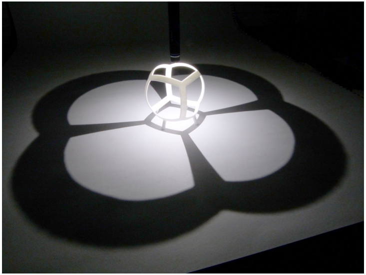
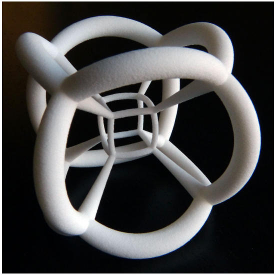
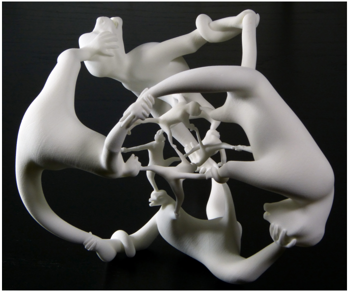



Henry Segerman
Oklahoma State University

How to make sculptures of 4-dimensional things

What is 4–dimensional space?

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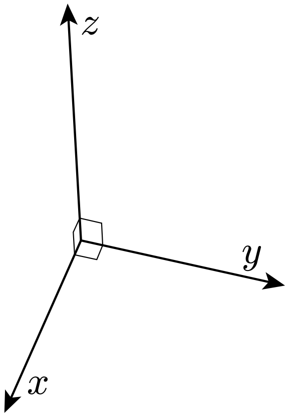
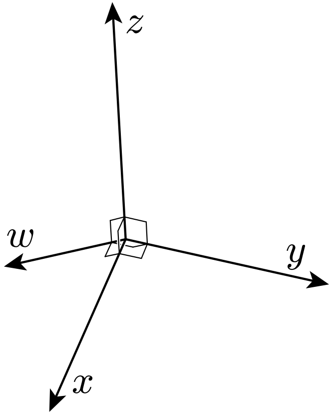
We describe a point in 3-dimensional space using three numbers, say (x, y, z).



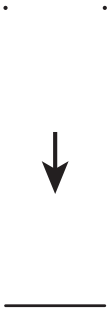
What is 4–dimensional space?

We describe a point in 3-dimensional space using three numbers, say (x, y, z).

A point in 4-dimensional space is given by four numbers, say (w, x, y, z).



Example: how to make a hypercube



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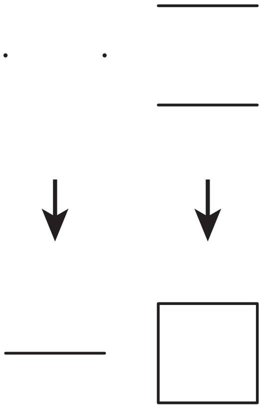

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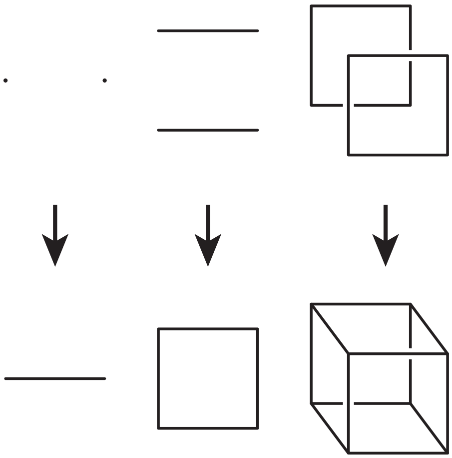

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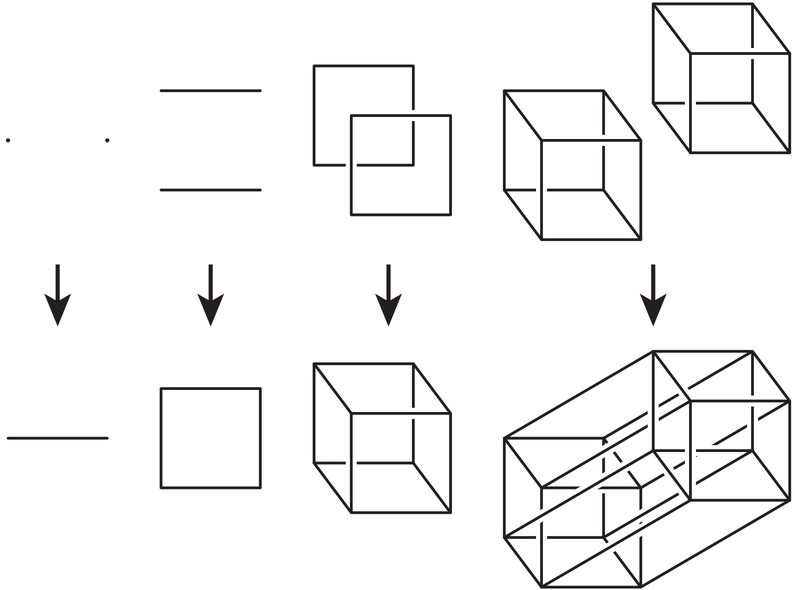

How can we see 4-dimensional things?

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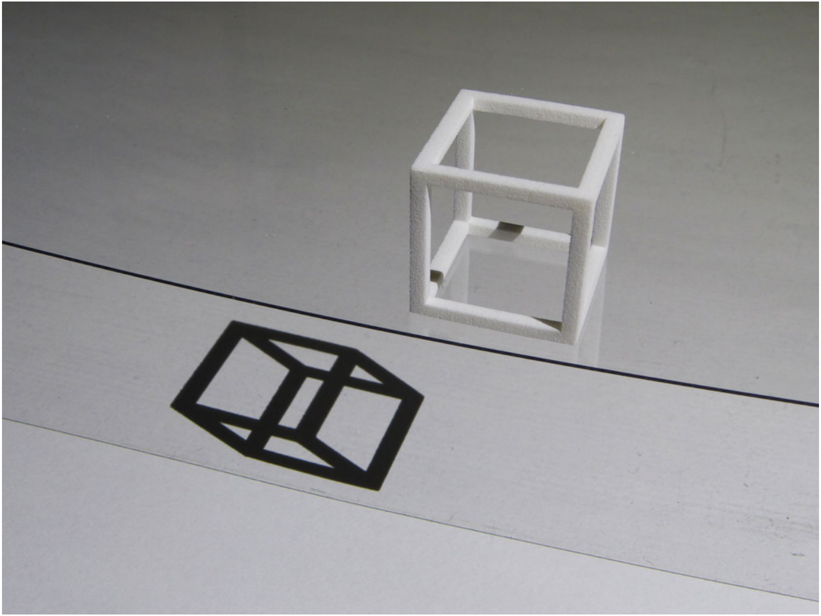
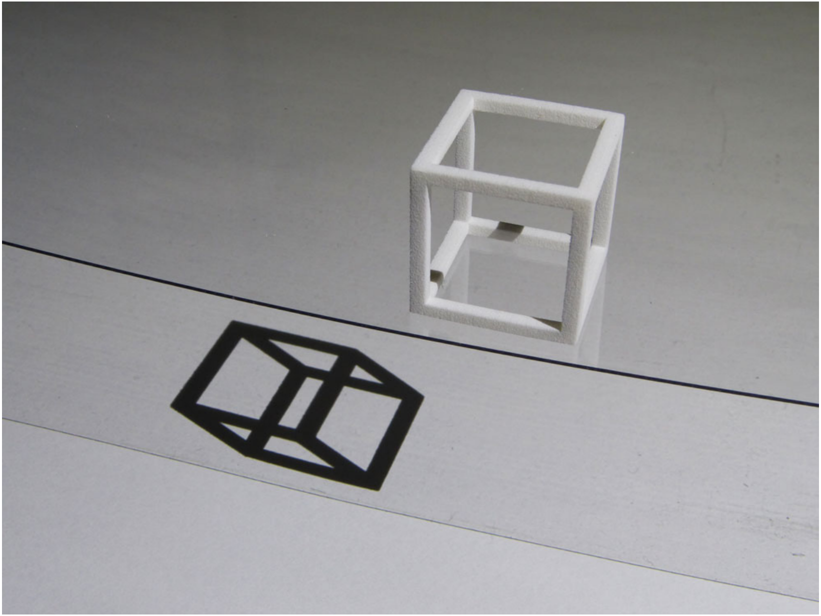


Parallel projection of a cube



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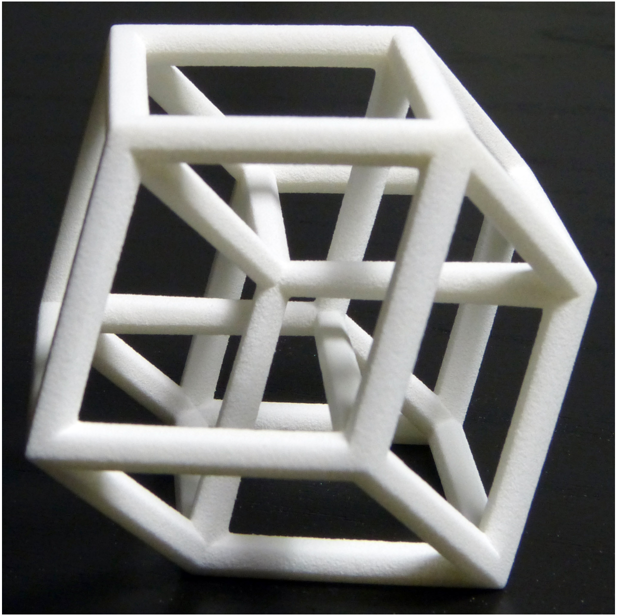
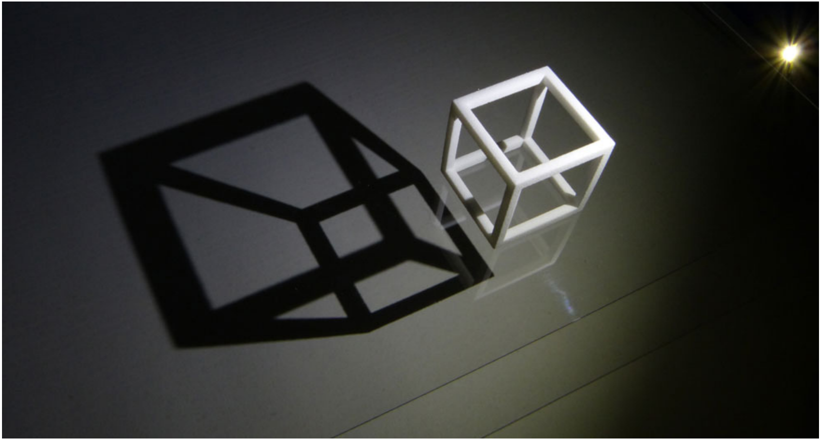
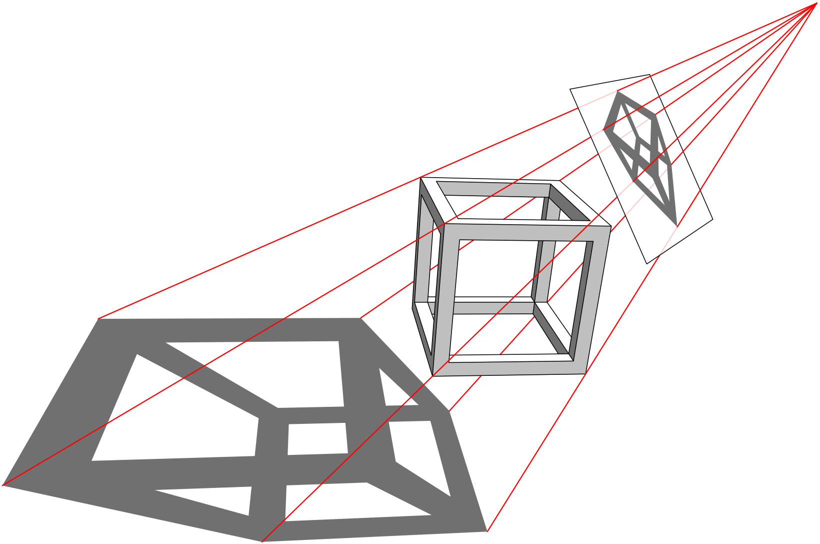


Hypercube B by Bathsheba Grossman.

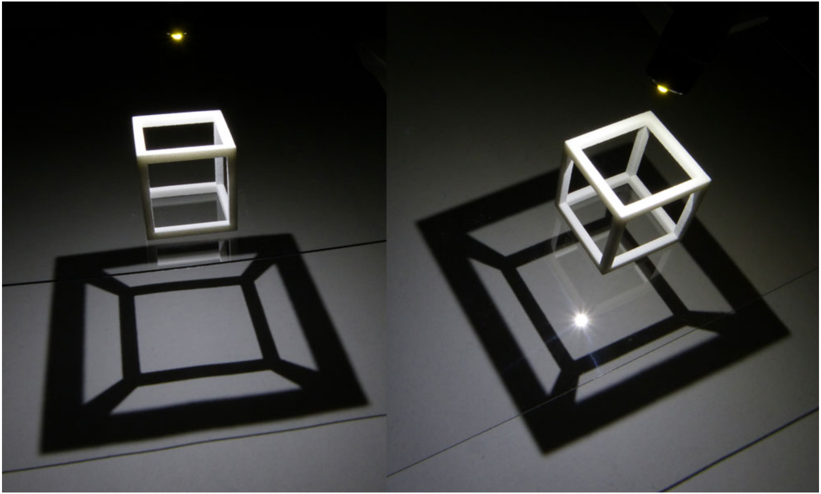
Perspective projection of a cube



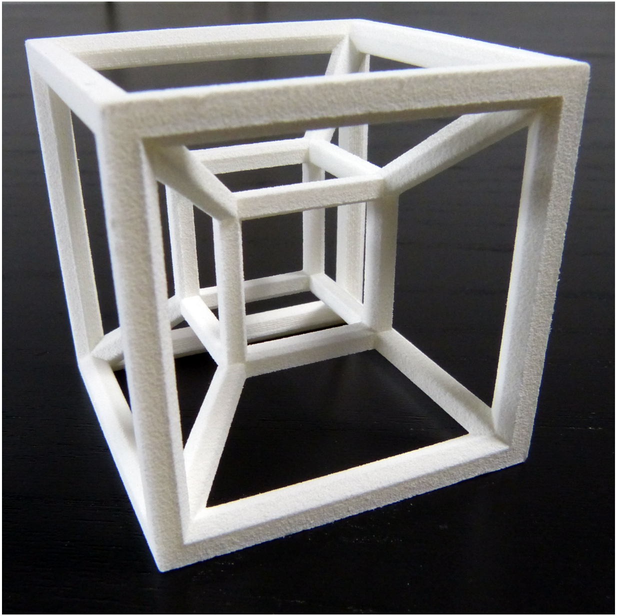
Perspective projection of a cube



Perspective projection of a cube



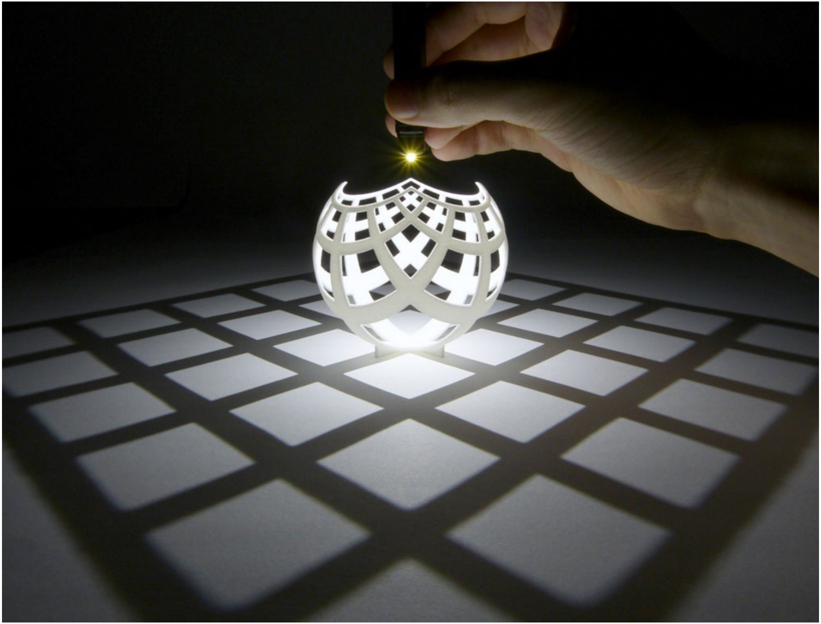
Perspective projection of a hypercube



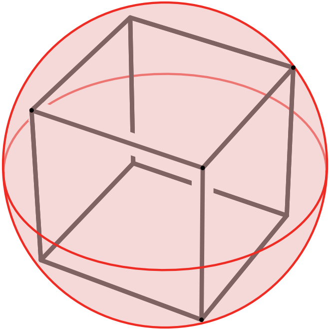
Hypercube A by Bathsheba Grossman.

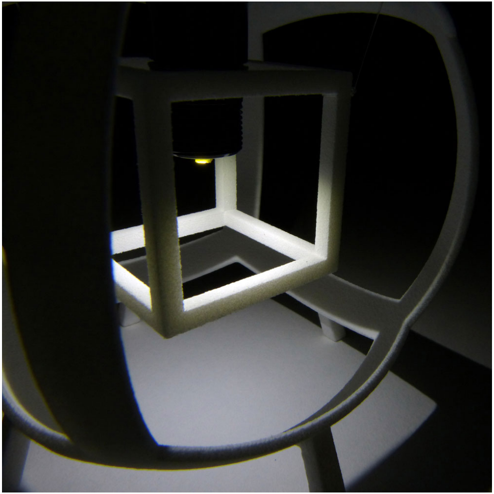
Stereographic projection

Stereographic projection



First radially project the cube to the sphere...





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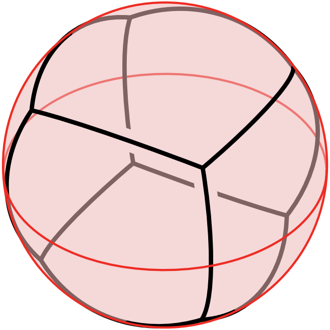
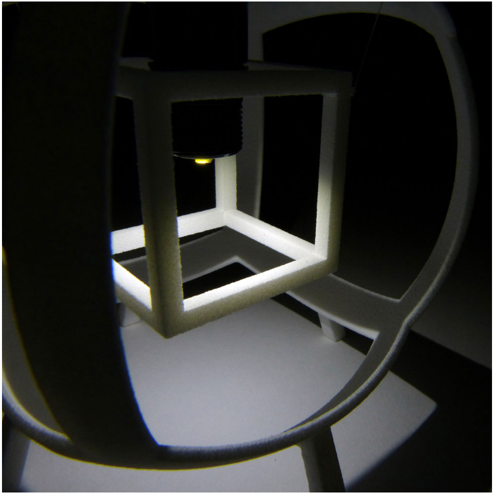
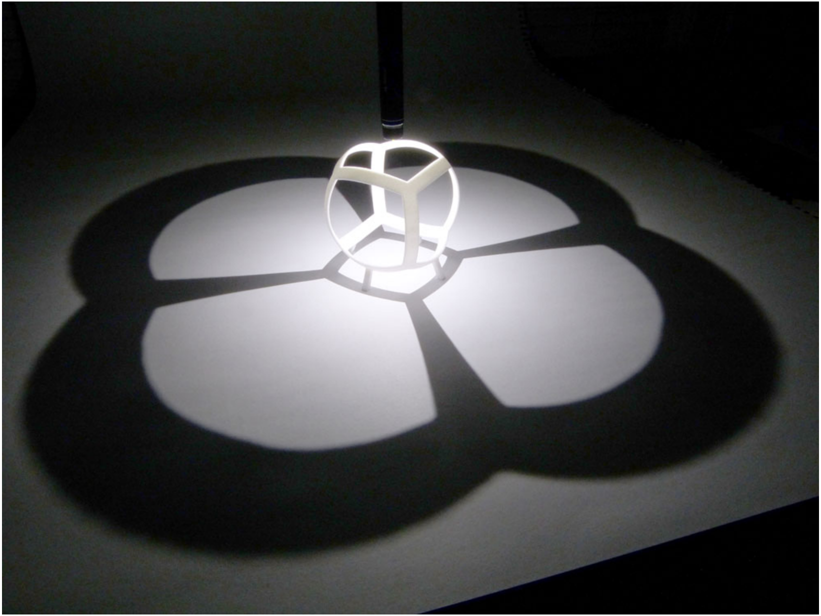




Then stereographically project to the plane



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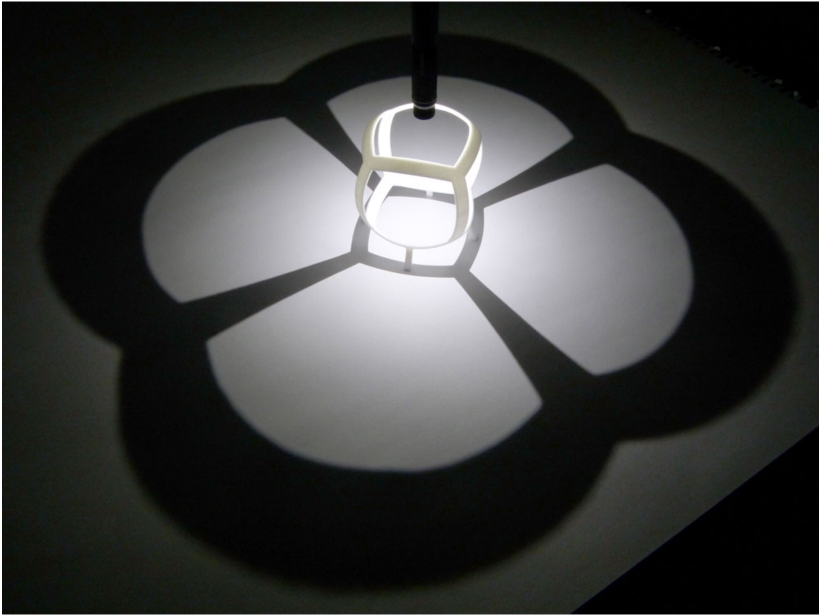
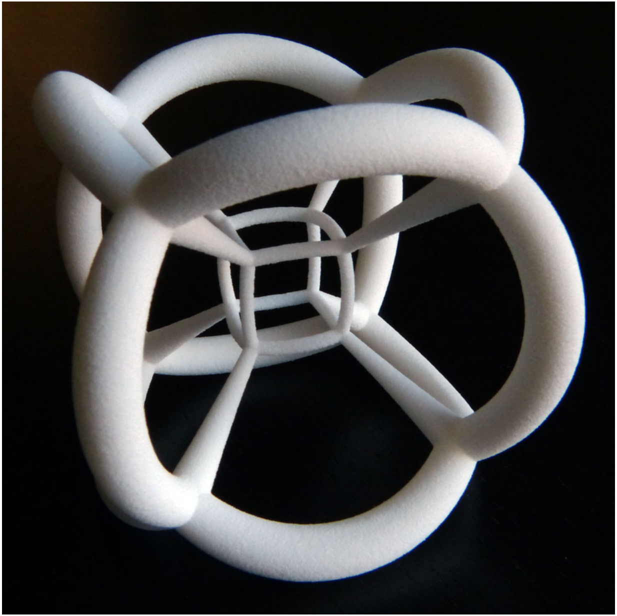


Do the same thing one dimension up to see a hypercube



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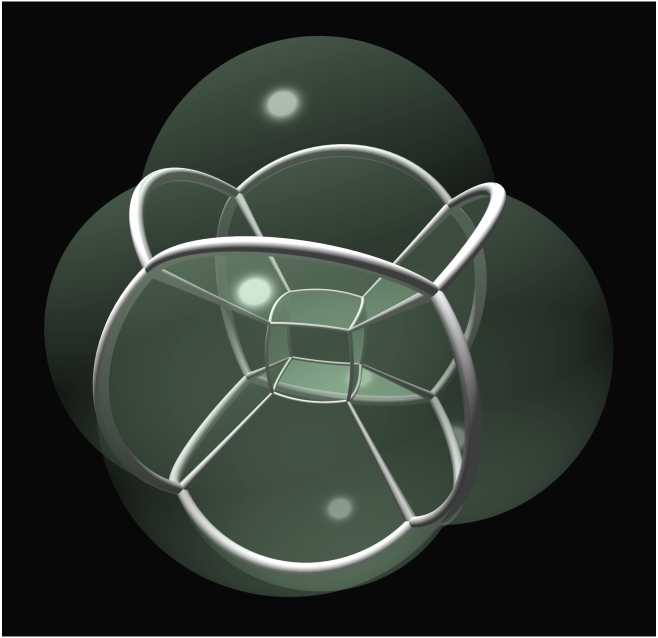
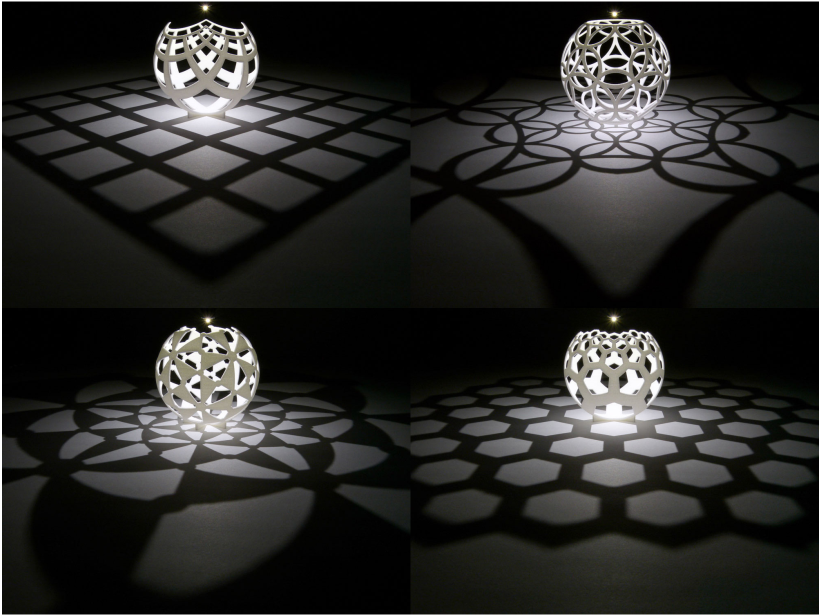


More amazing properties of stereographic projection



Visualising the sphere in 4-dimensional space

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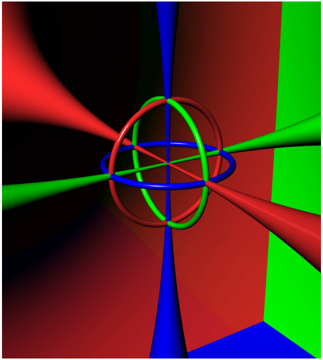
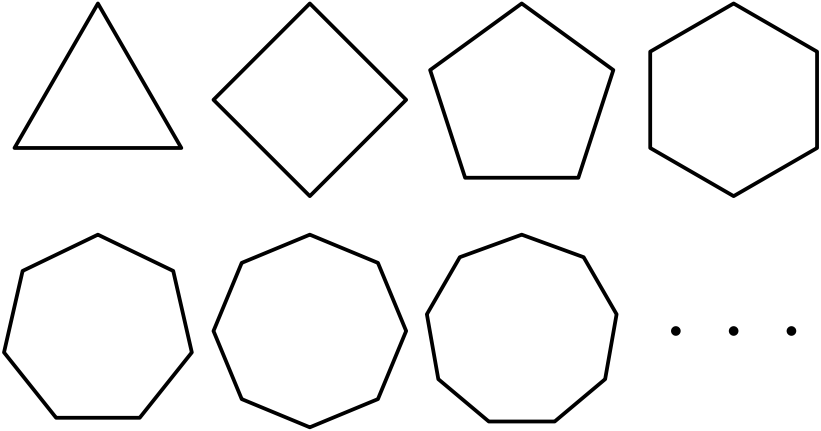
- The sphere in 4-dimensional space is "the same as" 3-dimensional space, plus a point.



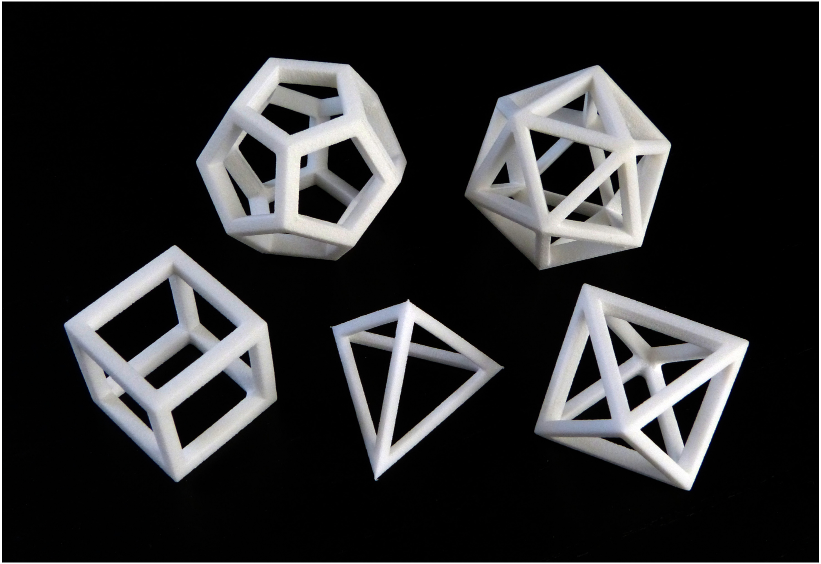
Regular Polytopes

In 2-dimensions: Regular polygons

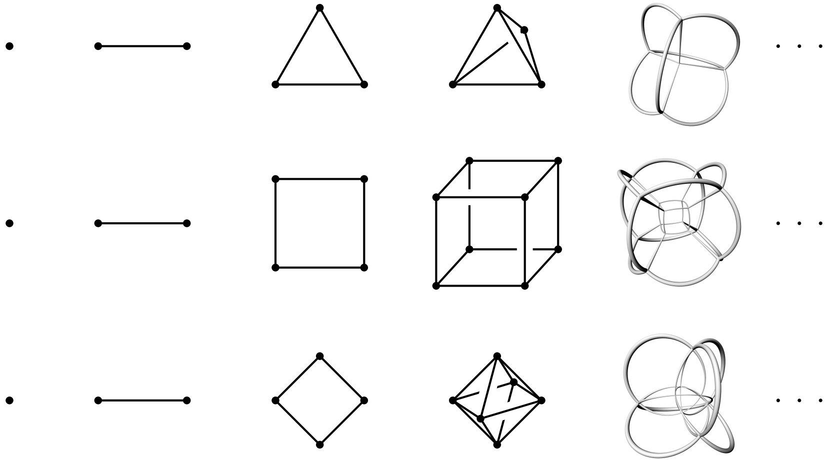


Regular Polytopes

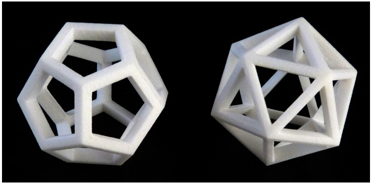
In 3-dimensions: Regular polyhedra



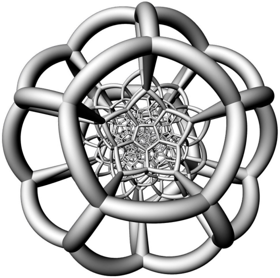
Three families of regular polytopes



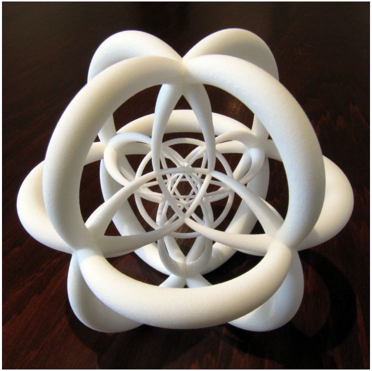
The only exceptions!



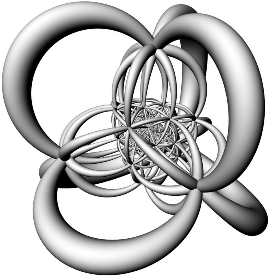
Dodecahedron Icosahedron



120-cell

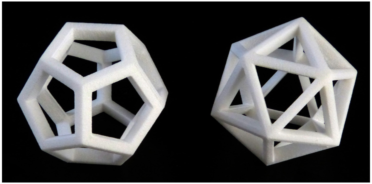


24-cell



600-cell

The only exceptions!



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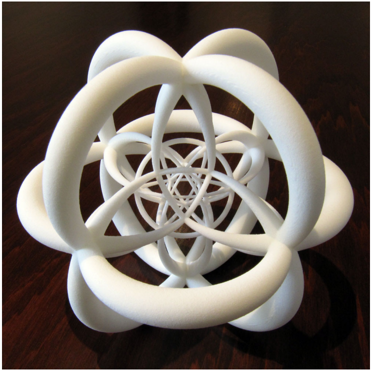
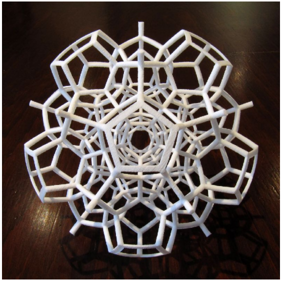
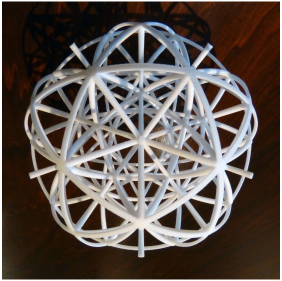


24-cell



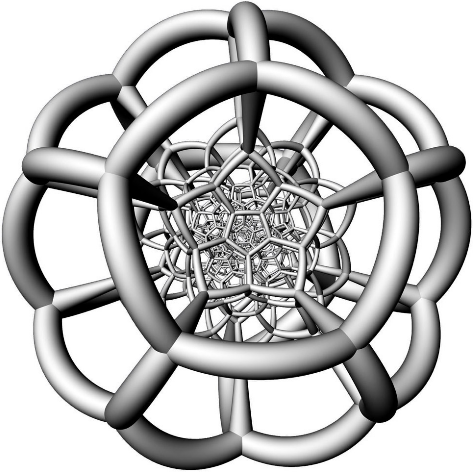
Half of a 120-cell



Half of a 600-cell

Puzzling the 120-cell

(Joint work with Saul Schleimer.)



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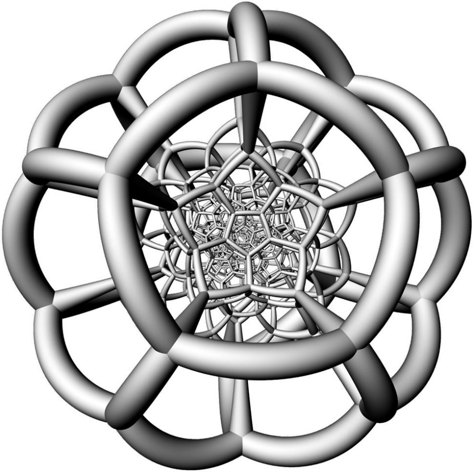

The 120–cell has

- 120 dodecahedral cells,
- 720 pentagonal faces,
- 1200 edges, and
- 600 vertices.

Spherical layers in the 120–cell

One way to understand the 120–cell is to look at the layers of dodecahedra around the central dodecahedron.

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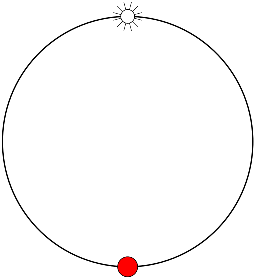
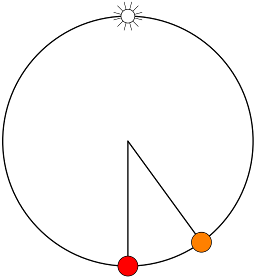


Spherical layers in the 120–cell

One way to understand the 120–cell is to look at the layers of dodecahedra around the central dodecahedron.

- 1 central dodecahedron
- 12 dodecahedra at angle $\pi/5$



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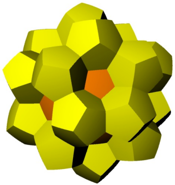

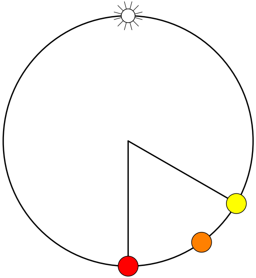

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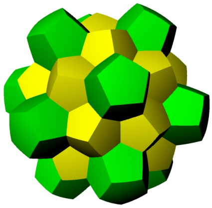

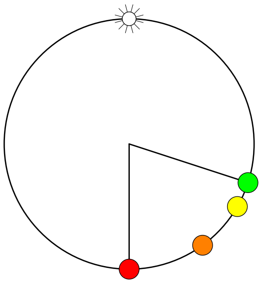

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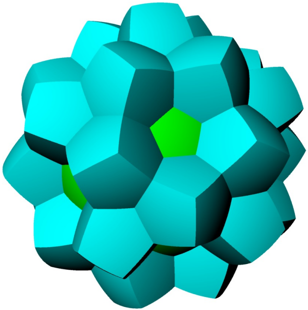
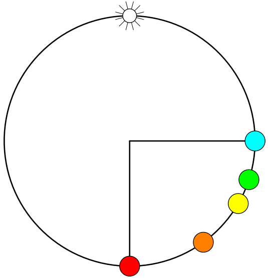
- 30 dodecahedra at angle $\pi/2$





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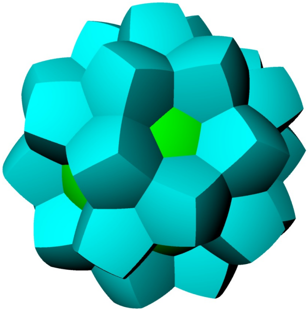
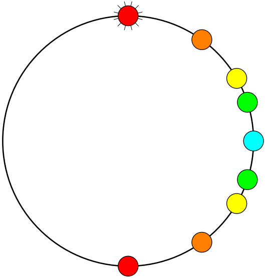
The pattern is mirrored in the last four layers.

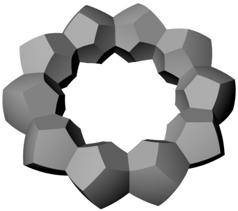
$1+12+20+12+30+12+20+12+1 = 120$





Rings of dodecahedra in the 120–cell

A second way to understand the 120–cell is by making it up out of rings of 10 dodecahedra.



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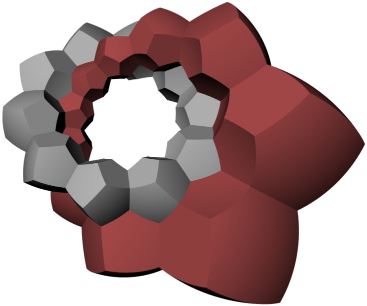
The rings wrap around each other.



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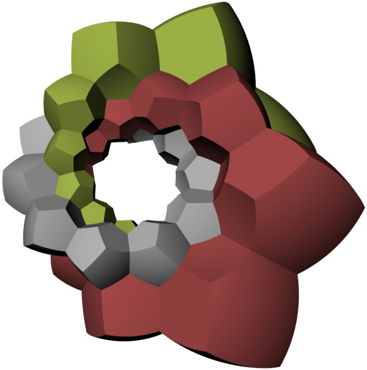

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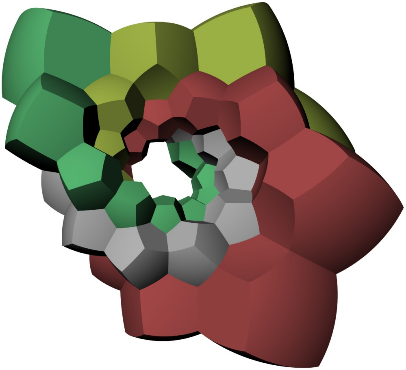

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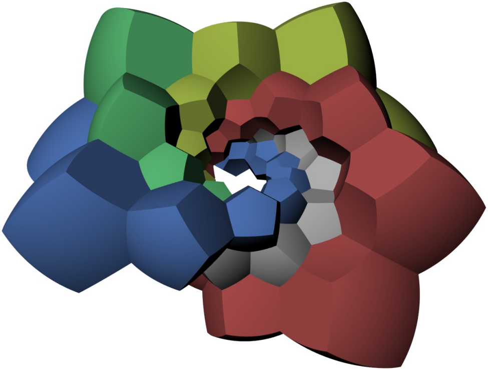
Each ring is surrounded by five others.



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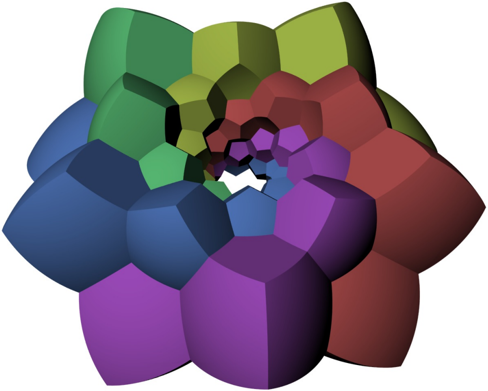

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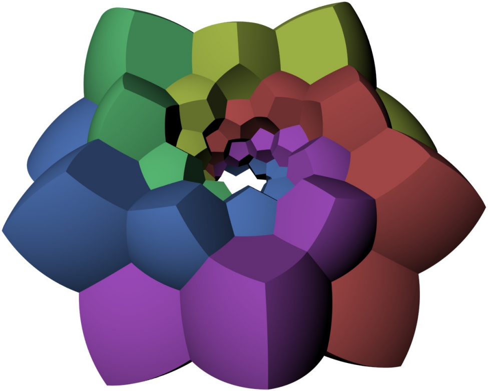

These six rings make up half of the 120–cell. The other half consists of five more rings that wrap around these, and one more ring "dual" to the original grey one.

$$1 + 5 + 5 + 1 = 12 = 120/10$$

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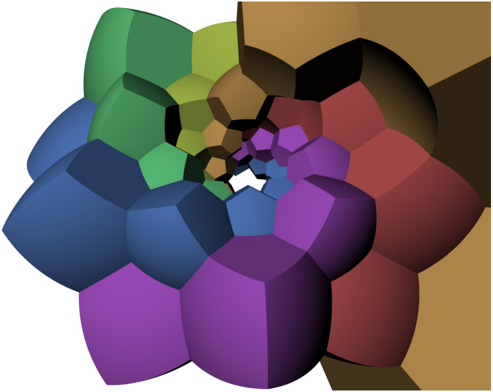

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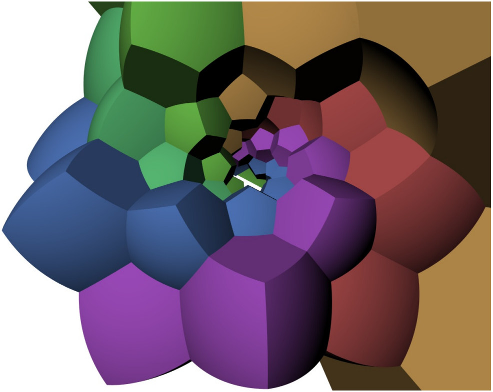

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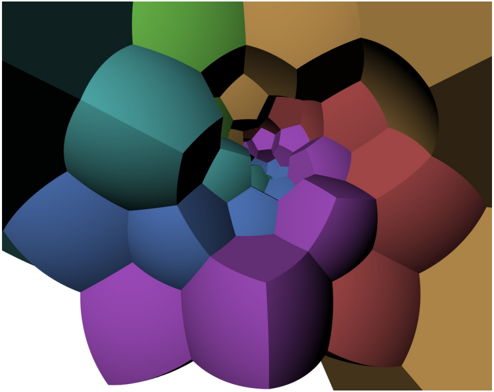


These six rings make up half of the 120–cell. The other half consists of five more rings that wrap around these, and one more ring "dual" to the original grey one.

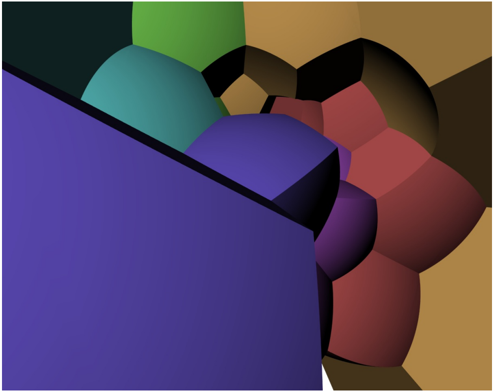
$$1 + 5 + 5 + 1 = 12 = 120/10$$

Rings of dodecahedra in the 120–cell

A second way to understand the 120–cell is by making it up out of rings of 10 dodecahedra.

The rings wrap around each other.

Each ring is surrounded by five others.



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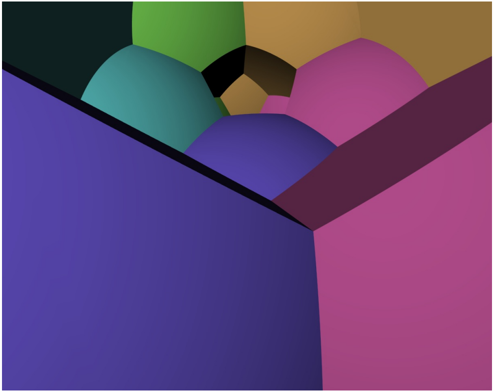

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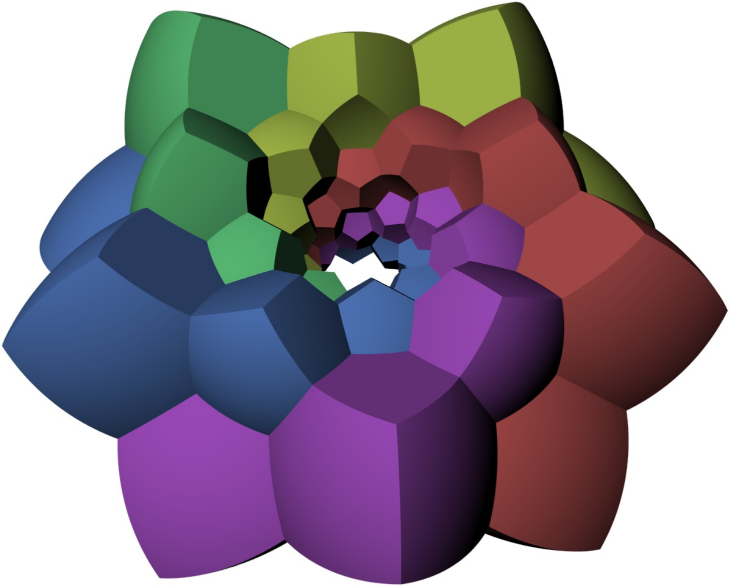
$$1 + 5 + 5 + 1 = 12 = 120/10$$

We wanted to 3D print all six of the inner rings together; it seems this cannot be done without them touching each other. (Parts intended to move must not touch during the printing process.)



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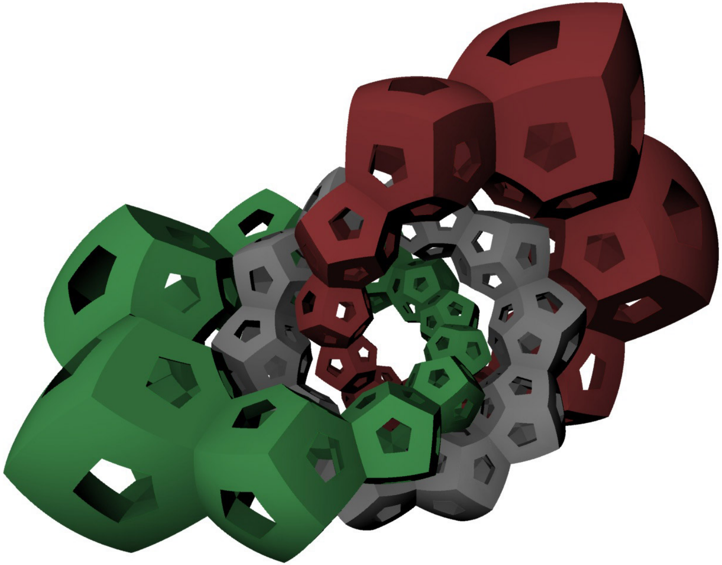
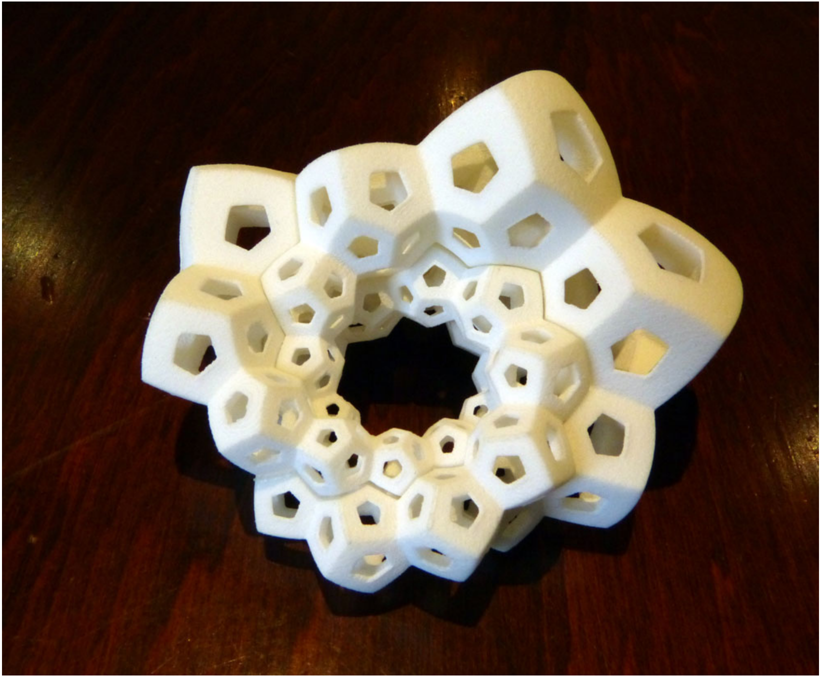
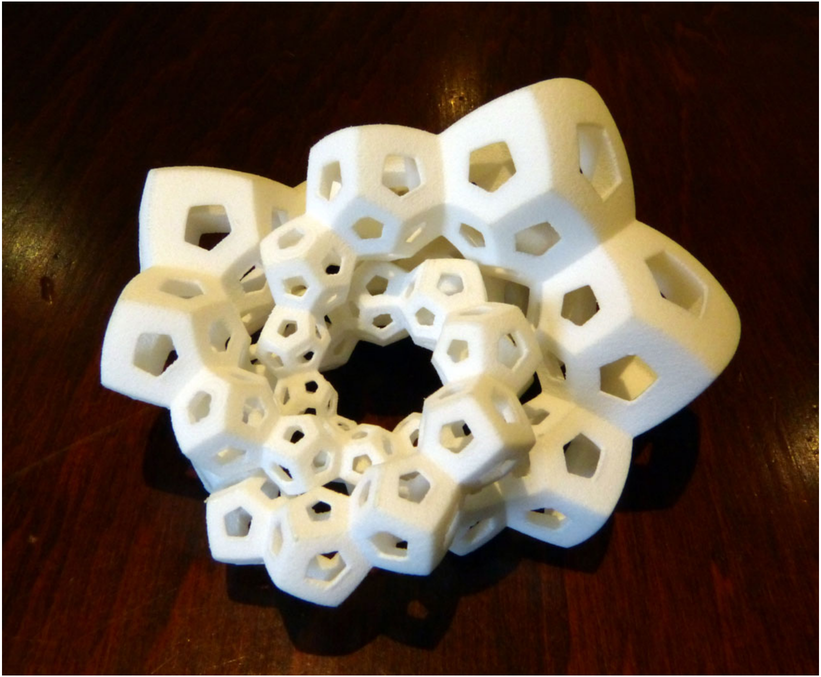
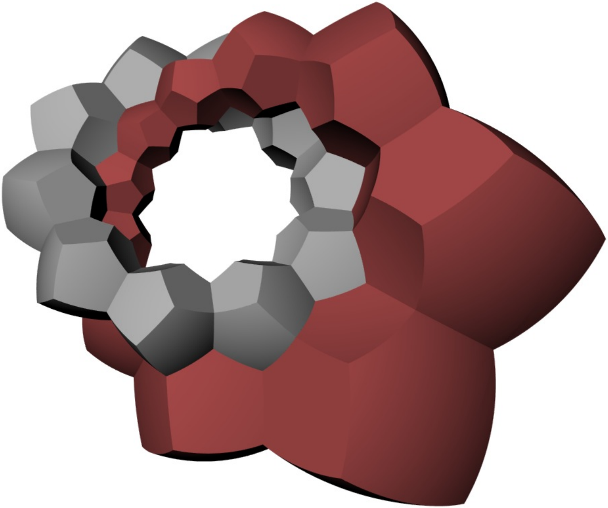


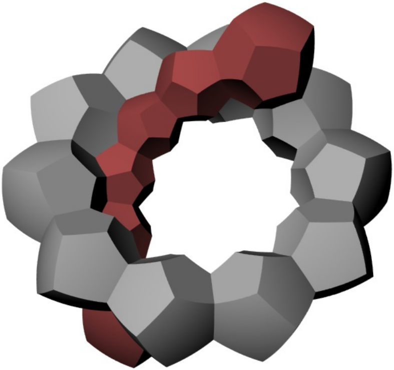




To print all five we use a trick...



To print all five we use a trick... don't print the whole ring. We call part of a ring a rib.



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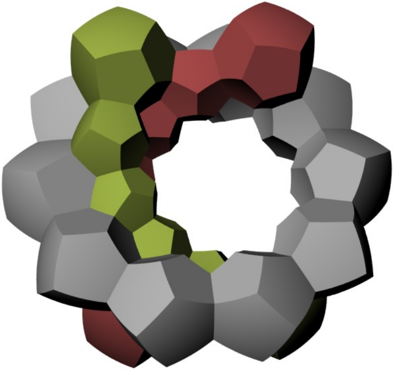

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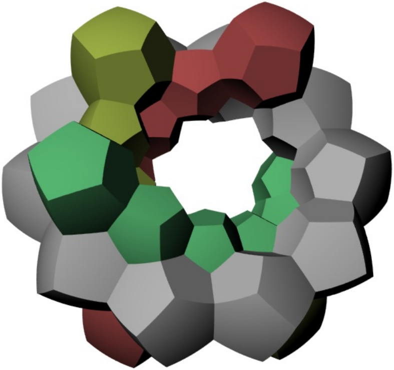

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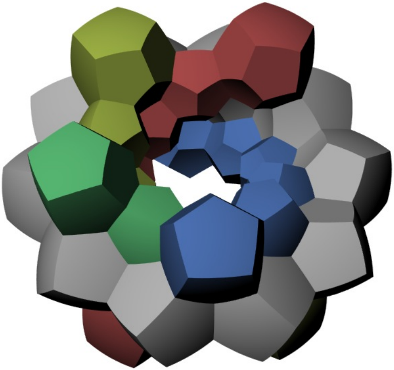

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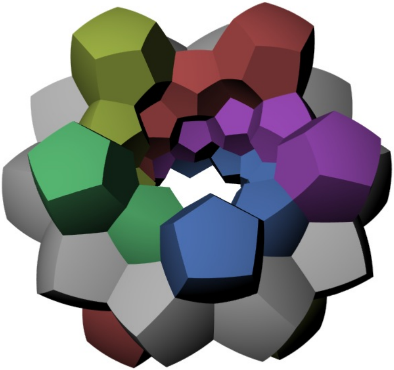

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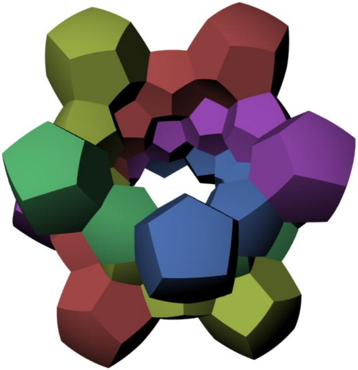

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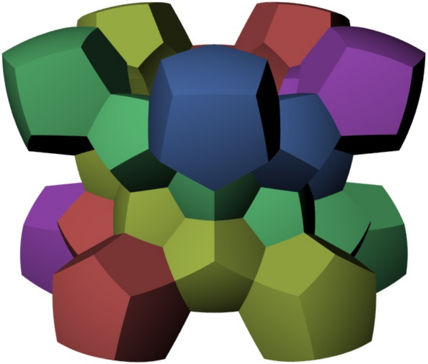
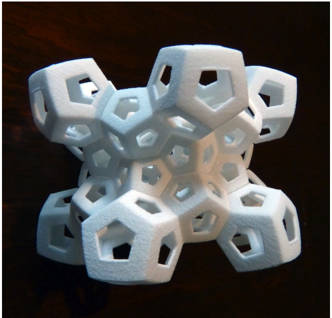
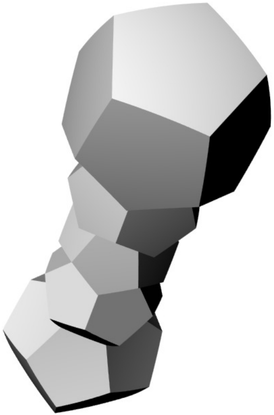


Dc30 Ring puzzle






Another decomposition, with even shorter ribs.



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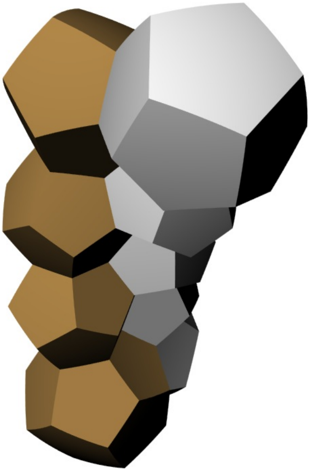

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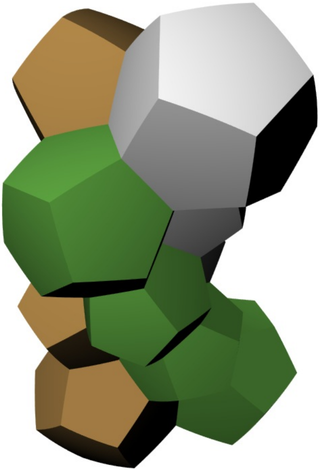

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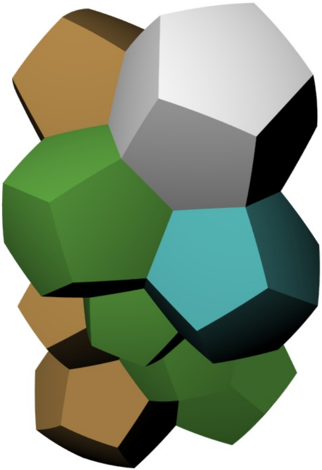

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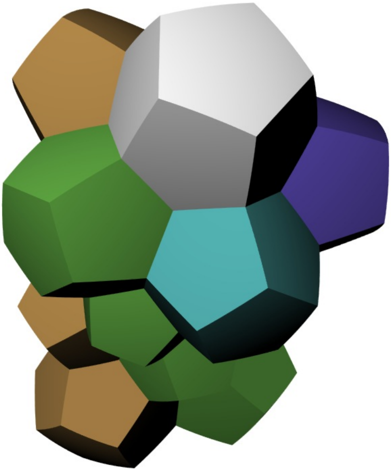

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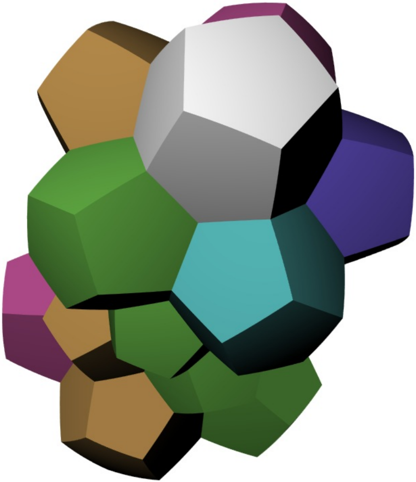

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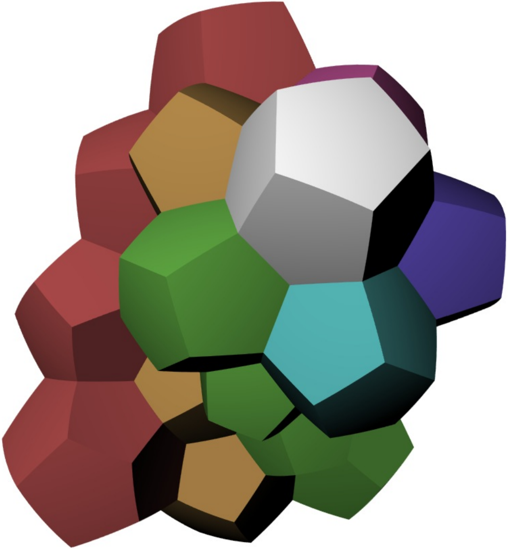

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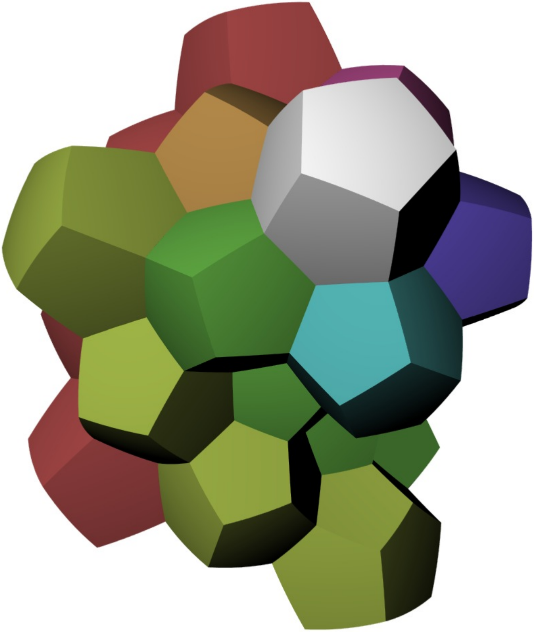

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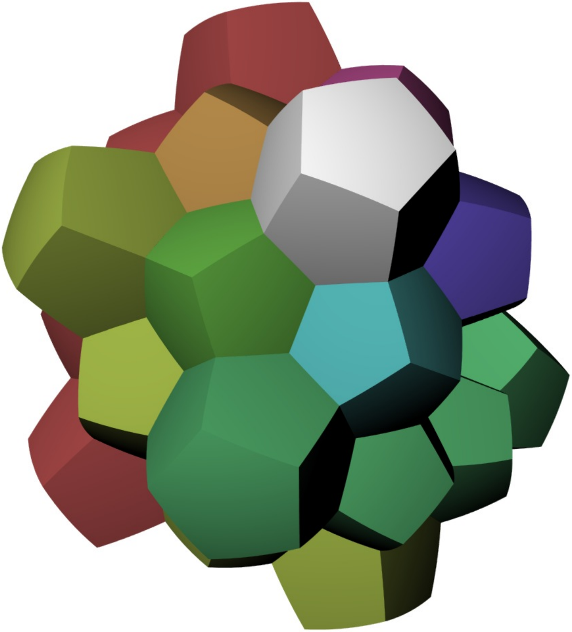

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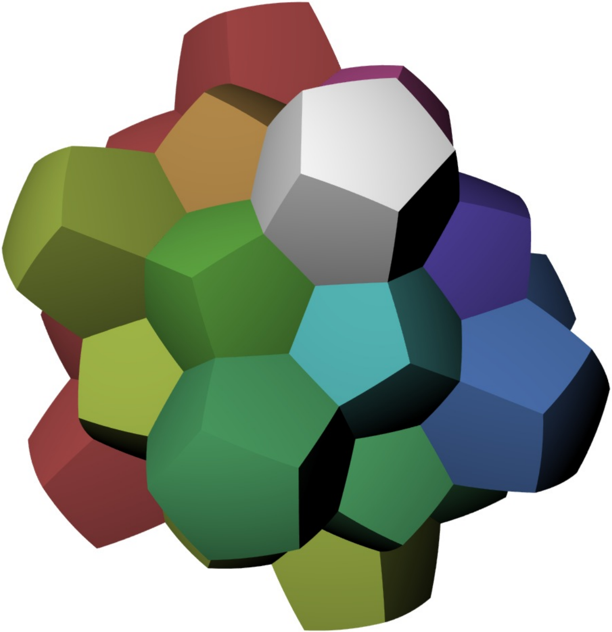

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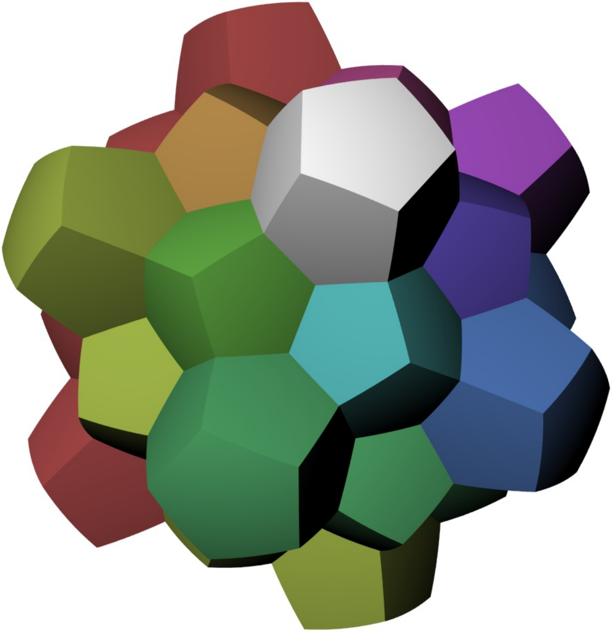
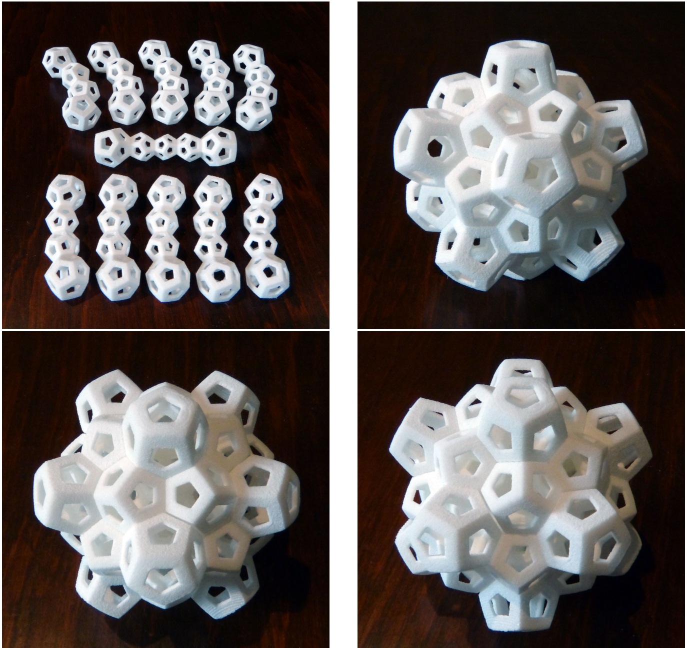
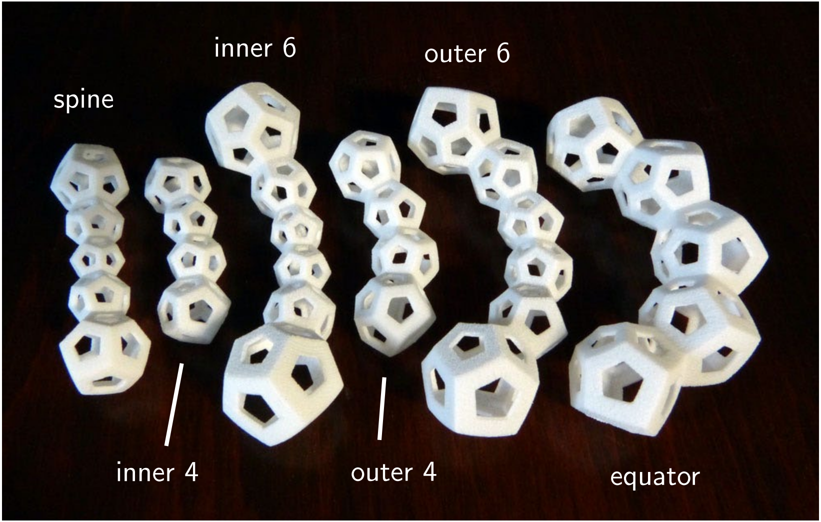


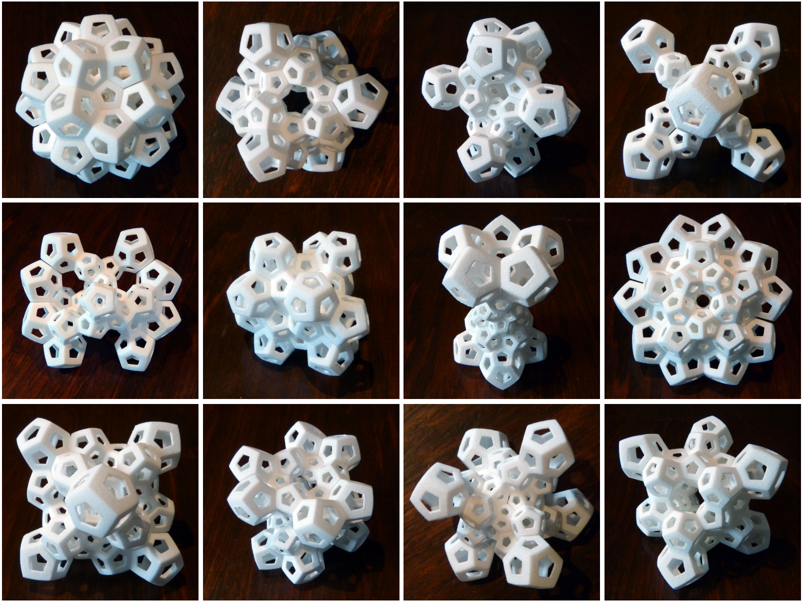
Dc45 Meteor puzzle



Six kinds of ribs

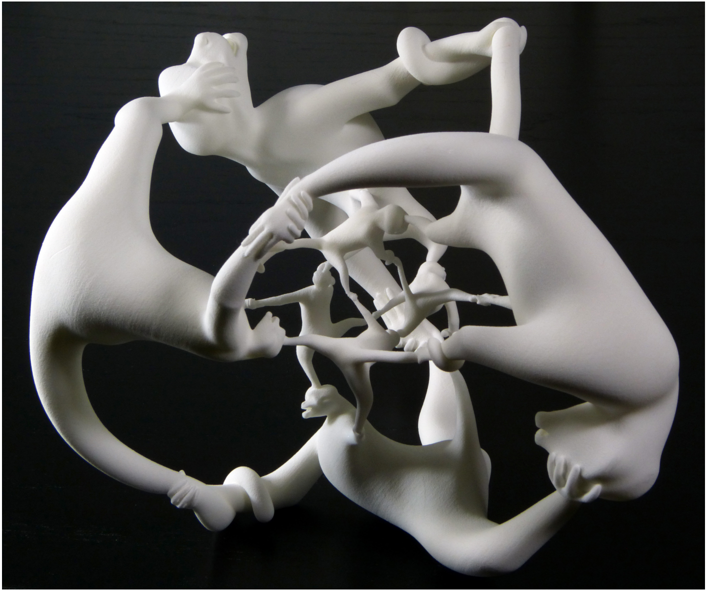


These make many puzzles, which we collectively call Quintessence.



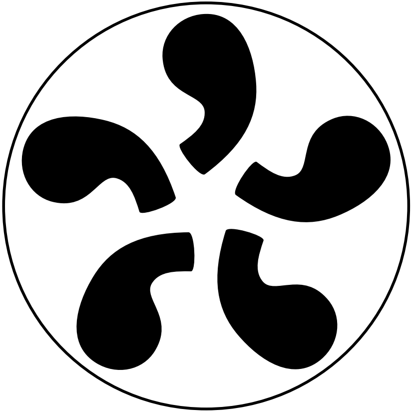
More fun than a hypercube of monkeys

(Joint work with Vi Hart and Will Segerman.)



Symmetry

A symmetry of an object is a motion that leaves the object looking the same.



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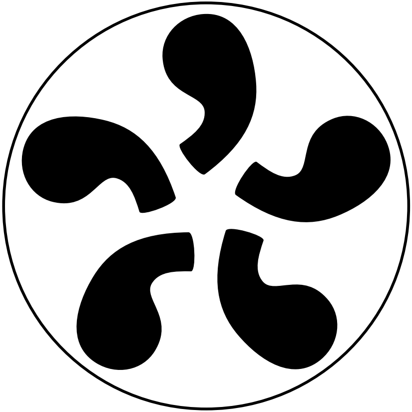
This object has five symmetries:

- Rotate by 1/5 of a turn,
- Rotate by 2/5 of a turn,
- Rotate by 3/5 of a turn,
- Rotate by 4/5 of a turn, and
- Do nothing.



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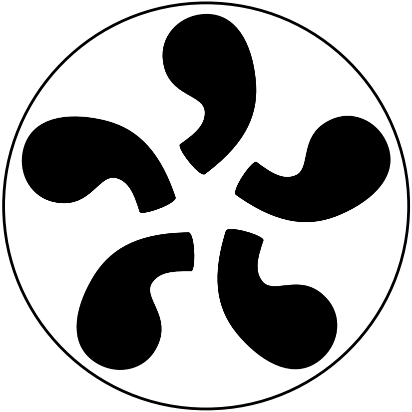
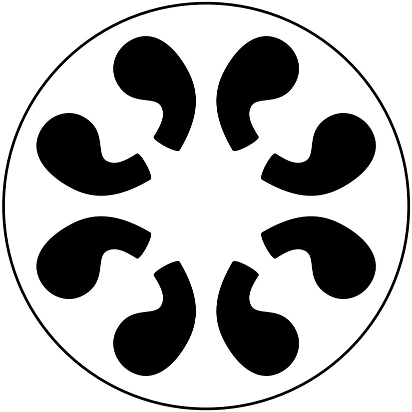


These symmetries can be "added together" by doing one motion followed by another.

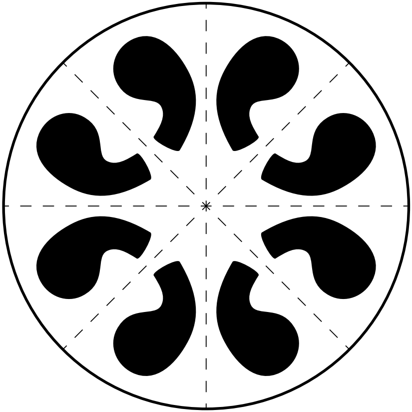
Symmetry



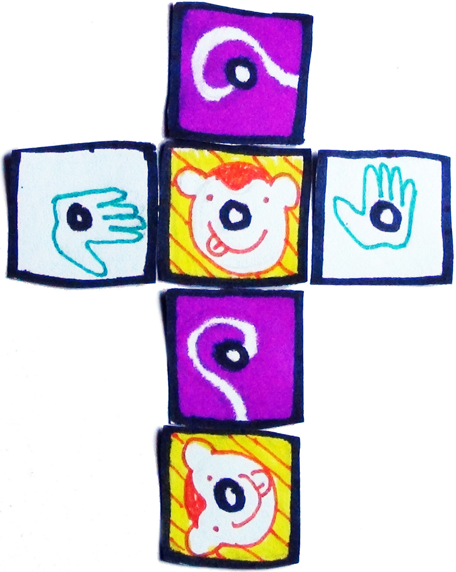
Symmetry

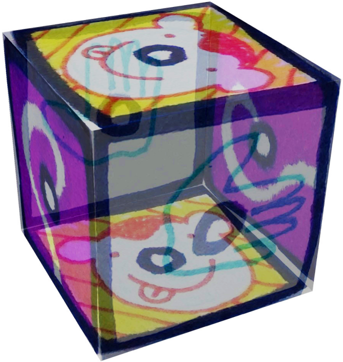
This object has eight symmetries:

- Four rotations (including do nothing), and
- Four reflections.



Monkey blocks





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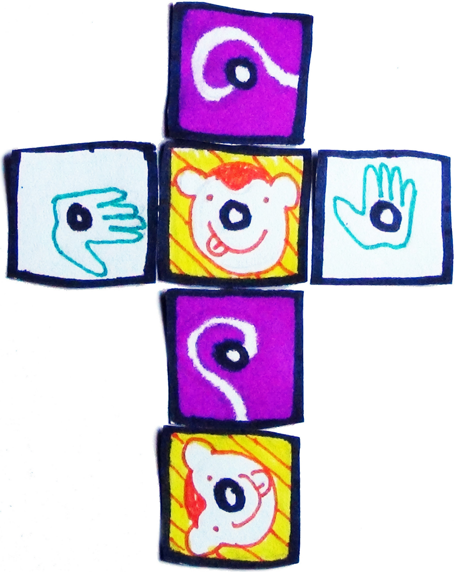

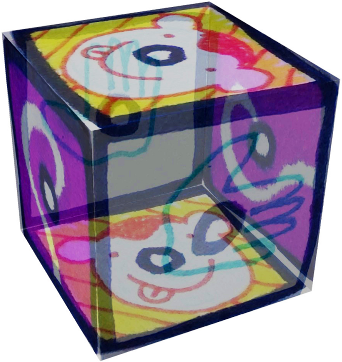

How can monkey blocks fit together so that the faces match?

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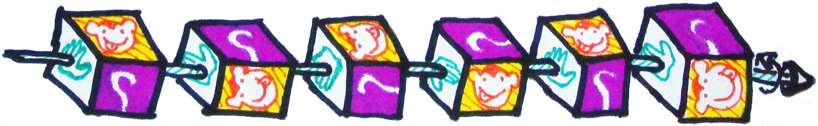

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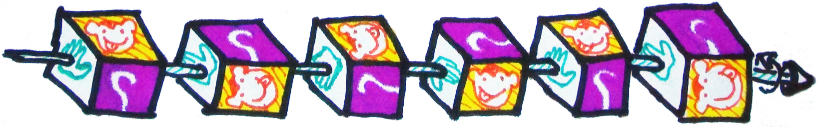

What are the symmetries of this infinite line of blocks?

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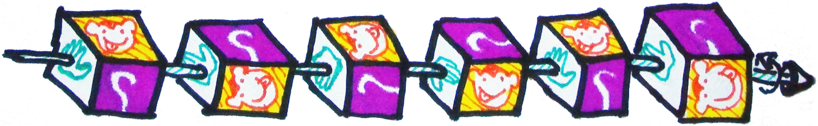

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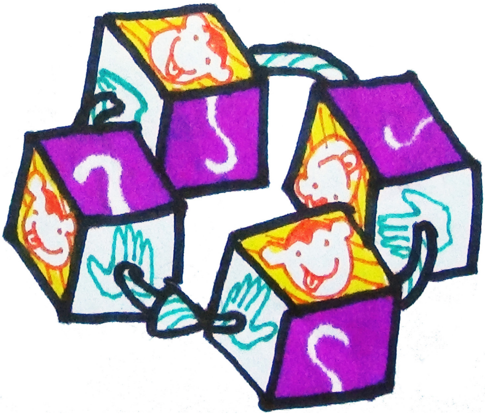

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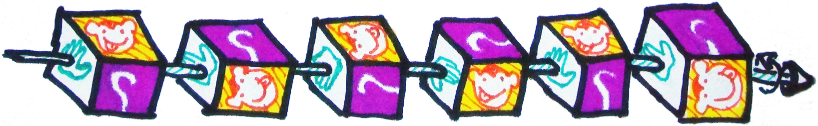
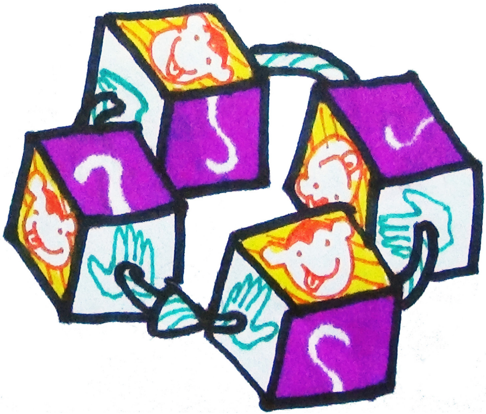


What are the symmetries of this infinite line of blocks?

What are the symmetries of this ring of four blocks?



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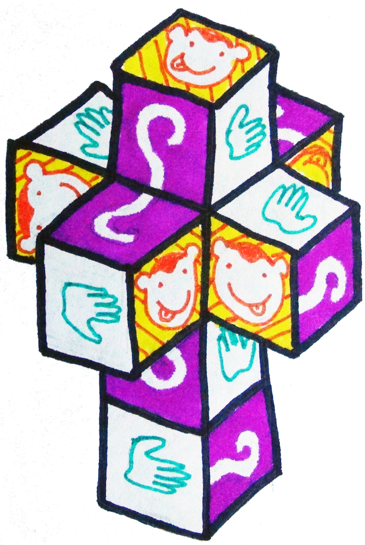

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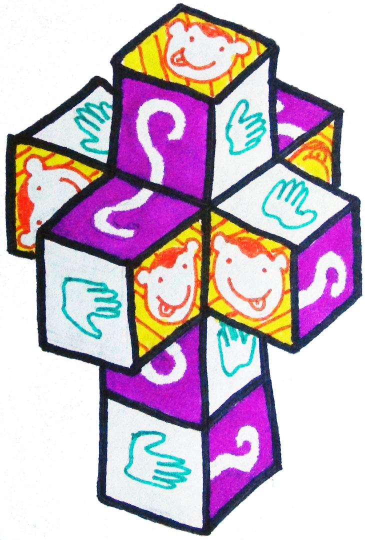

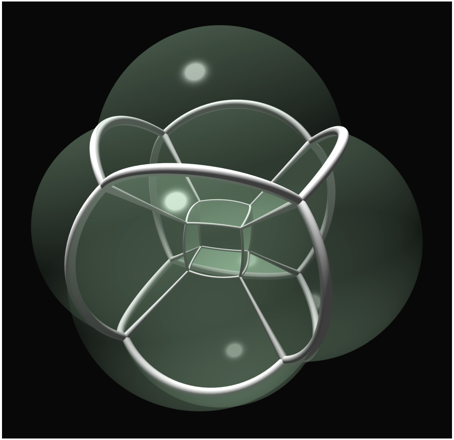

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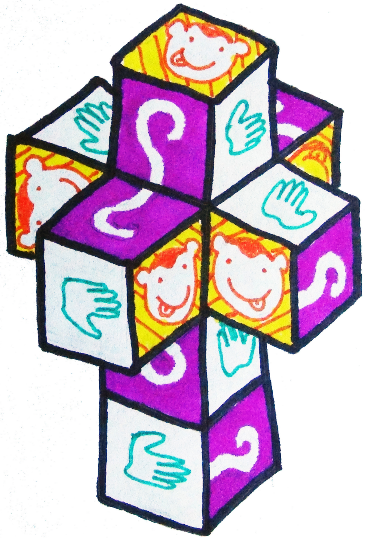

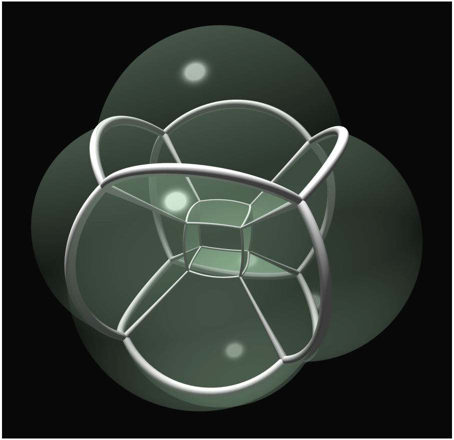

Eight monkey blocks glue together to make the cells of a hypercube!

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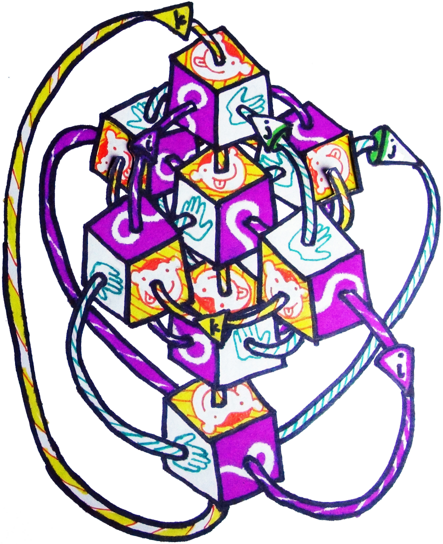
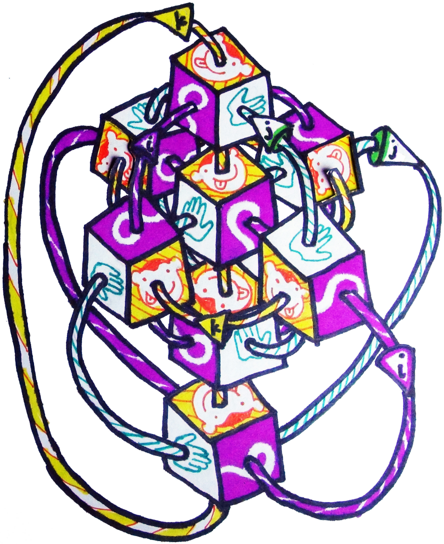


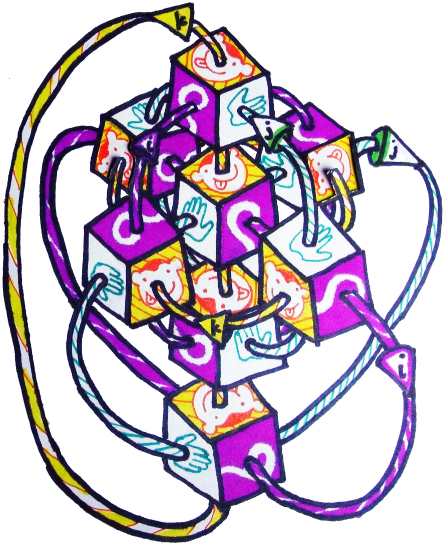
The quaternion group



There are eight symmetries of this decorated hypercube. These correspond to the eight elements of the quaternion group

$$Q_8 = \{1, i, j, k, -1, -i, -j, -k\}$$

The quaternion group



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$$Q_8 = \{1, i, j, k, -1, -i, -j, -k\}$$

- 1 is "do nothing",
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- -1 sends every cube to its "opposite".

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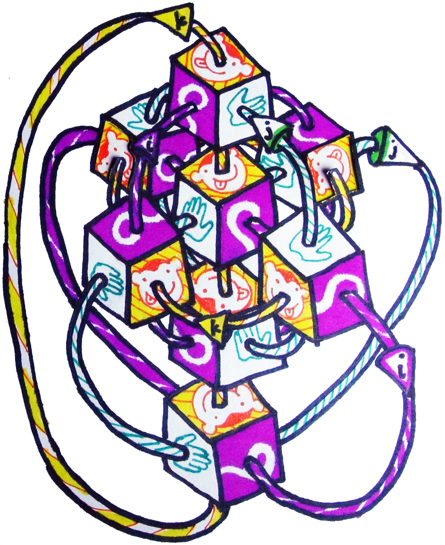

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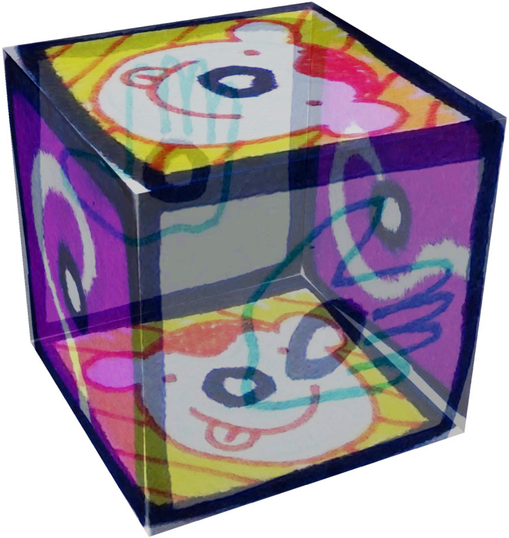
These satisfy
$i^2 = j^2 = k^2 = ijk = -1$.

A sculpture with Q_8 symmetry

Each monkey block itself has no symmetry.



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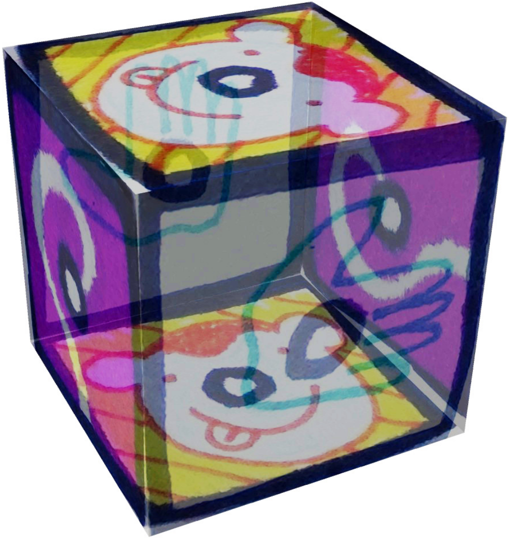

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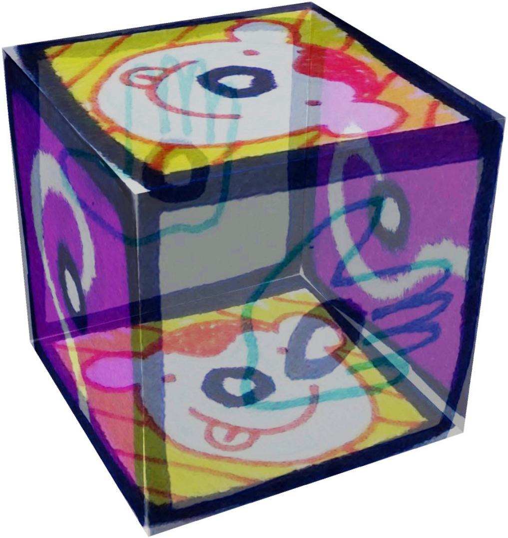
So to make a sculpture with Q_8 symmetry, we put a design with no symmetry into a cube.



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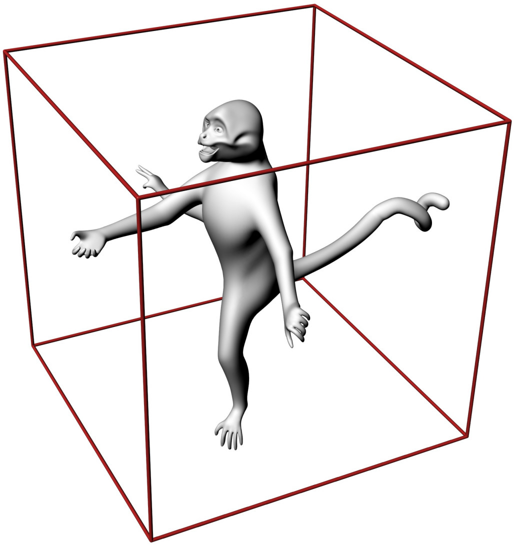

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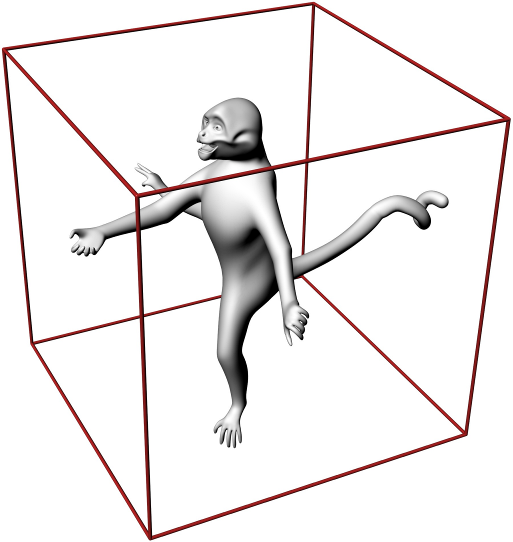

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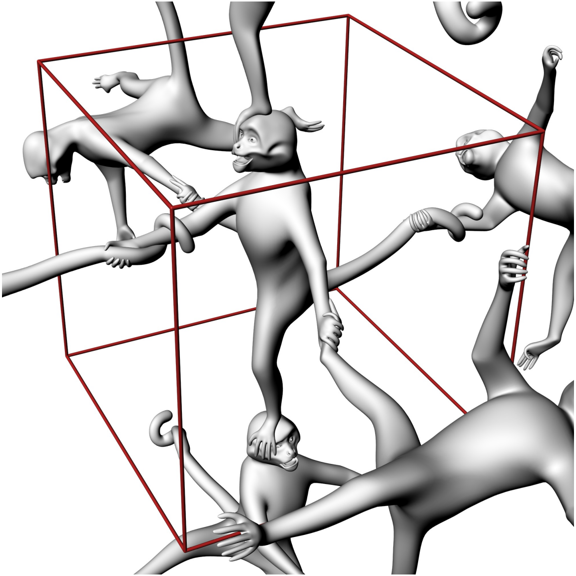
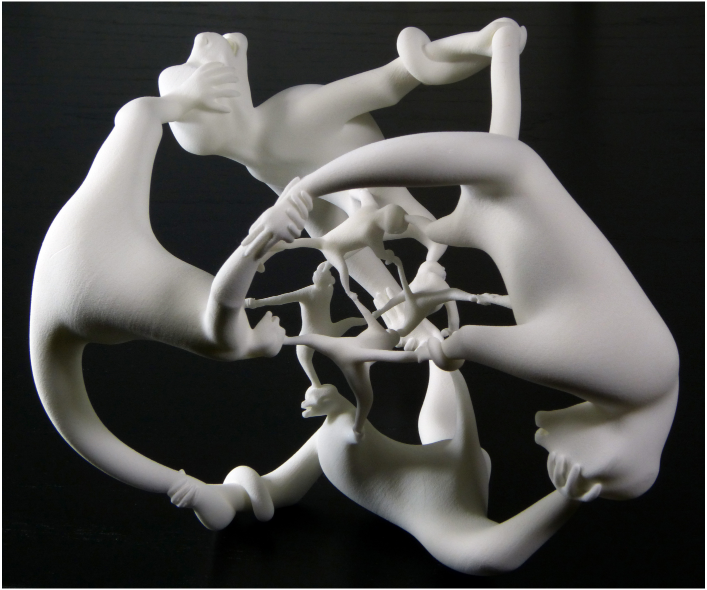


View these cubes as cells of the hypercube in 4-dimensional space, radially project and then stereographically project!



http://monkeys.hypernom.com

http://monkeys.hypernom.com

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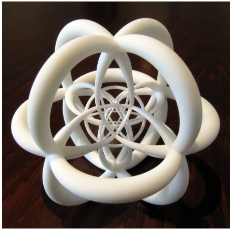
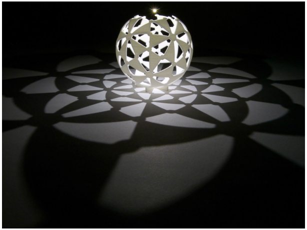
Thanks!





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