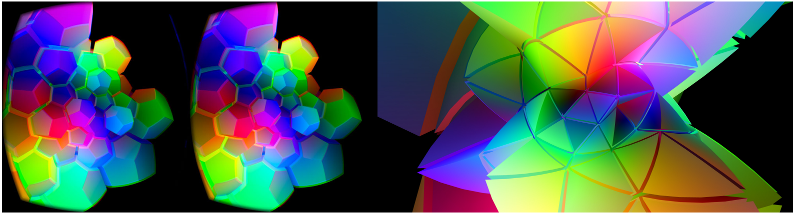

Navigating the three-sphere via quaternions

Henry Segerman
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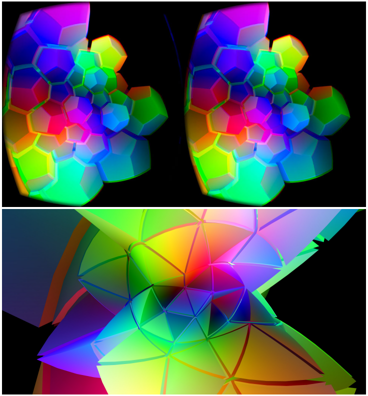
Based on Hypernom, joint work with Vi Hart, Andrea Hawksley and Marc ten Bosch.

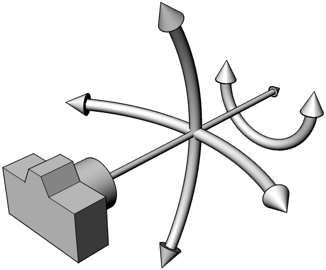
On your smartphone web browser, go to:

hypernom.com



(You may need to lock your phone's screen orientation.)

Hypernom is a mobile/virtual reality game/experience that uses the VR headset (or phone) orientation in an unusual way.



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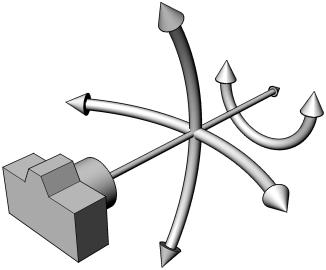
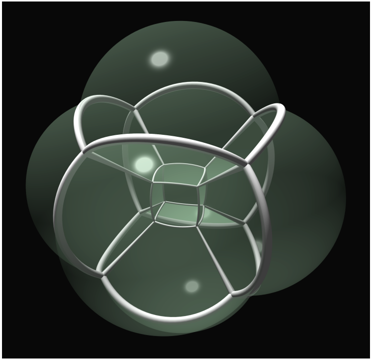


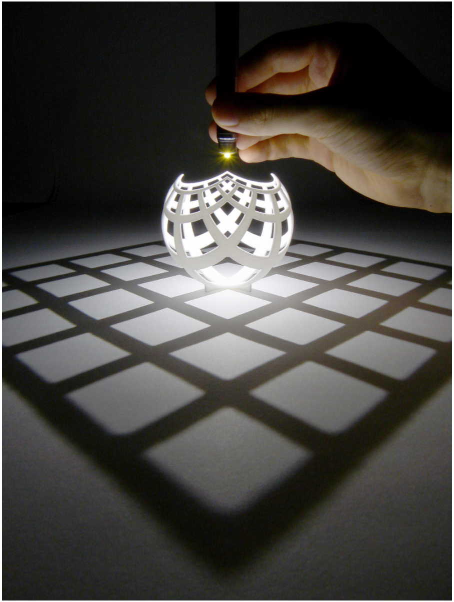
There are three degrees of freedom in the orientation of a headset (or phone, camera, etc.).

In fact, the set of possible device orientations is a three-dimensional manifold, $\mathrm{SO}(3) \cong \mathbb{R}\mathrm{P}^3$. The idea of Hypernom is to lift the device orientation to navigate through a three-dimensional space, namely the universal cover of $\mathbb{R}\mathrm{P}^3$, which is the three-sphere, S^3.

What to draw on screen

- Start with a regular polytope in $\mathbb{R}^4$
- Radially project it to S^3
- Then stereographically project it to $\mathbb{R}^3$





A camera positioned at $0 \in \mathbb{R}^3$ shows the result on screen.

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A Puzzle: Why does rotating the device seem to move us along the axis of the rotation?

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{I} \cong \mathbb{R}^4$ be the quaternions, where $\mathbb{I} = i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$ is the subspace of purely imaginary quaternions.

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So, rotation of the device around an axis u lifts to some quaternion in the direction of u as seen in the stereographic projection.

 

hypernom.com

- Works on iOS, Android and desktop
- Paper at archive.bridgesmathart.org/2015/bridges2015-387.html
- Source code at github.com/vihart/hypernom