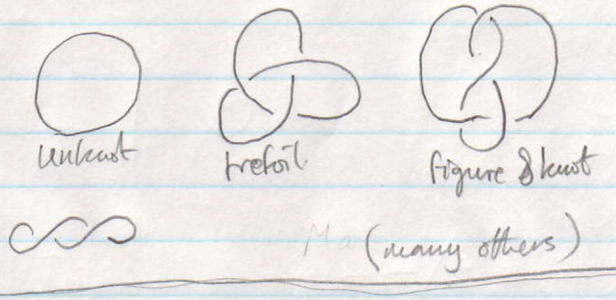
When is a knot not a knot?

Ropes

Defn A *knot* is a non-intersecting closed curve in space.
We consider two knots to be the same if one can be deformed into the other.

Examples

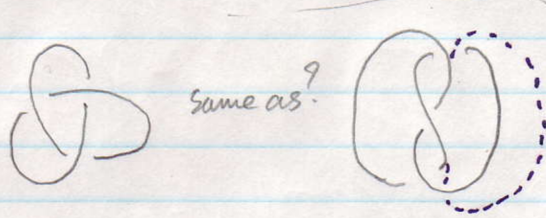


Knot diagrams for working with

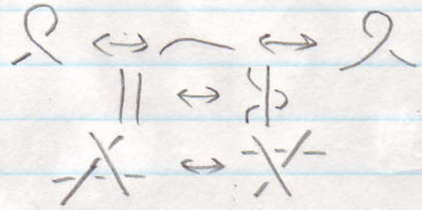
(Analogy with fractions)

Problem How can we know if diagrams D, D′ are the same knot?
different knots?

Ex

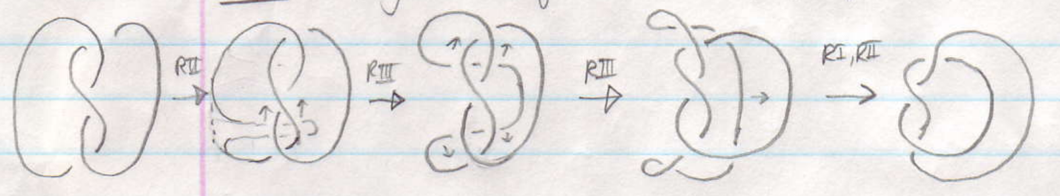


(Knot) Reidemeister Moves (1926)



(R0 deformation in the plane)

Fact Any two diagrams of the same knot are linked by a sequence of Reidemeister moves.

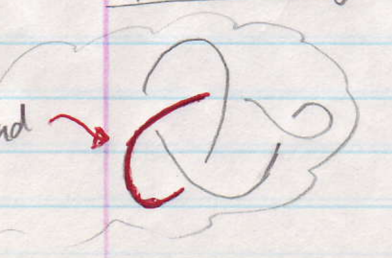


(Difficult to tell)

Idea an Invariant of a knot: I(Knot diagram) = Object
such that if we change the diagram by Reid moves then the Object doesn't change.

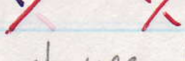
If $I(D_1) \neq I(D_2)$ then D_1, D_2 are different knots.

Tricolourability



Defn A knot diagram is *tricolourable* if we can colour the strands using 3 colours such that:

① At each crossing, either all 3 or only 1 colour is present



② We must use at least 2 colours

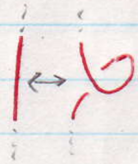
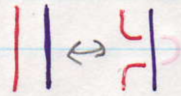
②

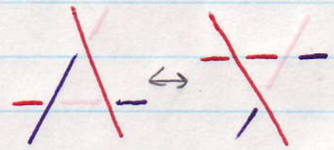
Exs {Trefoil yes,  no,  no}

Theorem Tricolorability is a knot invt (object is True/False)

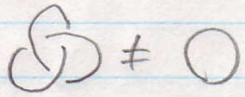
Proof Need to show: if D_1 is tricolorable then D_2 is also tricolorable + vice versa

$I(D_1) = \text{True}$ ⇕ $I(D_2) = \text{True}$

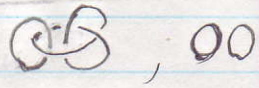
RI  RII  or 

RIII  Other cases similar

□

So  Now we know there are at least 2 knots...

(Need a better invariant)

First: A link is a collection of knots Ex 

(Louis) Kauffman Bracket (1987) (University of Illinois at Chicago) Input knot diagram, outputs a Laurent polynomial

Ex $\langle \text{Diagram} \rangle = A^5 + 3A^2 - 2A^{-3}$

Start with 3 variables, A, B, C

Rules 1: $\langle O \rangle = 1$

2: $\langle \text{link union } O \rangle = C \langle \text{link} \rangle$

3: $\langle \times \rangle = A\langle)(\rangle + B\langle \asymp \rangle$ ("over crossing turns **left** first") (Using these, reduce any link to a polynomial)

(Want to be an invt...)

RII $\langle || \rangle \overset{?}{=} \langle \text{RII diagram} \rangle = A\langle \cdot \rangle + B\langle \cdot \rangle$

$= A\left(A\langle \cdot \rangle + B\langle \cdot \rangle\right) + B\left(A\langle \cdot\, O \rangle + B\langle \cdot \rangle\right)$

$= (A^2 + B^2)\langle \asymp \rangle + AB\langle || \rangle + AB\underbrace{\langle \asymp\, O \rangle}_{= C\langle \asymp \rangle}$

$= (A^2 + B^2 + ABC)\langle \asymp \rangle + AB\langle || \rangle$

Need: $AB = 1 \Rightarrow B = A^{-1}$, $C = -A^2 - A^{-2}$

New Rules 1. $\langle O \rangle = 1$

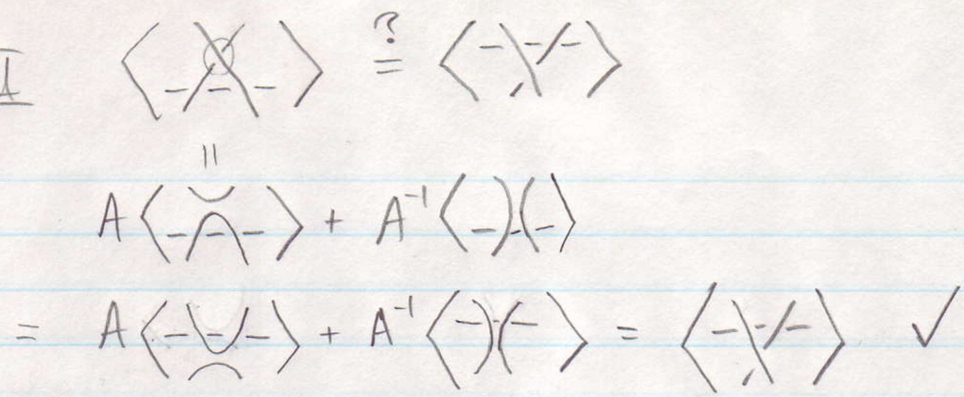
2. $\langle \text{link union } O \rangle = (-A^2 - A^{-2})\langle \text{link} \rangle$

3. $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

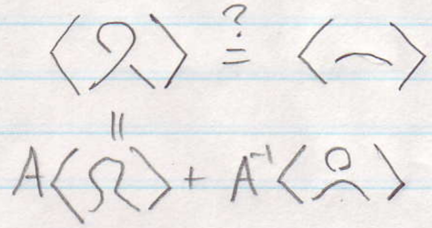
(Now invt under RII moves)

③

<u>RIII</u>

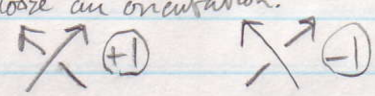


<u>RI</u>

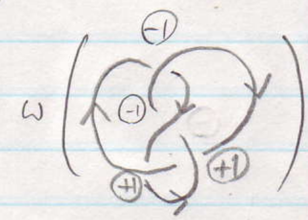


$= A\langle \frown \rangle + A^{-1}(-A^{2}-A^{-2})\langle \frown \rangle$
$= (A - A - A^{-3})\langle \frown \rangle$
$= (-A^{-3})\langle \frown \rangle$ (This looks bad!)

<u>Defn</u> <u>Writhe</u> of a knot diagram:
Choose an orientation.



add up all crossings



$\omega(\ldots) = -1 -1 +1 +1 = 0$

Not hard to check: – writhe doesn't depend on the choice of orientation
– writhe is invariant under RII, RIII
– $\omega(\text{♀}) = \omega(\frown) - 1$

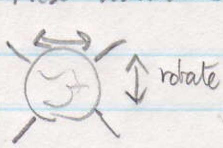
(Problems with writhe and Kauffman bracket will cancel out)

<u>Defn</u> <u>Kauffman polynomial X</u> $X(\text{knot}) = (-A^{-3})^{\omega(\text{knot})}\langle \text{knot} \rangle$

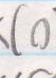
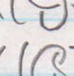
Then $X(\text{♀}) = (-A^{-3})^{\omega(\text{♀})}\langle \text{♀} \rangle$
$= (-A^{-3})^{\omega(\frown)-1}(-A^{-3})\langle \frown \rangle$
$= (-A^{-3})^{\omega(\frown)}\langle \frown \rangle = X(\frown)$

So X polynomial is an invariant!

(Not full invt: eg 5_1 and 10_{132}
Also mutants don't change X



)

<u>Eg</u> $X(\bigcirc) = 1$
X() $= -A^{16} + A^{12} + A^{4}$
X() $= A^{8} - A^{4} + 1 - A^{-4} + A^{-8}$