Squares that look round:
Transforming Spherical Images

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But first... Himmeli

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But first... Himmeli

Joint work with Marco Mahler.
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Equirectangular projection
Equirectangular projection
Stereographic projection $\rho : S^2 \to \mathbb{C}$

$$\rho(u, v, w) = \frac{u + iv}{1 - w}$$
Stereographic projection
Transform by $z \mapsto 2z$ (or pull back by $z \mapsto z/2$)
Pull back by $z \mapsto z^2$
Pull back by $z \mapsto z^2$
The Droste effect
The Droste effect
Droste annulus
Droste annulus

Apply log, then tile horizontally, apply exp.

....
Twisted Droste effect (Escher, De Smit-Lenstra)
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Other kinds of “twist”, in analogy with the Droste effect

The Weierstrass $\wp$–function (for the square lattice) can be given as

$$\wp_i(z) = \frac{1}{z^2} + \sum' \left( \frac{1}{(z - w)^2} - \frac{1}{w^2} \right),$$

where the sum ranges over the non-zero Gaussian integers $w \in \mathbb{Z}[i]$. 
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\]

where the sum ranges over the non-zero Gaussian integers \(w \in \mathbb{Z}[i]\).

The function is doubly periodic:

\[
\wp_i(z+1) = \wp_i(z+i) = \wp_i(z),
\]

so we can view it as a map from the torus to \(\hat{\mathbb{C}}\).
Charles Sanders Pierce used the Weierstrass $\wp$–function on spherical images in 1879.
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https://skfb.ly/NJRx
Our version of a torus Earth
Our version of a torus Earth

https://skfb.ly/MYpC
Tile, take a different square

Scale by $1 + i$
Tile, take a different square, then map back to the sphere using a Schwarz-Christoffel map.

Scale by $1 + i$, composition is $z \mapsto \frac{i}{2}(-z + 1/z)$. 
Tile, take a different square, then map back to the sphere using a Schwarz-Christoffel map.

Scale by 2, composition is $z \mapsto \frac{(z^2+1)^2}{4z(z^2-1)}$. 
Tile, take a different square, then map back to the sphere using a Schwarz-Christoffel map.

Scale by \(2 + i\), composition is \(z \mapsto z \frac{((-1+2i)+z^2)^2}{(-i+(2+i)z^2)^2}\).
Hexagonal variation

Instead, we can pull back by the Weierstrass function $\wp_\omega$, where $\omega = e^{\pi i / 3}$, giving a hexagonal torus.
Tile, take a different hexagon

Scale by $1 + \omega$
Tile, take a different hexagon, then map back to the sphere using a Schwarz-Christoffel map.

Scale by $1 + \omega$, composition is $z \mapsto \frac{z^3 + \sqrt{2}}{3\omega \cdot z^2}$. 

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Book: Visualizing Mathematics with 3D Printing

http://3dprintmath.com
Thanks!

▶ Videos at youtube.com/user/henryseg
▶ (Some) source code at github.com/henryseg/spherical_image_editing