Triangulations of hyperbolic 3-manifolds admitting strict angle structures

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January 4th 2012

Ideal triangulations of 3-manifolds

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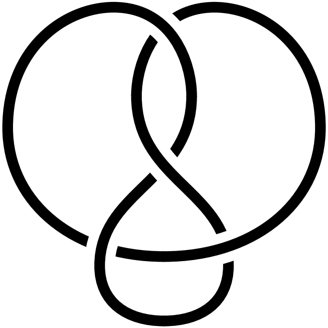

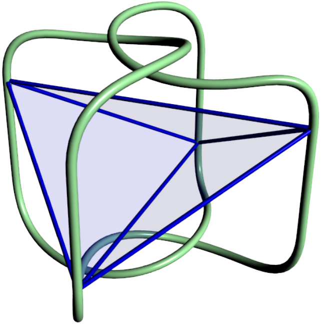

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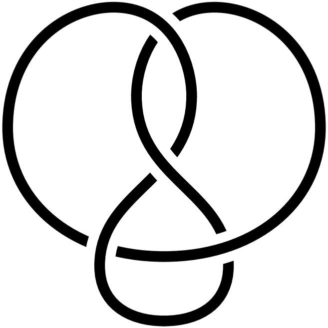

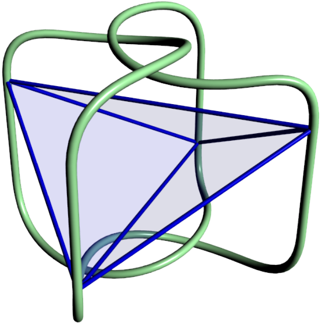

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Theorem (Epstein-Penner)

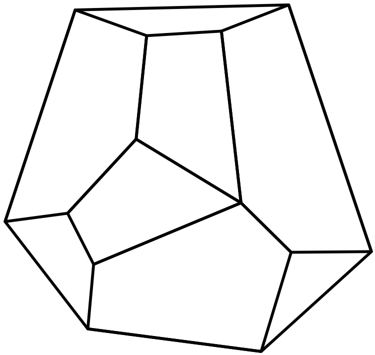
If M is a hyperbolic manifold with a single cusp then it has a canonical decomposition into convex ideal hyperbolic polyhedra.

If all of the polyhedra are tetrahedra then we are done. If not then all we have to do is cut up each polyhedron into ideal tetrahedra.

For example, we can cone the polyhedron from a chosen vertex.

This cuts the polyhedron into pyramids.

Then any triangulation of the base of a pyramid induces a triangulation of it.



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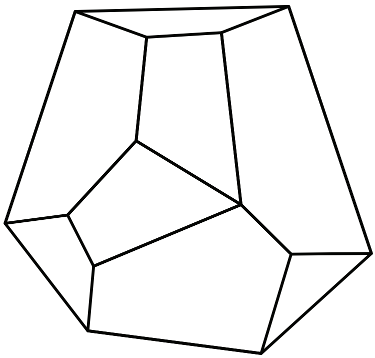

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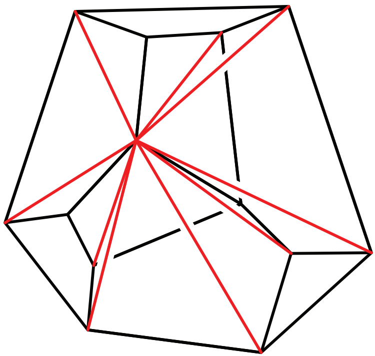

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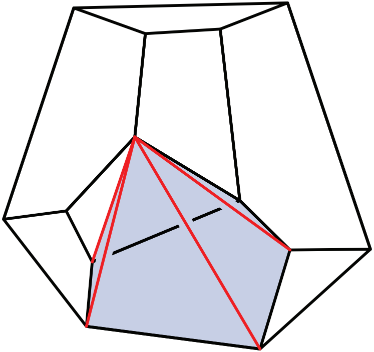

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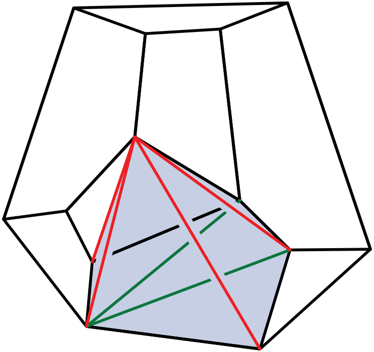
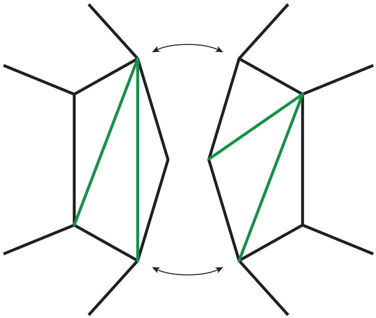


The problem

The triangulations of faces of the polyhedra may not match.



We can insert flat tetrahedra between the faces to “bridge” from one triangulation to the other, but then we don’t have a geometric triangulation!

With some strong assumptions on the combinatorics of the polyhedral decomposition, Wada, Yamashita and Yoshida show that the coning can be chosen so that this doesn’t happen, but the general problem remains open.

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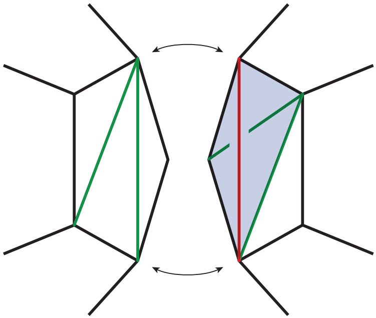

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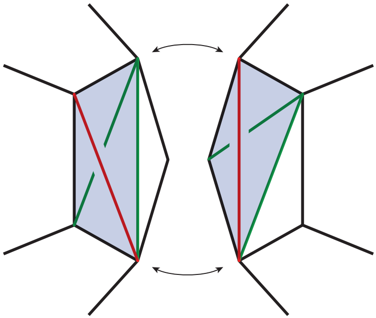

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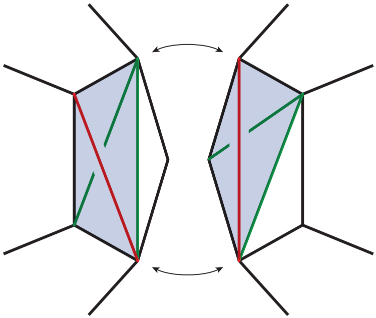

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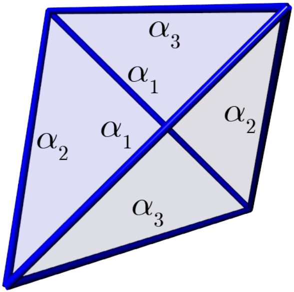
An easier problem: find a triangulation $\mathcal{T}$ with a strict angle structure

If we have a geometric triangulation, then for each ideal hyperbolic tetrahedron in $\mathcal{T}$, the dihedral angles (numbers in $(0, \pi)$) determine its shape. These angles satisfy:

- In each tetrahedron, angles at opposite edges are the same.
- In each tetrahedron, $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.
- Around each edge of $\mathcal{T}$, $\sum \alpha = 2\pi$.



A strict angle structure on a triangulation $\mathcal{T}$ is an assignment of angles that satisfies these properties.

The existence of an angle structure is a necessary condition for a triangulation to be geometric.

Theorem (Hodgson, Rubinstein, S)

Assume that M is the interior of a compact orientable irreducible hyperbolic manifold with incompressible tori boundary components. If $H_1(M;\mathbb{Z}_2) \to H_1(M,\partial M;\mathbb{Z}_2)$ is the zero map then M admits an ideal triangulation with a strict angle structure.

Corollary

If M is a hyperbolic link complement in S^3, then M admits an ideal triangulation with a strict angle structure.

Proof.

H_1 of a link complement is generated by meridians. $\square$

Outline of a sketch of a proof of the theorem

1. Start with the Epstein-Penner decomposition into convex ideal hyperbolic polyhedra.
2. Carefully choose a coning of each polyhedron.
3. When triangulations of faces of the polyhedra disagree, carefully bridge between them with flat tetrahedra.
4. Show that the resulting partially flat angle structure can be deformed into a strict angle structure.

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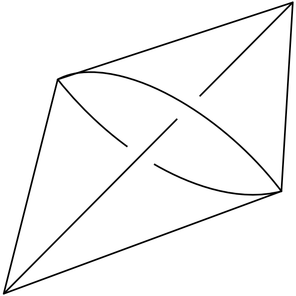
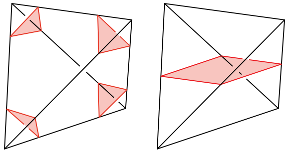
Example: When introducing flat tetrahedra in step 3, if we make a degree 2 edge then we will have no hope of deforming to make a strict angle structure.



Strict angle structures from normal surface theory

The tool we use for step 4 is a linear programming duality result between normal surface theory and the angle structure equations.

The normal surface solution space for a triangulation $\mathcal{T}$ is $C(M;\mathcal{T}) \subset \mathbb{R}^{7n}$. The coordinates represent weights of the 4 triangle and 3 quad disk types in the n tetrahedra.



The face gluings give compatibility equations on the $x \in C(M;\mathcal{T})$.

A quad type is vertical relative to a (partially flat) angle structure if the facing angles are 0.

Theorem (Kang-Rubinstein, Luo-Tillmann)

The triangulation $\mathcal{T}$ admits a strict angle structure if and only if there is no $x \in C(M;\mathcal{T})$ with all quadrilateral coordinates non-negative, all non-vertical quadrilateral coordinates zero and at least one quadrilateral coordinate positive.

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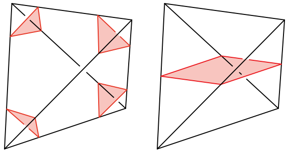

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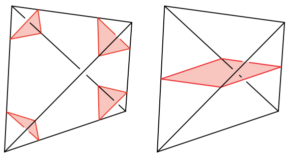

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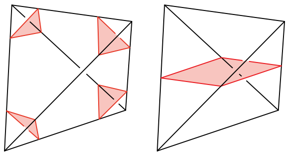

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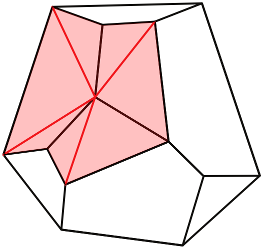
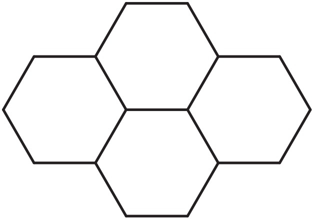
Any such "bad" $x \in C(M; \mathcal{T})$ can only have quads in the flat "bridge" regions between the polyhedra.

When we triangulate a polyhedron by coning, we never need to attach a bridge to faces that aren't next to the coning point.





We can choose our cone points so that the dual graph to the polyhedra contains a spanning tree of edges corresponding to face identifications without bridges.

Where we do have to insert bridges, they are of a very simple sort (in fact we show that only bridges on faces with either 4 or 6 sides can have any vertical quads in them).

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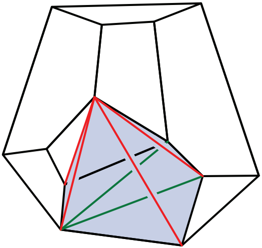

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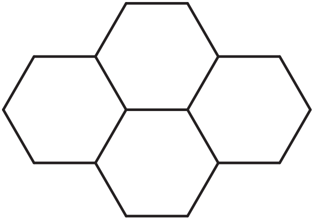

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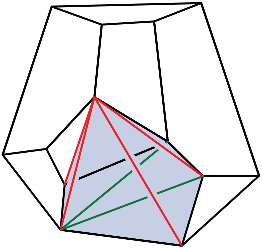

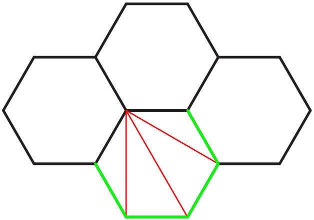

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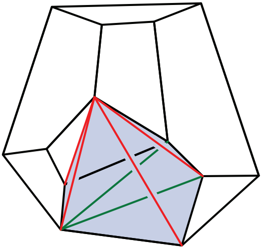

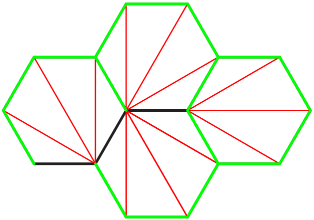

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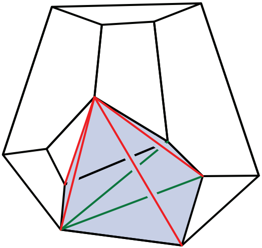

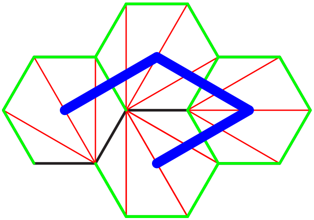

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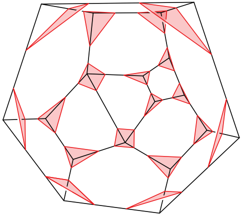
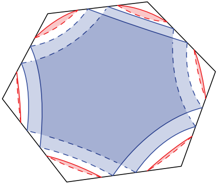
Assuming there is a bad $x \in C(M; \mathcal{T})$, we can replace the quads in the bridges with certain "twisted disk" surface parts to convert the normal surface solution class into an *embedded* closed surface.



Take a fundamental, non-peripheral component of the surface. It turns out that such a component must have an odd number of parallel twisted disks in some bridge region in order to be fundamental.



Then take a path through this bridge region. Complete the path to a loop using the spanning tree, and we get a non-peripheral element of $H_1(M; \mathbb{Z}_2)$ since no sum of peripheral loops can have odd intersection with a closed surface.

So $H_1(M; \mathbb{Z}_2) \to H_1(M, \partial M; \mathbb{Z}_2)$ is not the zero map, which is a contradiction to the hypothesis of the theorem.