega_2}+ \|e^n\|^2_{0,\Omega}+\frac{1}{2}\|e^n\|^2_{1,\Omega_2}+C\tau^2. \quad (3.11)$$

simplifying(3.11), we have

$$\left(\frac{1}{2\tau}-1\right)\|e^n\|^2_{0,\Omega}+\|e^n\|^2_{1,\Omega_1}\lesssim \frac{1}{2\tau}\|e^{n-1}\|^2_{0,\Omega}+\|e^n\|^2_{1,\Omega_2}+ \tau\sum_{j=0}^{n-1}\|e^j\|^2_{1,\Omega_2}+\tau^2. \quad (3.12)$$

Noticing that $\|e^n\|^2_{0,\Omega}\geq h^{-2}\|e^n\|^2_{1,\Omega}\geq h^{-2}\|e^n\|^2_{1,\Omega_2}$, for all $e^n\in H_h$. Therefore,

$$\left(\frac{1}{2\tau}-2\right)\|e^n\|^2_{0,\Omega}+\|e^n\|^2_{1,\Omega_1}+ (h^{-2}-1)\|e^n\|^2_{1,\Omega_2}\lesssim \frac{1}{2\tau}\|e^{n-1}\|^2_{0,\Omega}+ \tau\sum_{j=0}^{n-1}\|e^j\|^2_{1,\Omega_2}+\tau^2. \quad (3.13)$$

Summing over n from 1 to M and multiplied by τ on both side of (3.13), yields

$$\left(\frac{1}{2}-2\tau\right)\|e^M\|^2_{0,\Omega}+ \tau\sum_{n=1}^{M}\|e^n\|^2_{1,\Omega_1}+ (h^{-2}-1)\sum_{n=1}^{M}\tau\|e^n\|^2_{1,\Omega_2}\lesssim 2\sum_{n=1}^{M}\|e^{n-1}\|^2_{0,\Omega}+ \tau\sum_{n=1}^{M}\sum_{j=0}^{n-1}\tau\|e^j\|^2_{1,\Omega_2}+\tau. \quad (3.14)$$

Therefore,

$$\|e^M\|^2_{0,\Omega}+\tau\sum_{n=1}^{M}\|e^n\|^2_{1,\Omega_1}+ \sum_{n=1}^{M}\tau\|e^n\|^2_{1,\Omega_2}\lesssim \tau\lambda\sum_{n=1}^{M}(\|e^{n-1}\|^2_{0,\Omega}+ \sum_{j=0}^{n-1}\tau\|e^j\|^2_{1,\Omega_2})+\tau^2, \quad (3.15)$$

where $\lambda=\max\left\{\frac{2}{\frac{1}{2}-2\tau},\frac{1}{h^{-2}-1},1\right\}$

Applying the discrete Gronwall inequity: $a_n+b_n\lesssim c_n+\lambda\sum_{j=0}^{n-1}a_j\Rightarrow a_n+b_n\leq c_n\exp(n\lambda)$, yields

$$\|e^M\|^2_{0,\Omega}+\tau\sum_{n=1}^{M}\|e^n\|^2_{1,\Omega_1}+ \tau\sum_{n=1}^{M}\|e^n\|^2_{1,\Omega_2}\lesssim\tau^2. \quad (3.16)$$

Recalling the definition of e^n, and applying the triangle inequlity, we obtain,

$$\|\boldsymbol{u}_h(t_n)-\boldsymbol{u}_h^n\|_{1,\Omega_2}\leq \left\|\int_0^{t_n}\xi_h(s)\mathrm{d}s-L_{t_n}(\xi_h)\right\|_{1,\Omega_2}+ \|L_{t_n}(\xi_h)-L_n(\xi_h)\|_{1,\Omega_2}\lesssim \tau+\tau\sum_{n=1}^{M}\|e^n\|_{1,\Omega_2}\lesssim\tau. \quad (3.17)$$

$$\sum_{n=1}^{M}\|\boldsymbol{v}_h(t_n)-\boldsymbol{v}_h^n\|^2_{1,\Omega_1}\lesssim \sum_{n=1}^{M}\|e^n\|^2_{1,\Omega_1}\lesssim\tau \quad (3.18)$$

$$\|\boldsymbol{v}_h(t_n)-\boldsymbol{v}_h^n\|_{0,\Omega_1}+\|\boldsymbol{u}_h(t_n)-\boldsymbol{u}_h^n\|_{0,\Omega_2}\lesssim \tau\sum_{n=1}^{M}\|e^n\|_{0,\Omega}\lesssim\tau \quad (3.19)$$

Combining (3.17)-(3.19), (2.22), with the triangle inequality, we obtain (3.9).

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