MATH 2233 (Differential Equations) Lecture 2

Section 1.2 Solutions and Initial Value Problems

Goal of this section

1. understand the explicit and implicit solution of a differential equation.
2. understand the initial value problem for a differential equation.

1. Explicit Solution

The general form of n-th order ODEs with x independent, y dependent, can be expressed as

In many cases, we can isolate the highest-order term and write the equation as

Definition. A function $\phi(x)$ is called an ____________________ of an ODE if

Example 1. *Verify that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the differential equation*

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0,$$

but $\psi(x) = x^3$ is not.

Example 2. *Show that for any choice of the constant c_1 and c_2, the function*

$$\phi(x) = c_1 e^{-x} + c_2 e^{2x}$$

is an explicit solution to the linear equation

$$y'' - y' - 2y = 0.$$

2. Implicit Solution

Example 3. *Show that the relation*

$$y^2 - x^3 + 8 = 0$$

implicitly defines a solution to the nonlinear equation

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

on the interval $(2, \infty)$.

Definition. A relation $G(x, y)$ is said to be an ____________________ of an ODE on the interval I if it defines one or more explicit solutions on I.

Example 4. *Show that the relation*

$$x + y + e^{xy} = 0$$

is an implicit solution to the nonlinear equation

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0.$$

Example 5. *Verify that for every constant C the relation $4x^2 - y^2 = C$ is an implicit solution to*

$$y\frac{dy}{dx} - 4x = 0$$

Remark. For brevity, from now on we use **solution** to mean either explicit or implicit solution.

3. Initial Value Problem

As indicated in Example 2 and Example 4, a differential equation usually has infinitely many solutions. To uniquely determine a solution, we often impose additional conditions.

Example 6.

- *Find <u>all</u> solutions of the differential equation* $\dfrac{dy}{dt} = y$.
- *If, in addition to the differential equation, we also require* $y(0) = 3$*, what can we say about the solution*

<u>Remark.</u>

- The additional condition $y(0) = 3$ is often called **initial condition (IC)**, since the independent variable t often represents time in many physical applications.
- A differential equation together with an appropriate initial condition is called an **initial value problem (IVP)**.

<u>Remark.</u>

- The IVP for a 1st-order differential equations is
- The IVP for a 2nd-order differential equations is

Example 7. *As shown in Example 2, the function* $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ *is a solution to*

$$y'' - y' - 2y = 0$$

for any choice of constants c_1 *and* c_2*. Determine* c_1 *and* c_2 *so that the initial conditions*

$$y(0) = 2 \quad \textit{and} \quad y'(0) = -3$$

are satisfied.