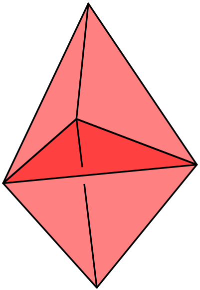
Veering Dehn surgery

Henry Segerman

Oklahoma State University

(Joint work with Saul Schleimer)

Ideal triangulations of 3-manifolds



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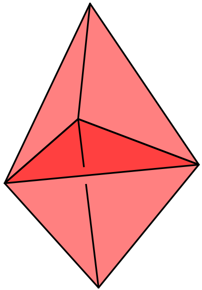

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Remove the vertices from the tetrahedra to get an ideal triangulation, the boundary is where the vertices were.

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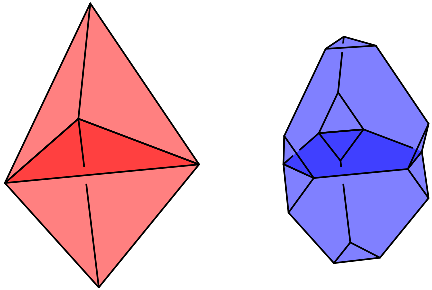

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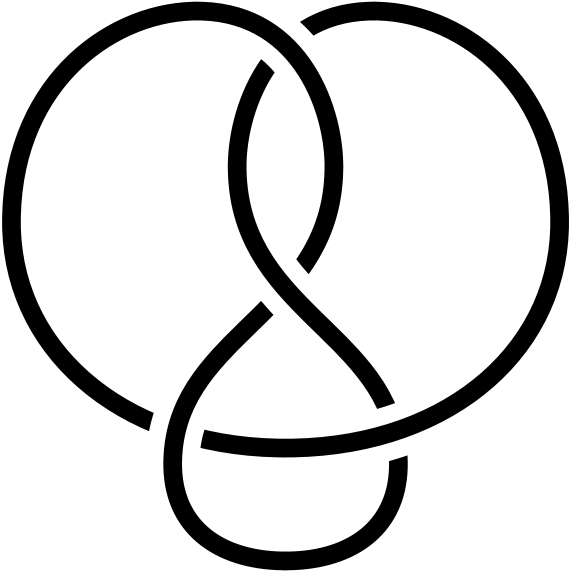
Alternatively we can think of the tetrahedra as truncated, and the truncated ends glue to form the boundary tori.

Ex: Figure 8 knot complement



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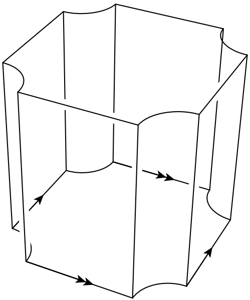

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Glue the bottom to the top via

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathsf{SL}_2\mathbb{Z}.$$

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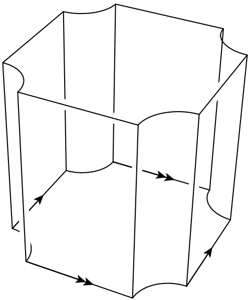

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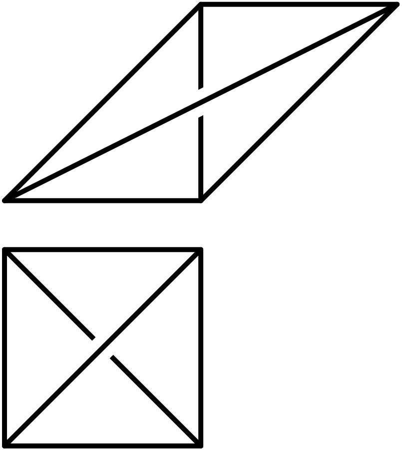

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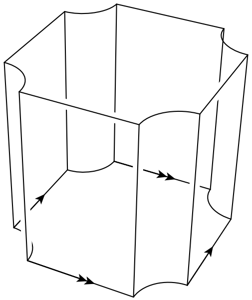

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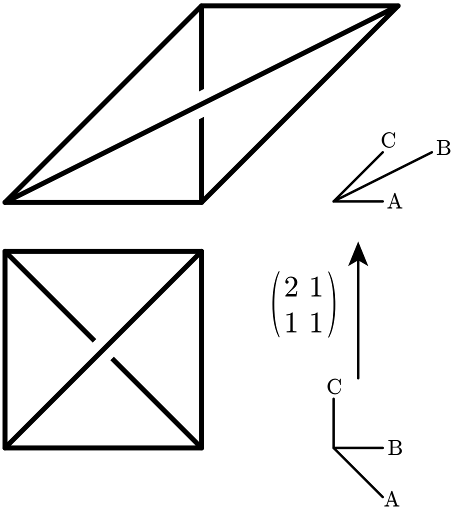

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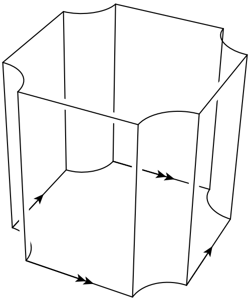

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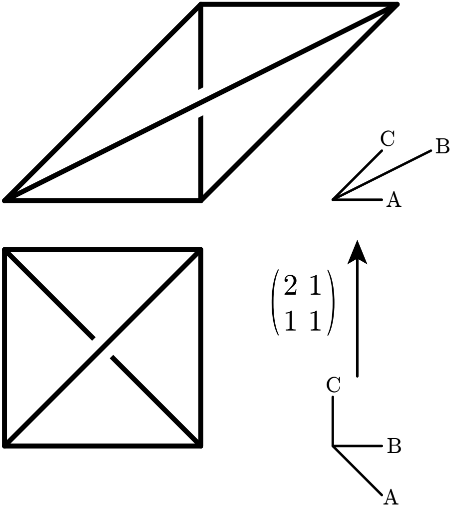
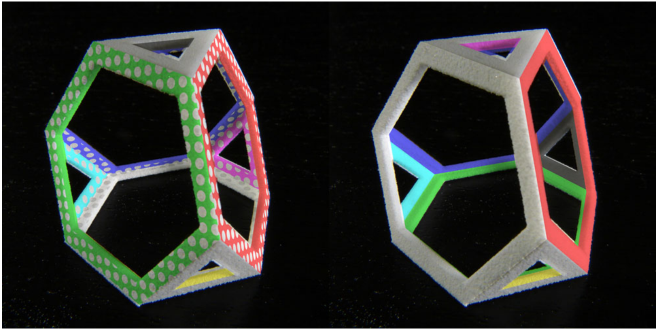


Let's call this triangulation $\mathcal{T}_{fig8}$.



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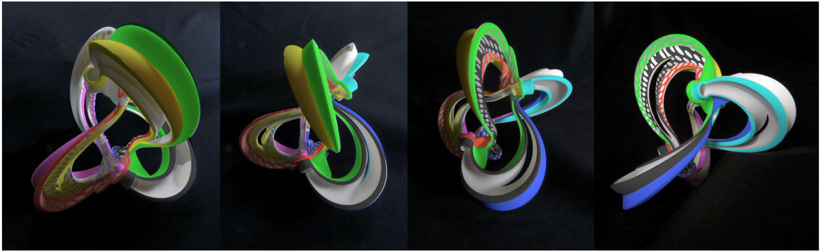

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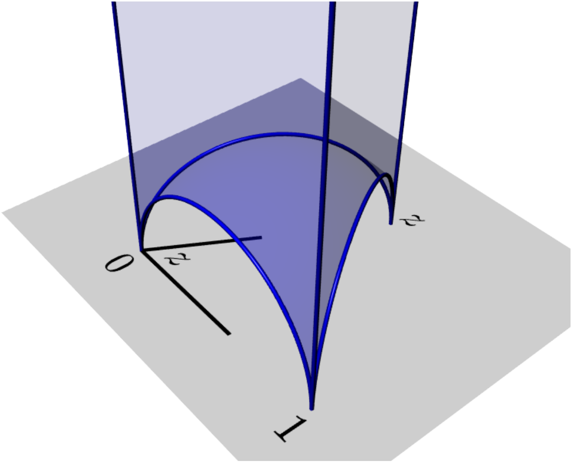
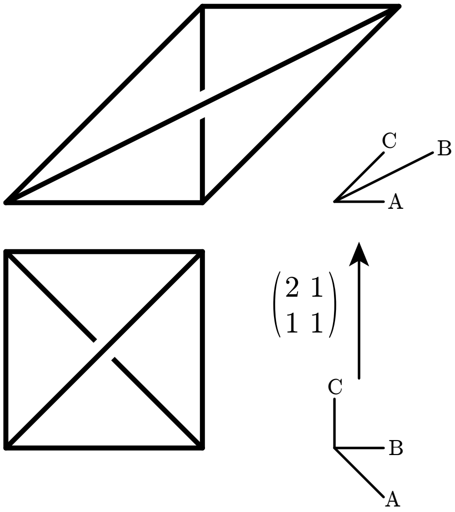
Geometric



Defn: A triangulation of a hyperbolic 3-manifold M is geometric if we can give the tetrahedra positive volume ideal hyperbolic shapes which fit together to give the complete hyperbolic structure on M.

$\mathcal{T}_{fig8}$ is geometric

In $\mathcal{T}_{fig8}$, all (both) edges are degree six. Set both tetrahedron shapes to the regular ideal tetrahedron, which has all dihedral angles $\pi/3$.



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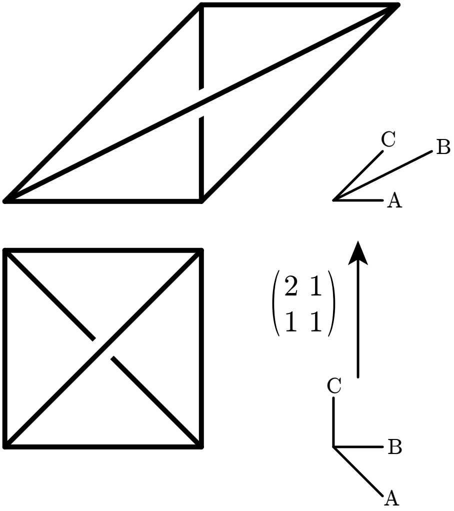
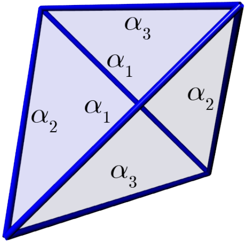
Conjecture: (Everyone) Every hyperbolic 3-manifold with torus boundary components has a geometric triangulation.



Angle structures

Defn: Associate angles (real numbers) to the edges of the tetrahedra of $\mathcal{T}$, so that:

1. In each tetrahedron, angles at opposite edges are the same.
2. In each tetrahedron, $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.
3. Around each edge of $\mathcal{T}$, $\sum \alpha = 2\pi$.



If all angles are in $(0, \pi)$ then this is a strict angle structure on $\mathcal{T}$.

If all angles are in $\{0, \pi\}$ then this is a taut angle structure on $\mathcal{T}$.

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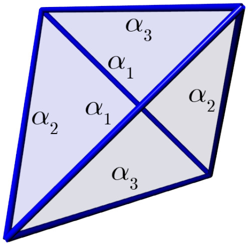

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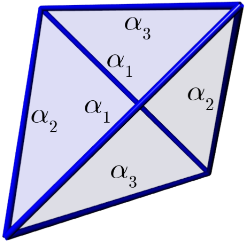

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Theorem: (Casson, Thurston)
Strict angle structure $\implies$ M is hyperbolic.

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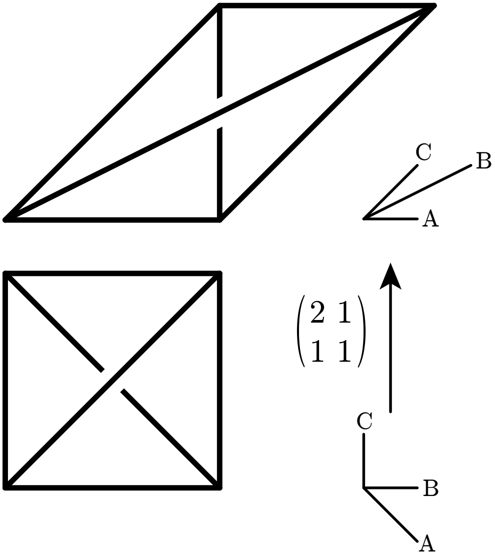
Since $\mathcal{T}_{fig8}$ is geometric, it also admits a strict angle structure
(all angles are $\pi/3$).



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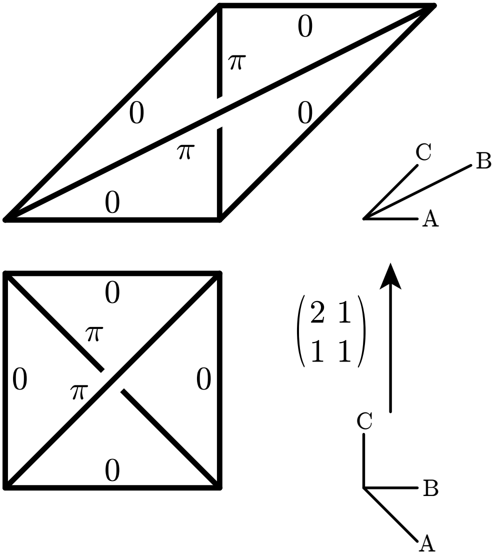
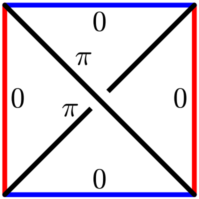
$\mathcal{T}_{fig8}$ also admits a taut angle structure – we can see this from the layered construction.



Veering structures

Defn: A veering structure on an oriented triangulation is a taut angle structure, with the property that the edges of the triangulation can be coloured red and blue so that the 0-angle edges of each tetrahedron are coloured as below.



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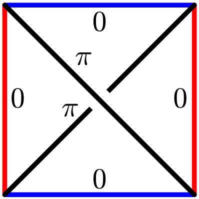

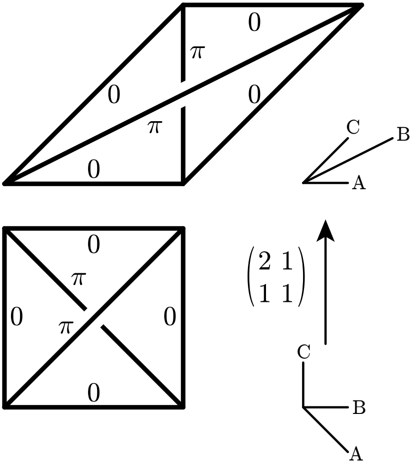

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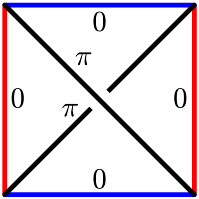

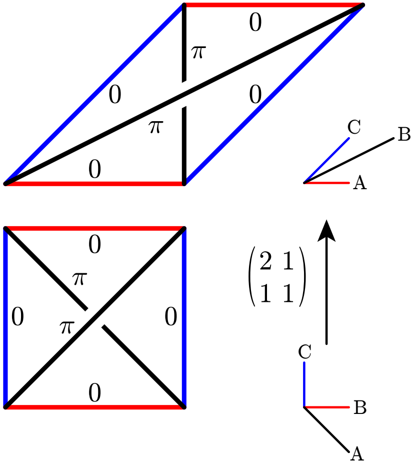

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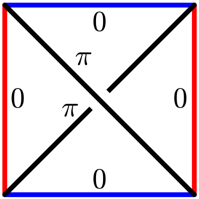

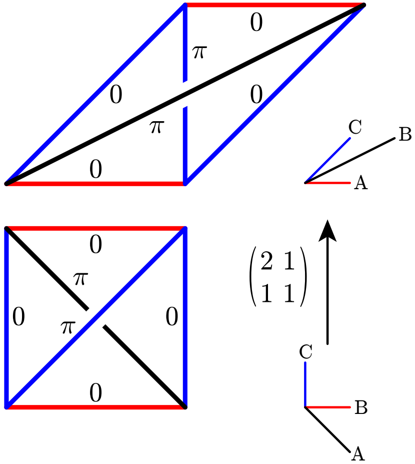

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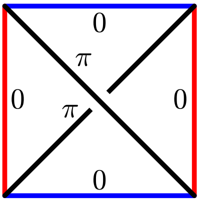

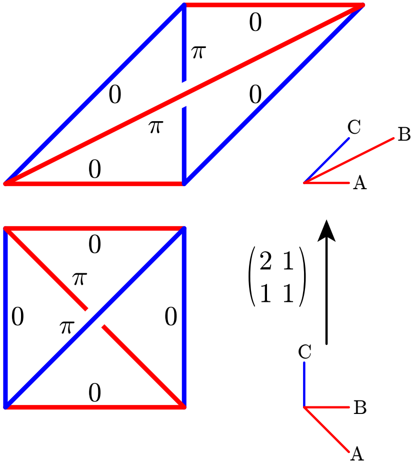

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Theorem: (Agol, 2010) Every pseudo-Anosov surface bundle, drilled along singular points of the stable and unstable foliations, has an ideal triangulation with a veering structure.

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Theorem: (Hodgson-Issa-S, 2015) There exist non-geometric triangulations with veering structures. (Found by implementing Agol's construction, smallest example with 13 tetrahedra. Current record is 9 tetrahedra.)

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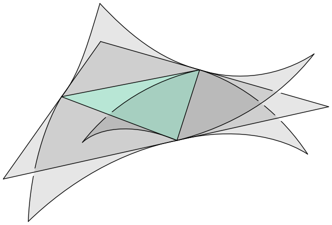
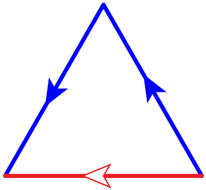


In a layered triangulation, the branched surface carries a cyclic sequence of fiber surfaces, built out of triangles of the triangulation.

"Flipping" such a surface up through the tetrahedra moves it around the cycle.

1. We start with a triangulation with a veering structure, and that contains a Möbius strip triangle.



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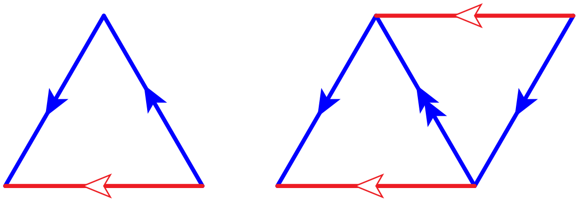

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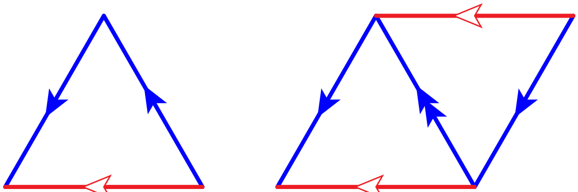

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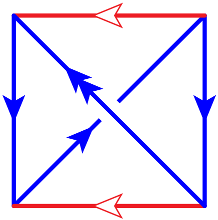

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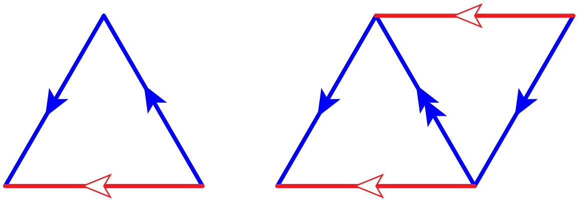

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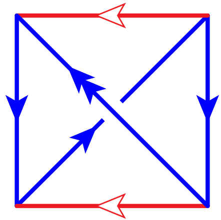

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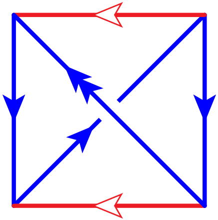
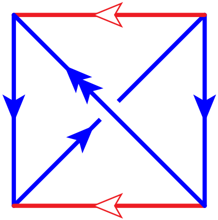
It turns out to always be possible to assign $0, \pi$ angles to the new tetrahedron so that we get a veering structure on the new triangulation.



The solid torus has another Möbius strip triangle at its core, so we can repeat the operation to construct a sequence of veering triangulations of different manifolds.



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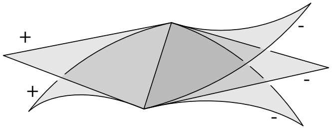
Starting with a non-geometric triangulation with a veering structure, we found a non-geometric sequence.

(The drilled but unfilled manifold has a triangulation closely related to our sequence, which is non-geometric. Away from the surgery, tetrahedron shapes in our sequence of triangulations approach the drilled triangulation's non-geometric shapes.)

Starting with a triangulation of a non-fibered manifold with a veering structure, we expected to find a sequence of triangulations of non-fibered manifolds with veering structures. But we didn't – instead we got fibered manifolds, but the veering triangulations are all non-layered.



Surfaces carried by the branched surface associated to a taut angle structure correspond to positive "face vectors" in the kernel of the edge equation matrix.

The columns of the edge equation matrix correspond to the faces of the triangulation, the rows to the edges, with faces on opposite sides of an edge appearing with different signs.

Using Farkas' lemma, the existence of certain positive "edge vectors" rule out positive face vectors, showing that the branched surface carries no surfaces.

Questions:

Are there other ways to construct veering triangulations?

How can we find more veering triangulations of non-fibered manifolds?

What is the geometric significance of non-layered veering triangulations?

Thanks!