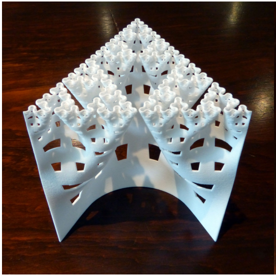
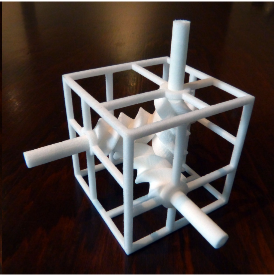


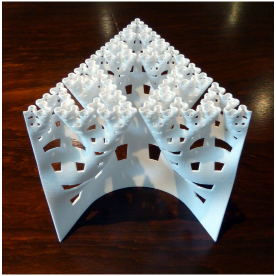


Henry Segerman

Oklahoma State University

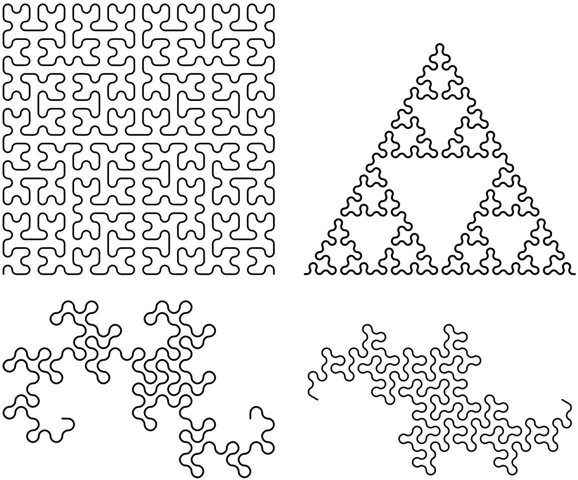
Fractal curves, 4-dimensional puzzles and unlikely gears





Developing Fractal Curves (joint with Geoffrey Irving)

Fractal curves



L-systems

An *L-system grammar* consists of an *alphabet*, an *axiom* and a set of *replacement rules*. For example:

$$G_{\text{terdragon}} = (\{F, -, +\}, F, \{F \mapsto F{-}F{+}F\})$$

Start with the axiom, and repeated apply the rule:

$$F$$

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Start with the axiom, and repeated apply the rule:

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & F & & & & & & & & \\
 & & F & & & - & & & F & & & + & & & F & & \\
F & - & F & + & F & - & F & - & F & + & F & + & F & - & F & + & F
\end{array}$$

Now interpret the strings as "turtle graphics" instructions:

F	move forward one unit
$-$	turn right 120°
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$$F$$

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F	move forward one unit
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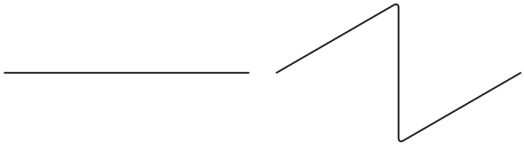
F

$F \quad - \quad F \quad + \quad F$



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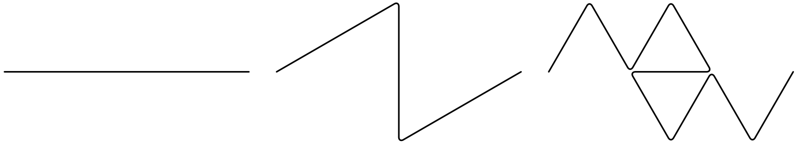
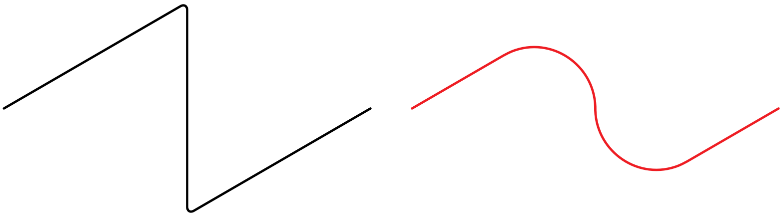
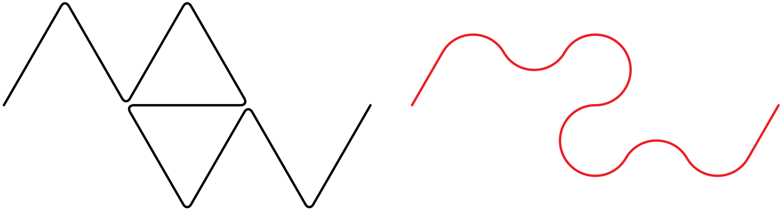
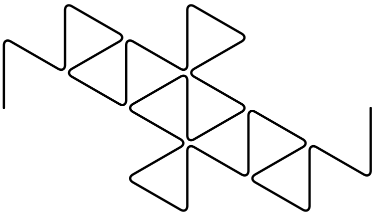
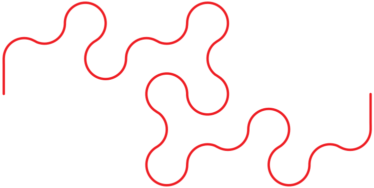
								F								
		F			$-$			F			$+$			F		
F	$-$	F	$+$	F	$-$	F	$-$	F	$+$	F	$+$	F	$-$	F	$+$	F

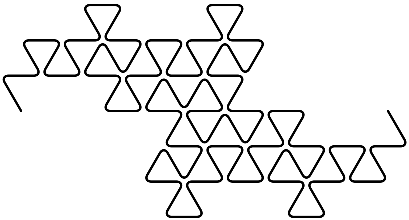


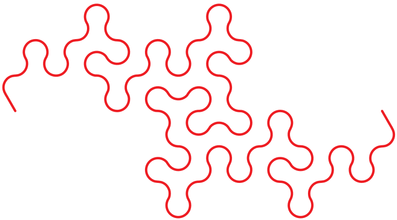


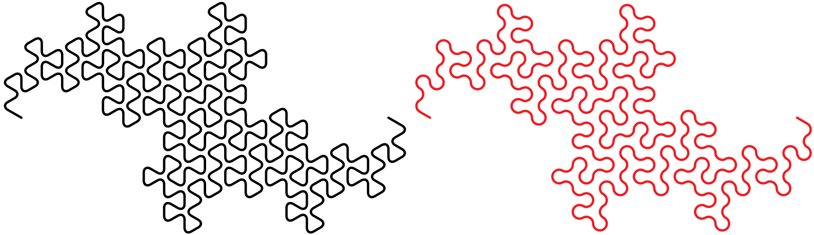




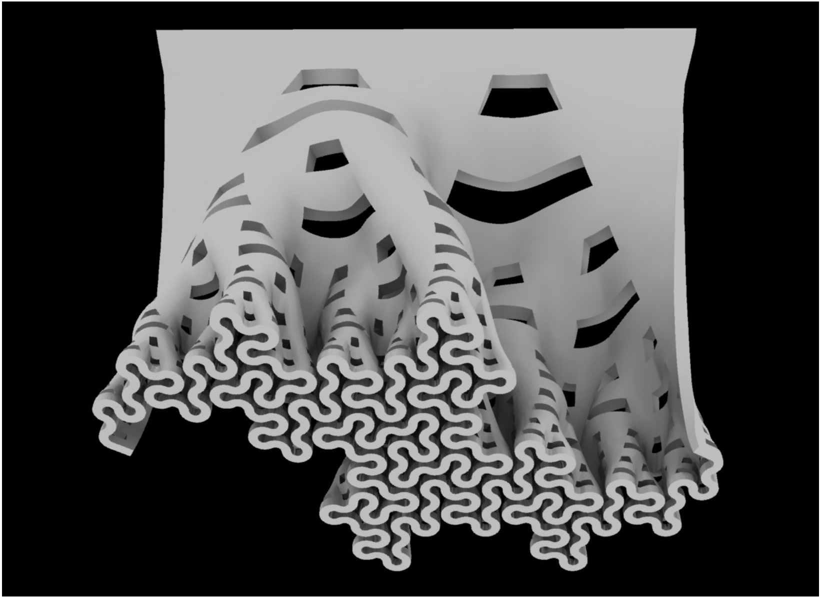


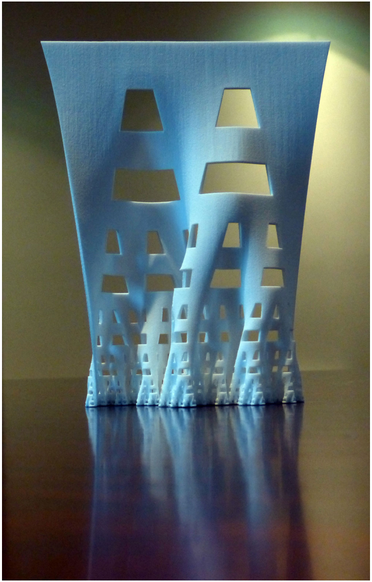
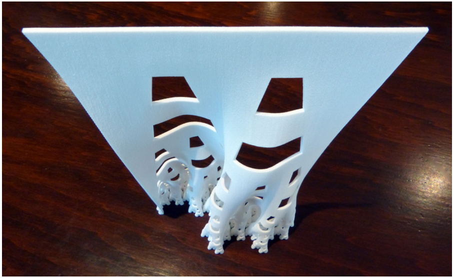
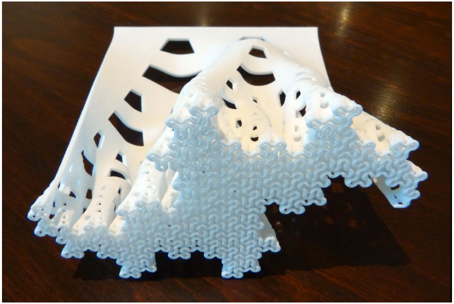




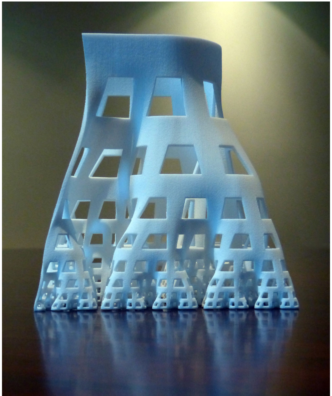
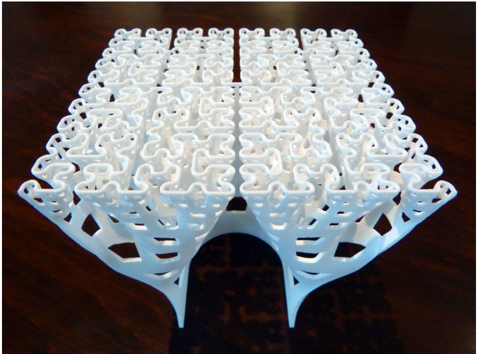


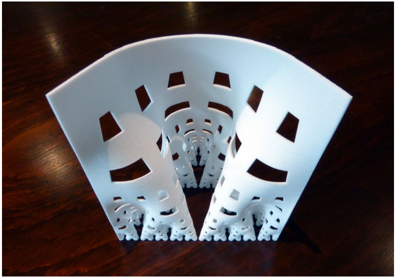
Now arrange these in space rather than time!

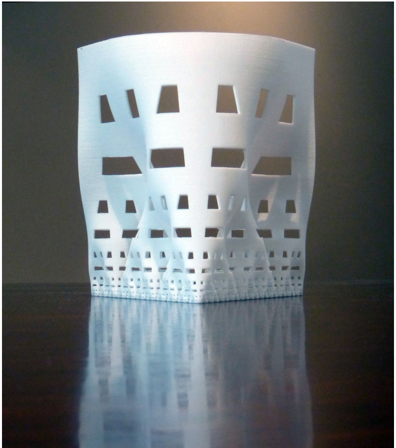


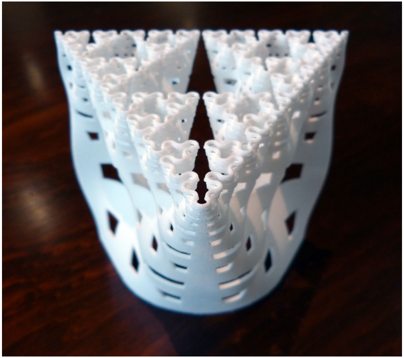




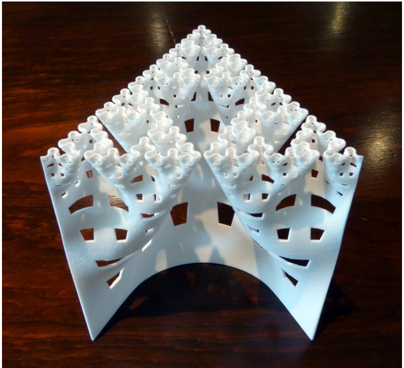


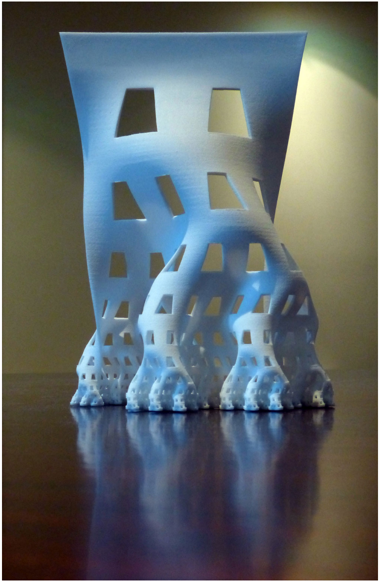
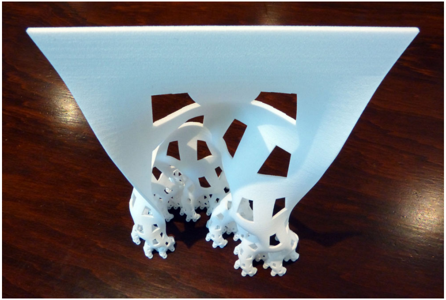
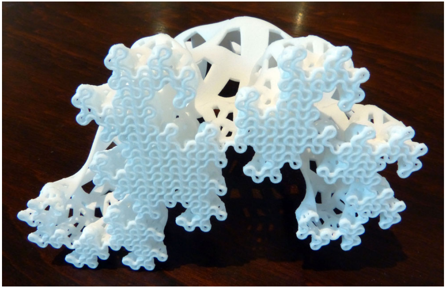



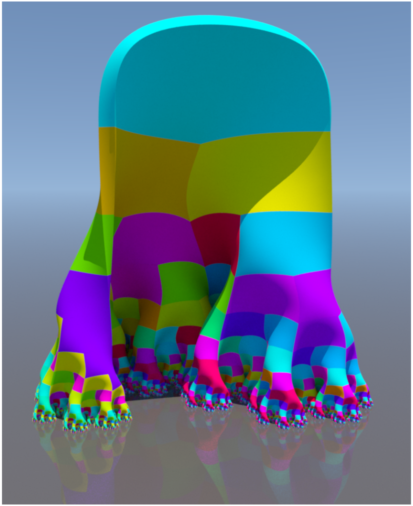


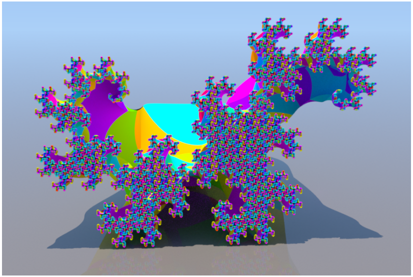


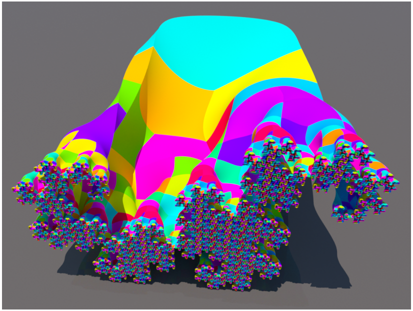




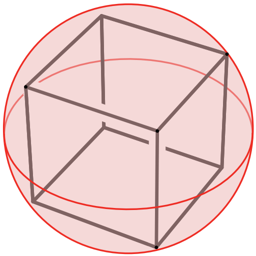






4-dimensional puzzles (joint with Saul Schleimer)

Projecting a cube into $\mathbb{R}^2$



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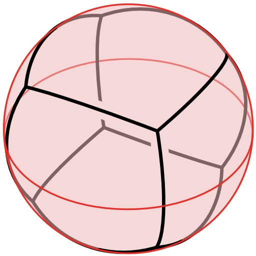

Radial projection

$$\mathbb{R}^3 \setminus \{0\} \to S^2$$

$$(x, y, z) \mapsto \frac{(x, y, z)}{|(x, y, z)|}$$

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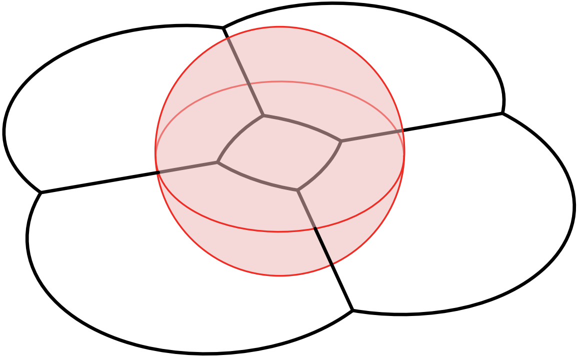

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$\mathbb{R}^3 \setminus \{0\} \to S^2$

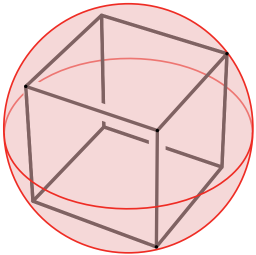
$$(x, y, z) \mapsto \frac{(x, y, z)}{|(x, y, z)|}$$

Stereographic projection

$S^2 \setminus \{N\} \to \mathbb{R}^2$

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

Vertex-centered versus cell-centered projection



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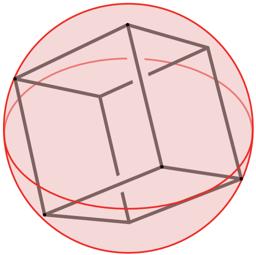

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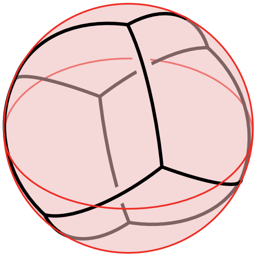

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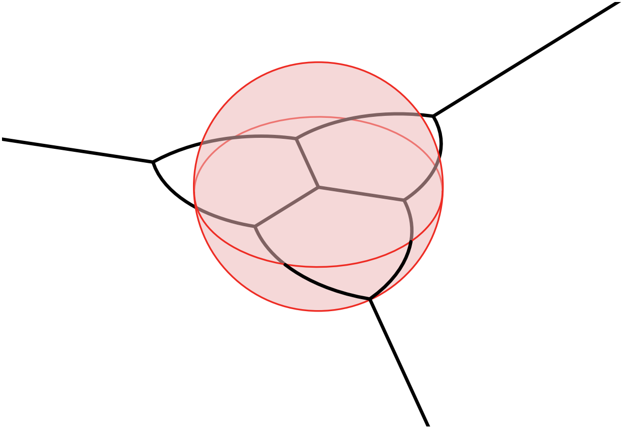

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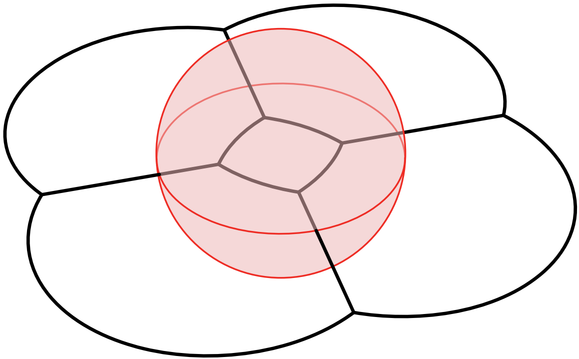
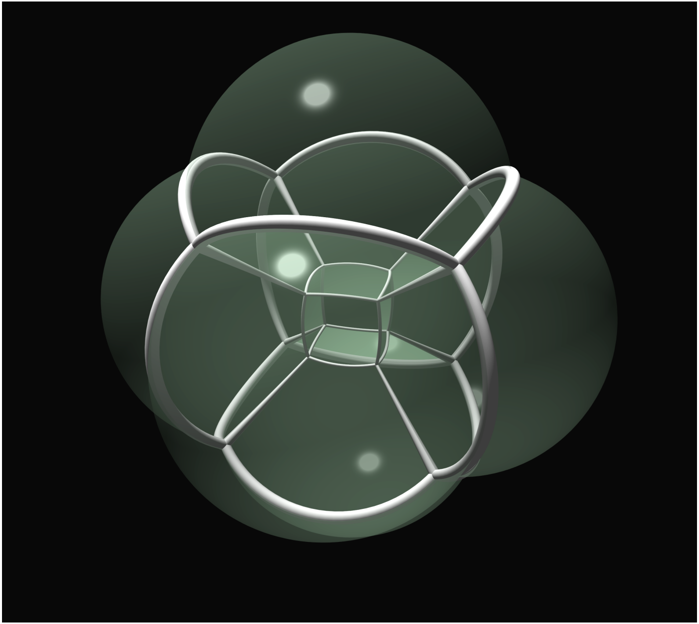
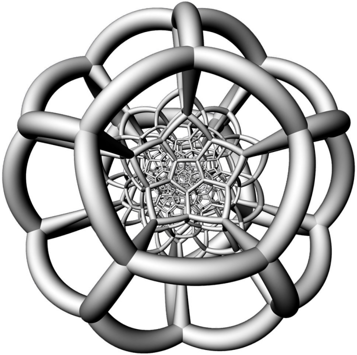


Do the same one dimension up to see a hypercube



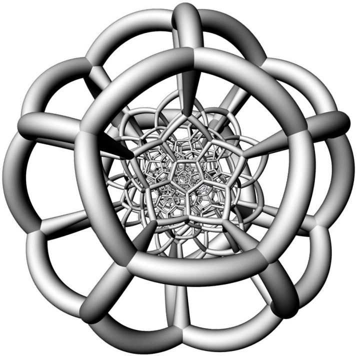
Another 4-dimensional polytope: the 120-cell



Another 4-dimensional polytope: the 120-cell

The 120–cell has

- 120 dodecahedral cells,
- 720 pentagonal faces,
- 1200 edges, and
- 600 vertices.



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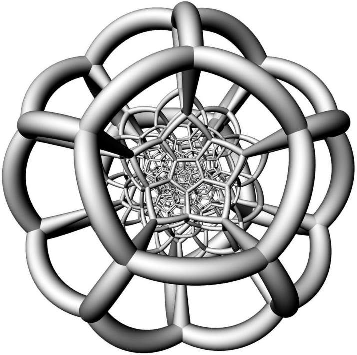
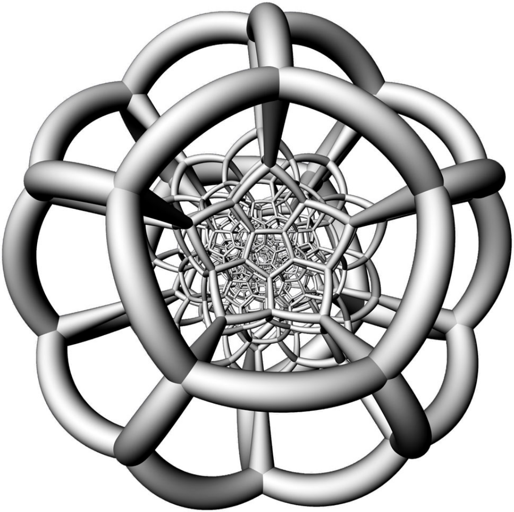


We use radial projection followed by stereographic projection to help us visualise the 120–cell.

$$\mathbb{R}^4 \setminus \{0\} \to S^3 \subset \mathbb{R}^4 \qquad S^3 \setminus \{N\} \to \mathbb{R}^3$$

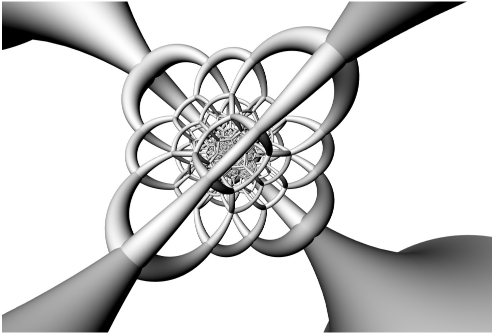
$$(w,x,y,z) \mapsto \frac{(w,x,y,z)}{|(w,x,y,z)|} \qquad (w,x,y,z) \mapsto \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}\right)$$

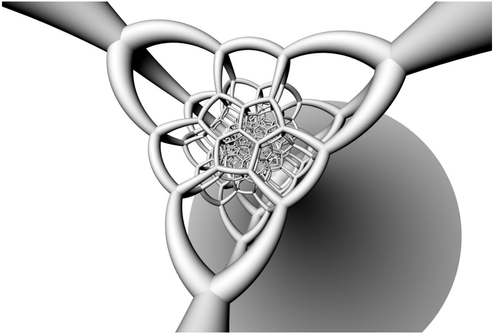
This is the
cell-centered projection
of the 120-cell; it has
dodecahedral symmetry
in $\mathbb{R}^3$.



The vertex-centered projection has tetrahedral symmetry in $\mathbb{R}^3$ and so has fewer possibilities for puzzle making.

Other choices have even less symmetry, and so have even fewer interesting ways to combine pieces.





Spherical layers in the 120–cell

A first way to understand the combinatorics of the 120–cell is to look at the layers of dodecahedra at fixed distances from the central dodecahedron.

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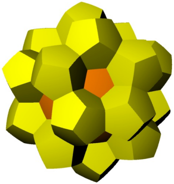
- 12 dodecahedra at distance $\pi/5$
- 20 dodecahedra at distance $\pi/3$



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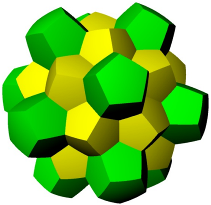
- 20 dodecahedra at distance $\pi/3$
- 12 dodecahedra at distance $2\pi/5$



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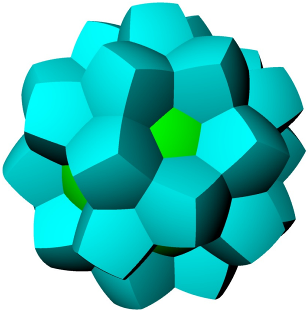

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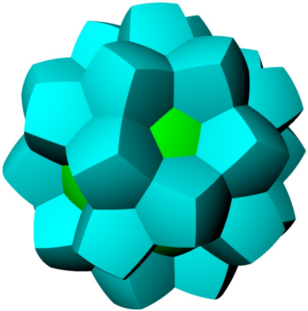

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The pattern is mirrored in the last four layers.

$$1 + 12 + 20 + 12 + 30 + 12 + 20 + 12 + 1 = 120$$

Hopf fibers in the 120–cell

A second way to understand the 120–cell is via a combinatorial version of the Hopf fibration.

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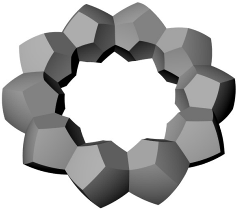
Each fiber is a "ring" of 10 dodecahedra.



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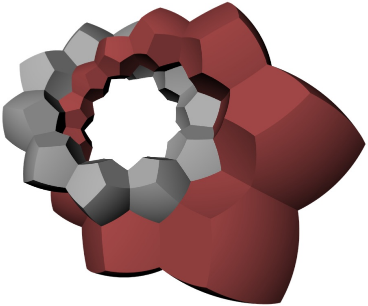
A second way to understand the 120–cell is via a combinatorial version of the Hopf fibration.

Each fiber is a "ring" of 10 dodecahedra.

The rings wrap around each other.



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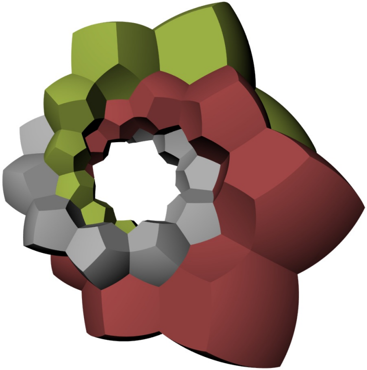

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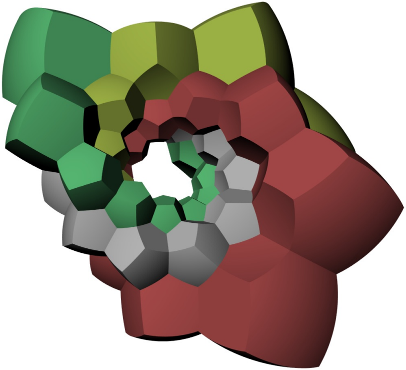

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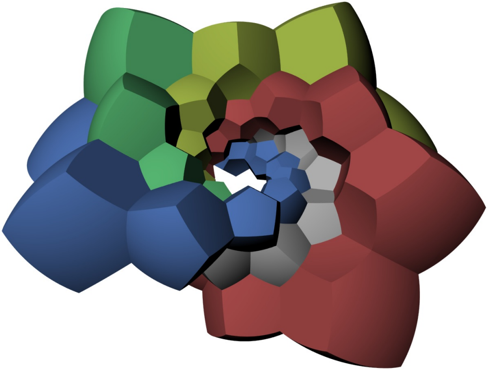
A second way to understand the 120–cell is via a combinatorial version of the Hopf fibration.

Each fiber is a “ring” of 10 dodecahedra.

The rings wrap around each other.

Each ring is surrounded by five others.



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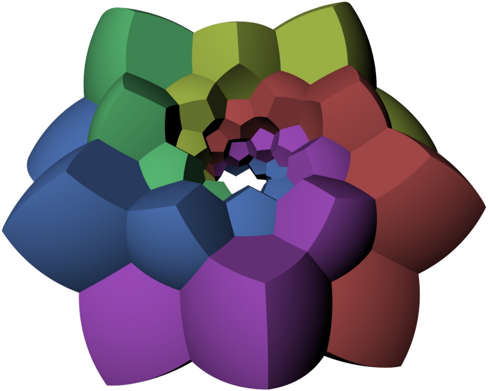

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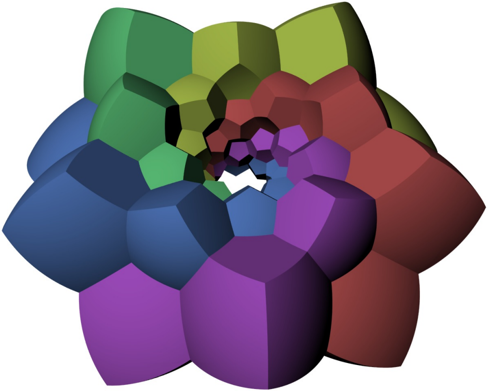


These six rings make up half of the 120–cell. The other half consists of five more rings that wrap around these, and one more ring "dual" to the original grey one.

$$1 + 5 + 5 + 1 = 12 = 120/10$$

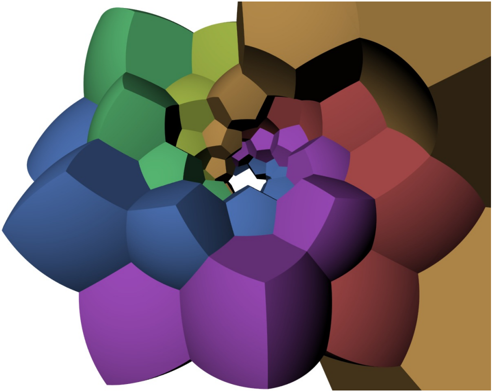
Hopf fibers in the 120–cell

A second way to understand the 120–cell is via a combinatorial version of the Hopf fibration.

Each fiber is a "ring" of 10 dodecahedra.

The rings wrap around each other.

Each ring is surrounded by five others.



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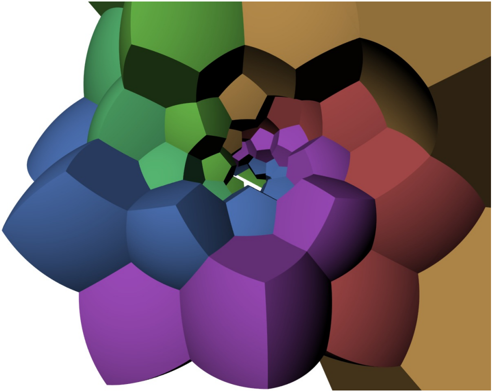

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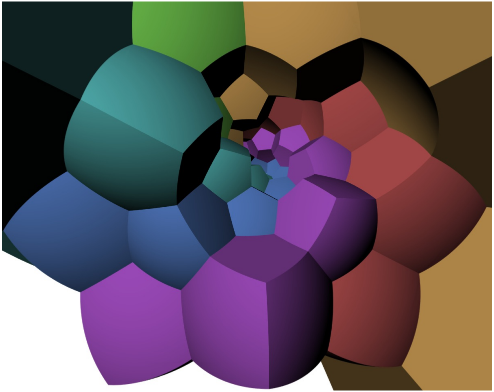

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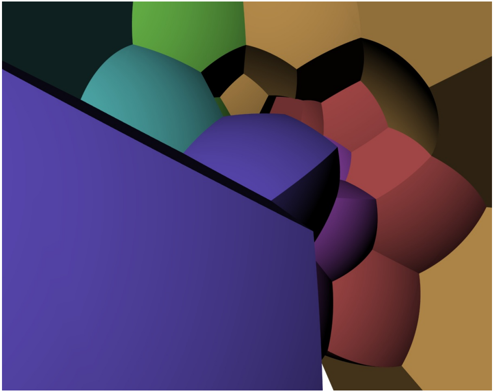

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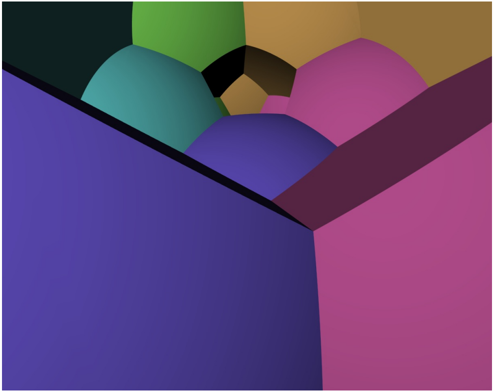

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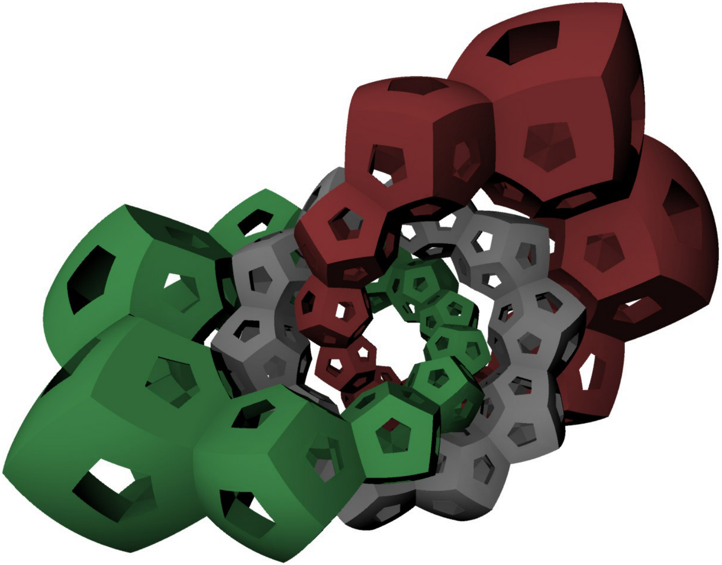
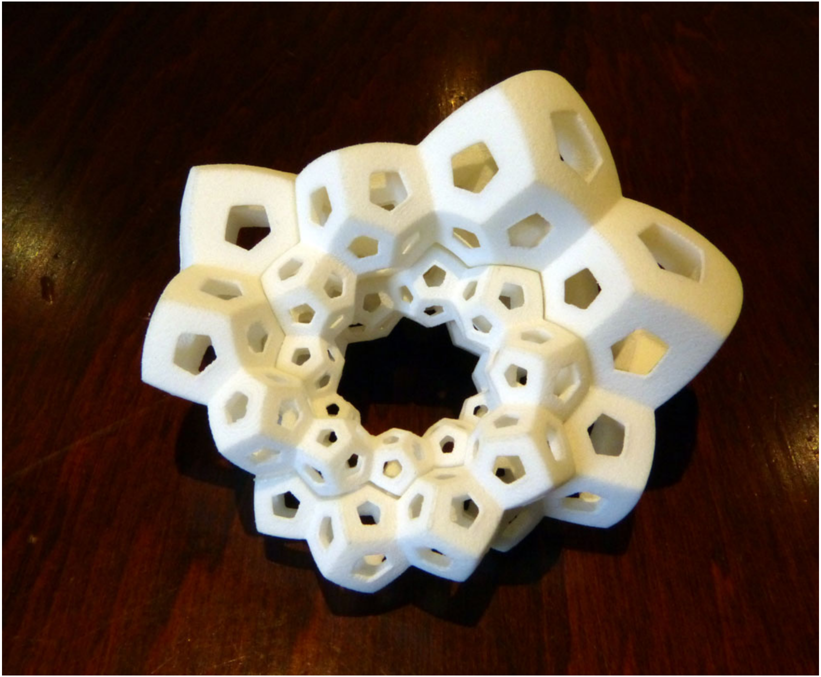
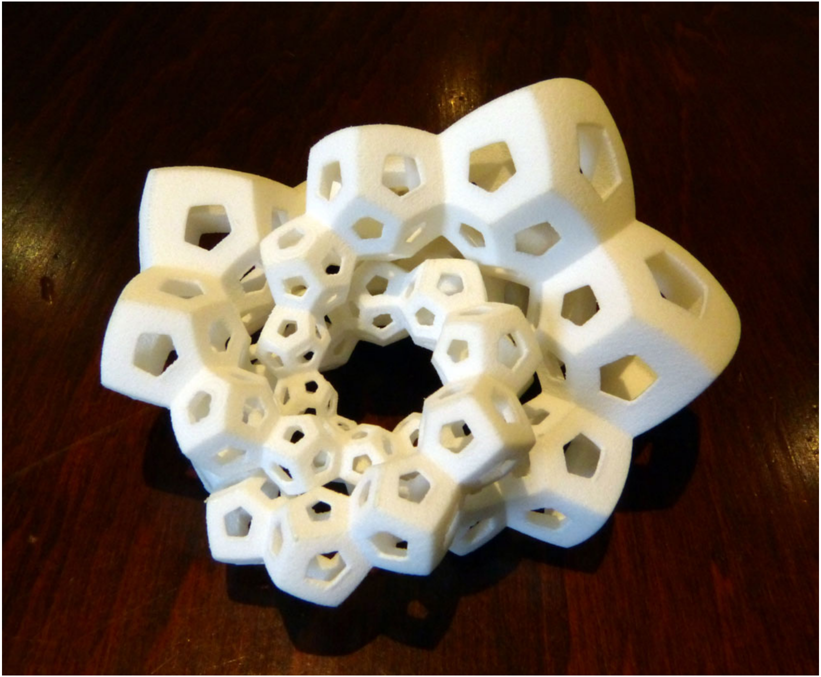
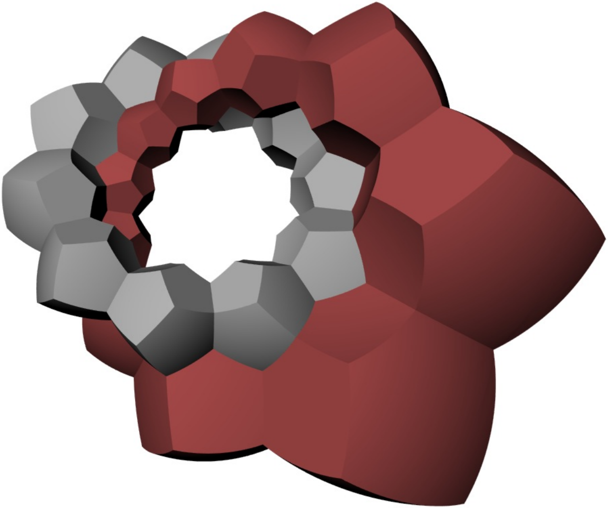
We wanted to 3D print all six of the inner rings together; it seems this cannot be done without them touching each other. (Parts intended to move must not touch during the printing process.)



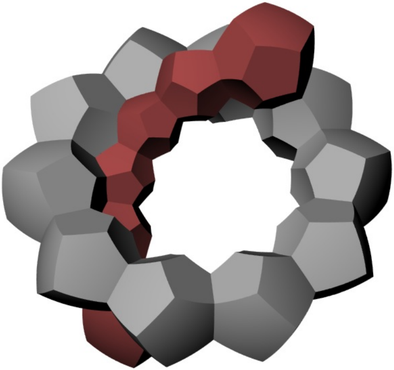




To print all five we use a trick...



To print all five we use a trick... don't print the whole ring. We call part of a ring a rib.



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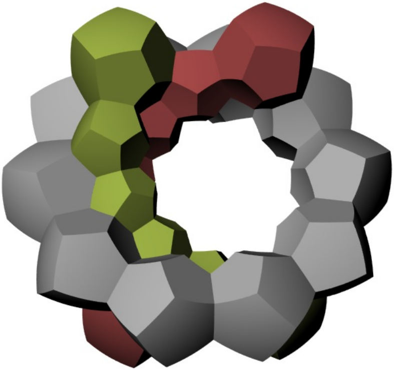

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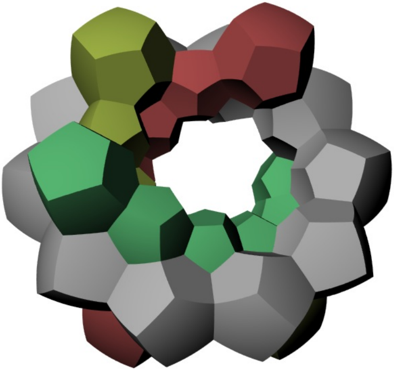

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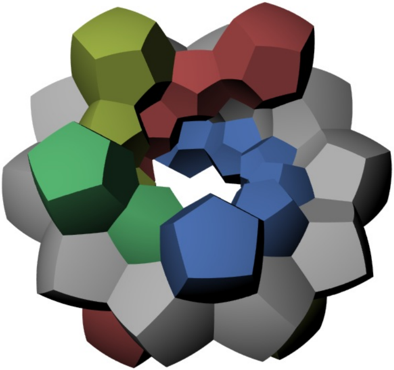

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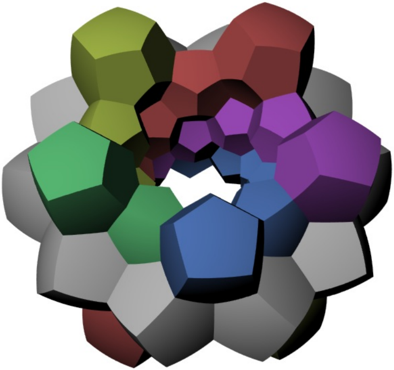

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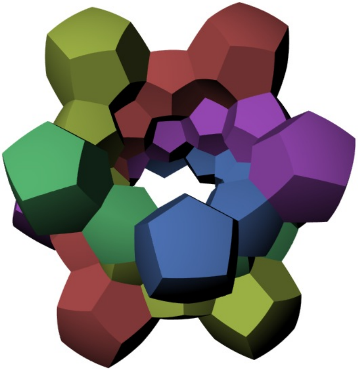

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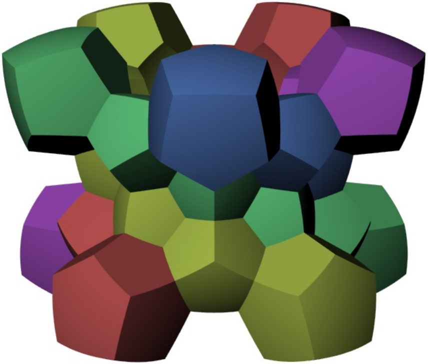
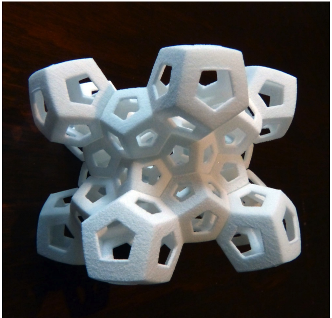
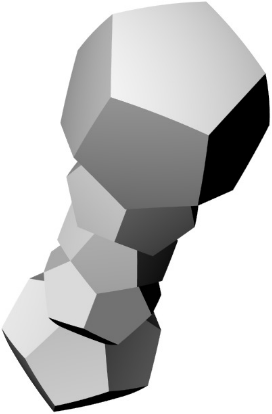


Dc30 Ring puzzle






Another decomposition, with even shorter ribs.



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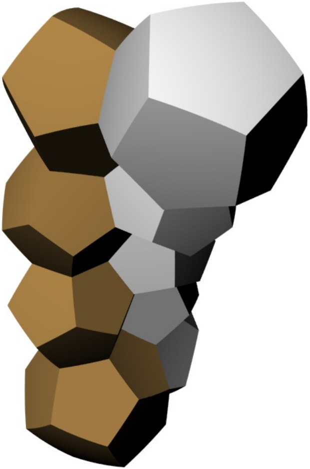

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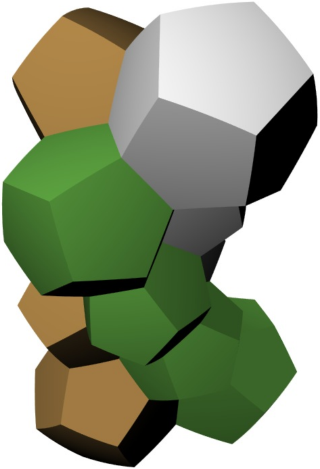

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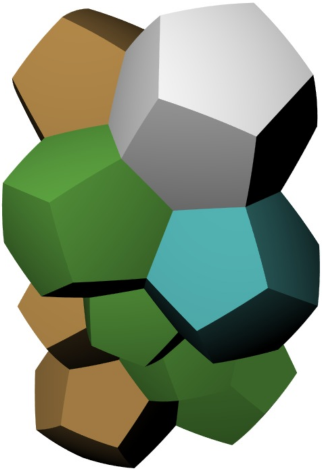

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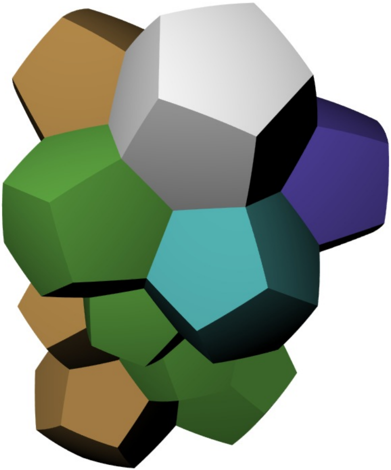

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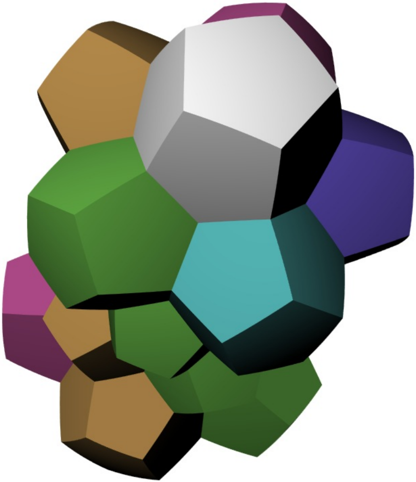

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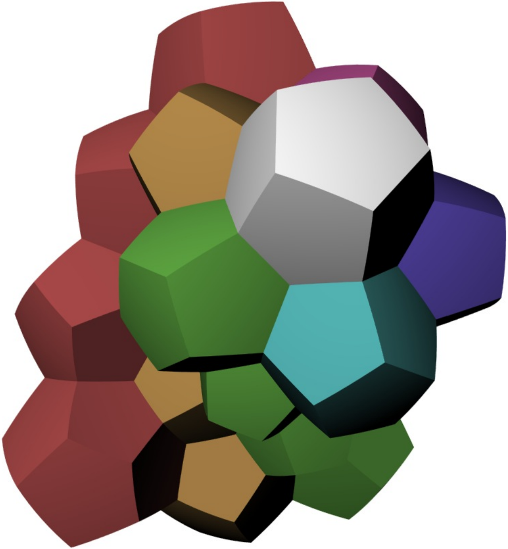

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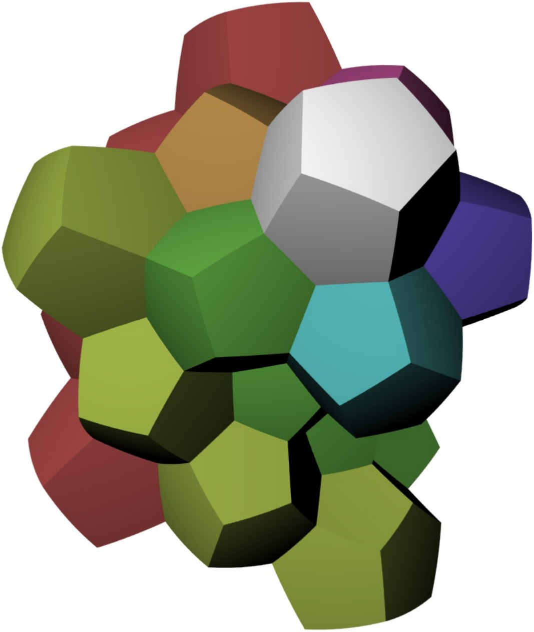

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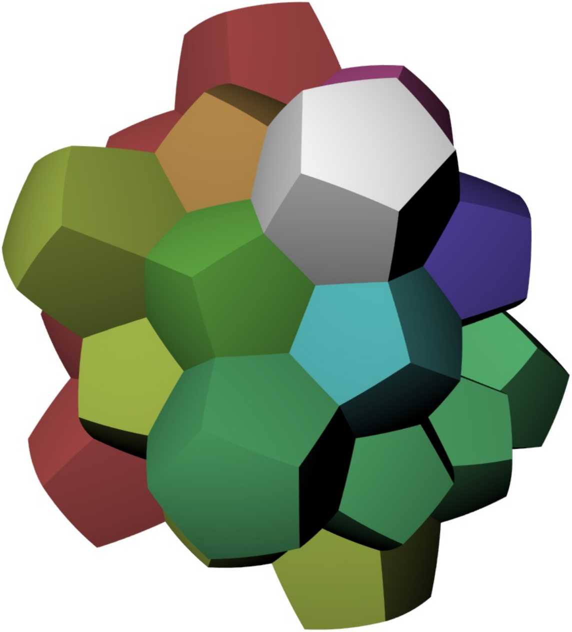

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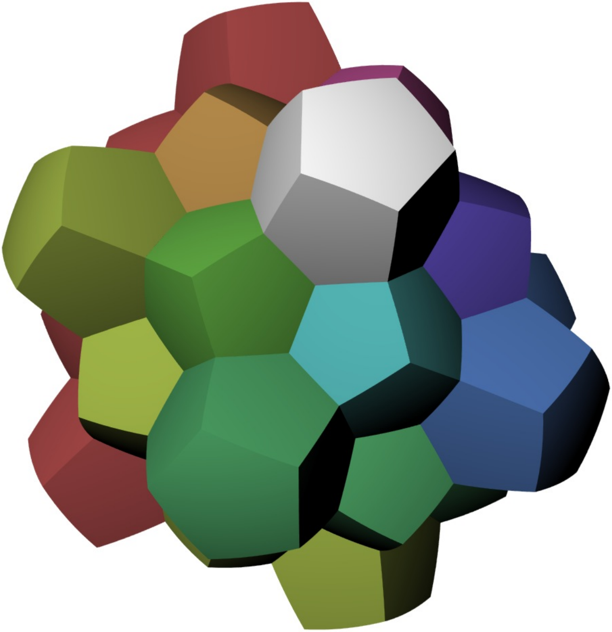

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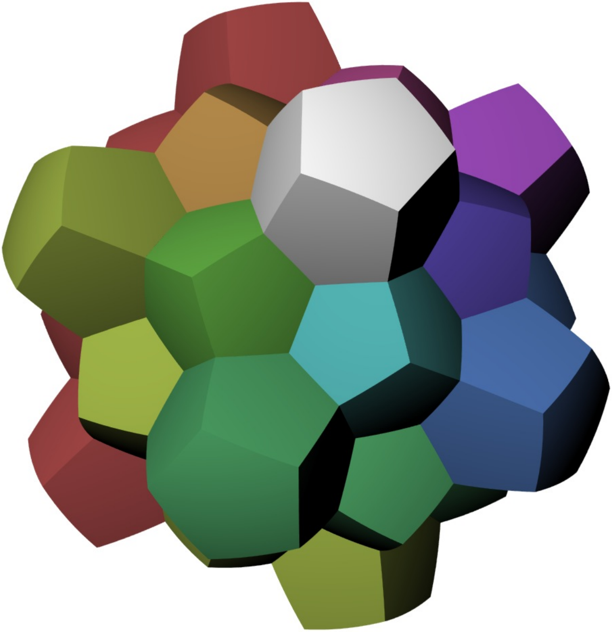
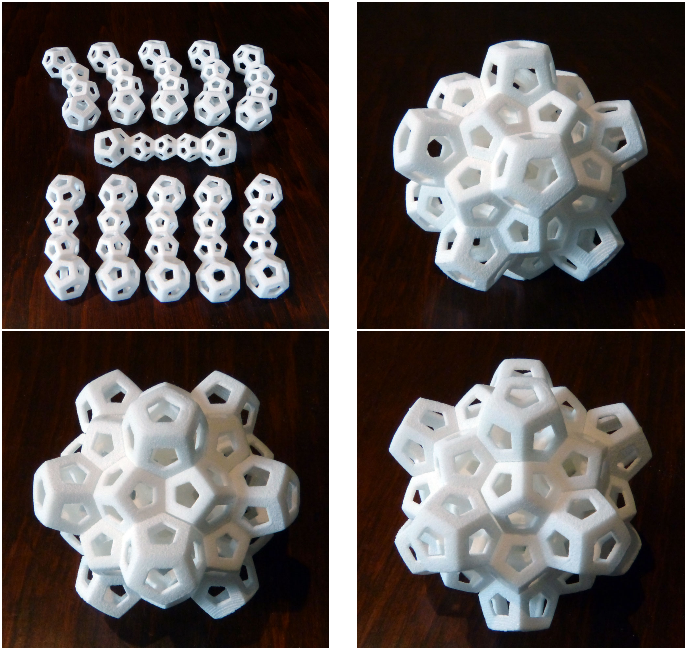
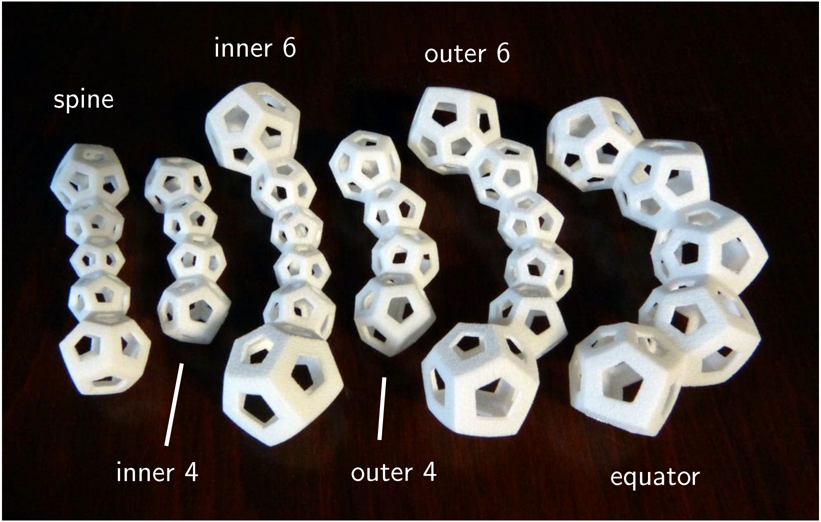


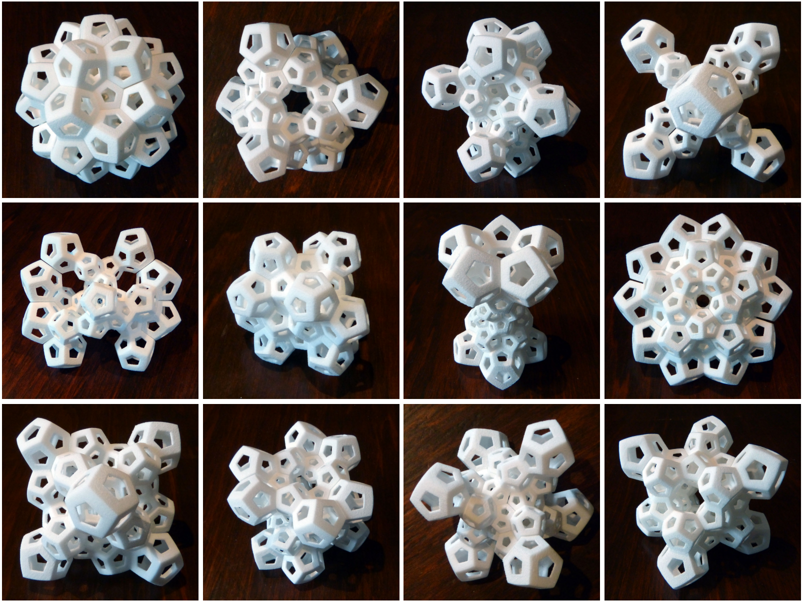
Dc45 Meteor puzzle



Six kinds of ribs



These make many puzzles, which we collectively call Quintessence.



Theorem

- *At most six inner ribs are used in any puzzle.*
- *At most six outer ribs are used in any puzzle.*
- *At most ten inner and outer ribs are used in any puzzle.*

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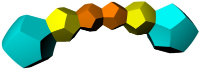

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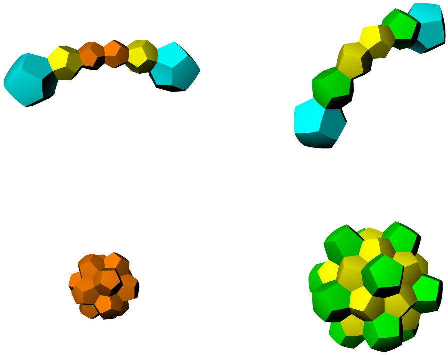

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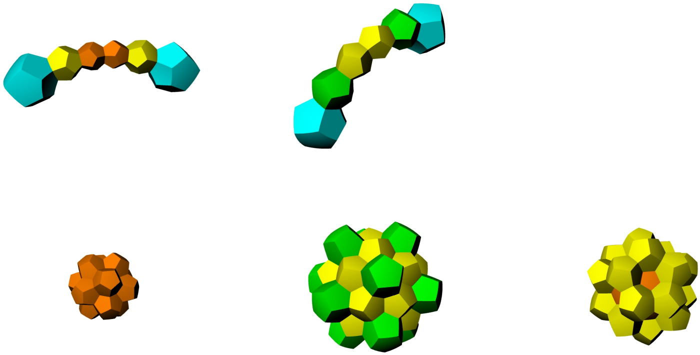


□

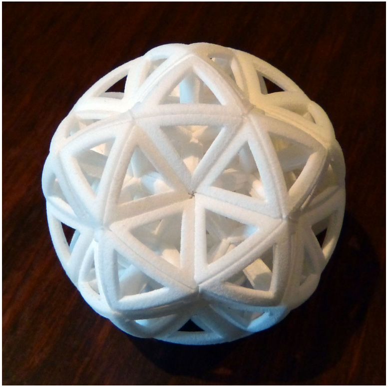
Further possibilities: vertex centered projection

Dv30 Asteroid puzzle

Further possibilities: other polytopes

The 600-cell works, although the ribs now have handedness.

Tv270 Meteor puzzle



Further possibilities: other polytopes

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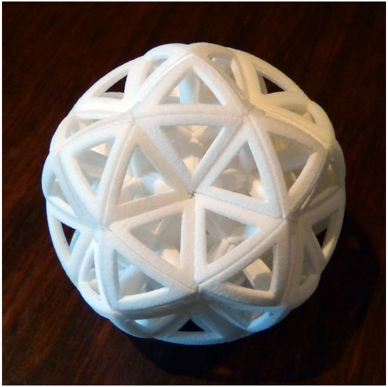
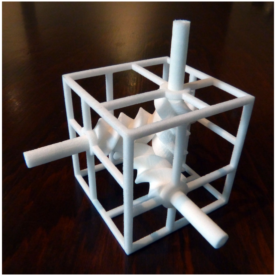
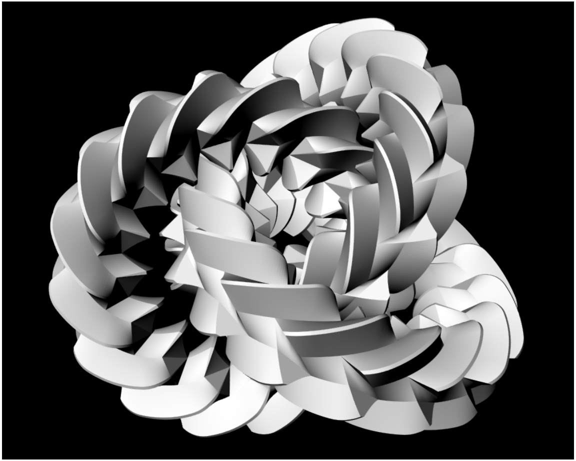
Tv270 Meteor puzzle



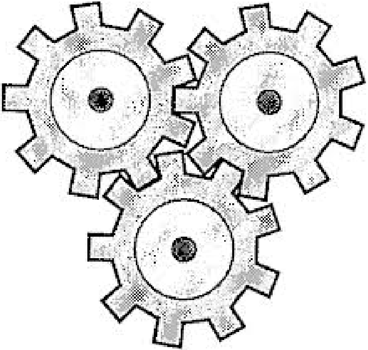
The other regular polytopes seem to have too few cells to make interesting puzzles.



Unlikely gears (joint with Saul Schleimer)

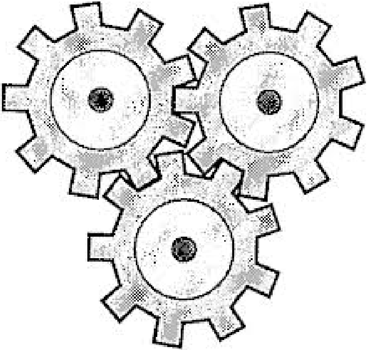


Manchester Metroshuttle advertisement, photo credit: Bill Beaty



Cooperative learning logo from the University of Saskatchewan.

Manchester Metroshuttle advertisement, photo credit: Bill Beaty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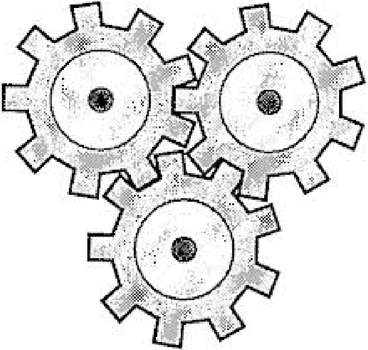
Three pairwise meshing gears are usually frozen...

Manchester Metroshuttle advertisement, photo credit: Bill Beaty


Cooperative learning logo from the University of Saskatchewan.

Three pairwise meshing gears are usually frozen...

A challenge: Find a triple of pairwise meshing gears that moves!

“Umbilic Rolling Link” by Helaman Ferguson.

“Knotted Gear” by Oskar van Deventer.

Our solution is inspired by these “linked” gears.

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Our solution is inspired by these “linked” gears.

They have two “gears”; we want to do the same with three.

But we need to say what “the same” means...

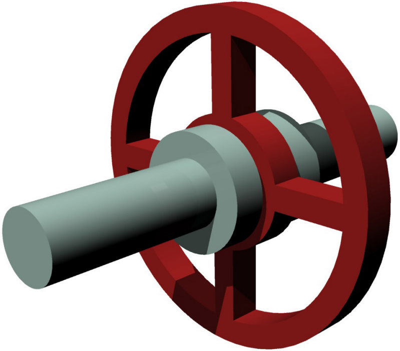
"Umbilic Rolling Link" by Helaman Ferguson.

"Knotted Gear" by Oskar van Deventer.

In both examples the gears are

Tracked: The gears can move relative to each other, but basically in only one way.

“Knotted Gear” by Oskar van Deventer.



A wheel on an axle.

Also they have no “gearbox”; everything is a gear.

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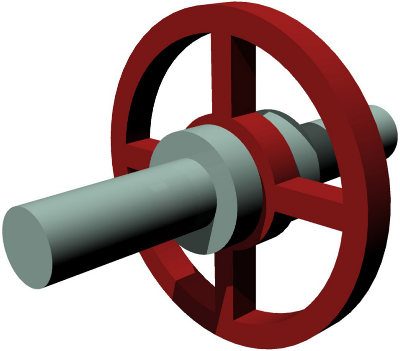

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For a wheel on an axle, the axle acts as a gearbox.

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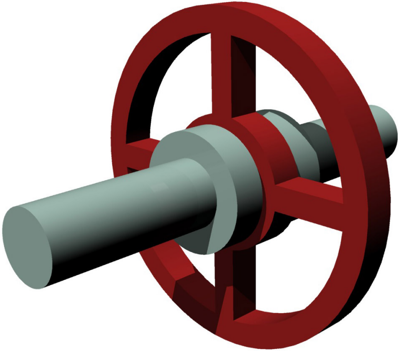

A wheel on an axle.

Also they have no "gearbox"; everything is a gear.

For a wheel on an axle, the axle acts as a gearbox.

We rule this out via

Epicyclic: The movement of one gear in the frame of reference of another is not a rotation.

Axioms

So far we have

- Tracked: The gears in the mechanism can move relative to each other, but basically in only one way.
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To simplify our search, we also impose

- Symmetry: Any of the gears may be taken to any other by a rigid motion preserving the mechanism.

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- Tracked: The gears in the mechanism can move relative to each other, but basically in only one way.
- Epicyclic: The movement of one gear in the frame of reference of another is not a rotation.

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- Symmetry: Any of the gears may be taken to any other by a rigid motion preserving the mechanism.

We want to construct a mechanism with three gears that satisfies these axioms.

If the gears could be separated, there would be too many ways for them to move - violating Tracked. So they have to be linked somehow.

They also have to be rings, that is round, so that when they rotate their shapes don't change too much.

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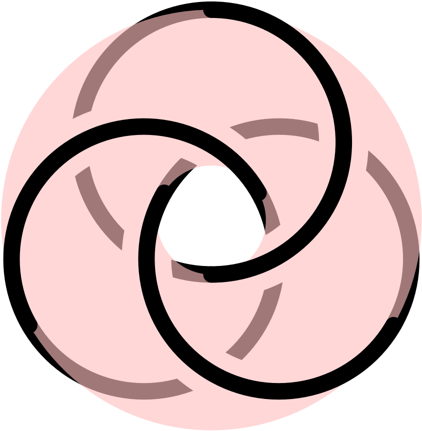

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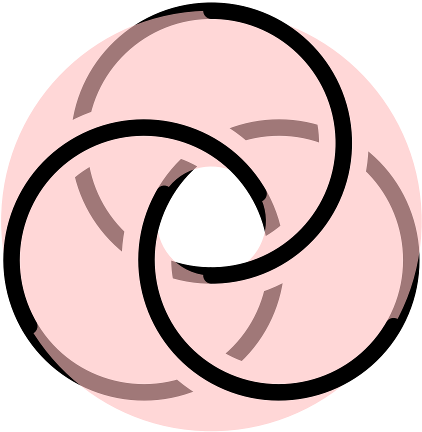
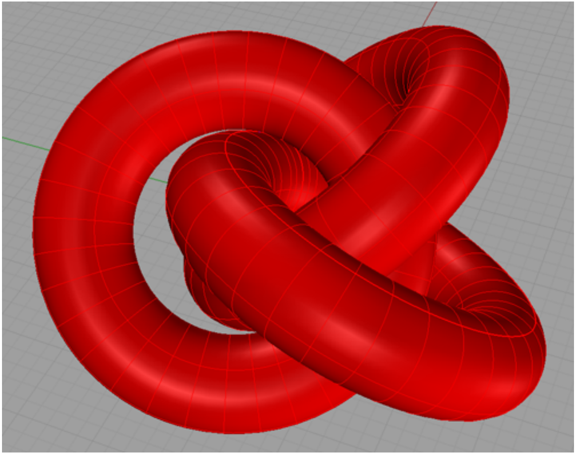


In fact there is only one symmetric way to do this: the three component Hopf link.

Try it! Take three round key-rings and link them all pairwise. Then you will have made the three-component Hopf link. Nothing else is possible!

To satisfy Tracked, the gears must remain in contact. To enforce this, we gradually inflate the three rings, letting them bump against each other while preserving the 3-fold symmetry, until they reach maximum thickness.



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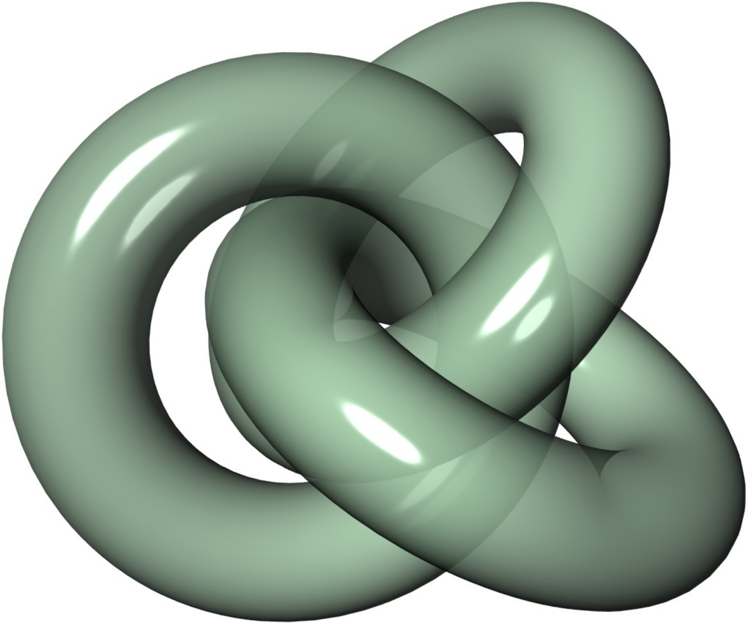
To stop them moving out of place, we design gear teeth.

To design the teeth, we investigate how the rings touch each other.



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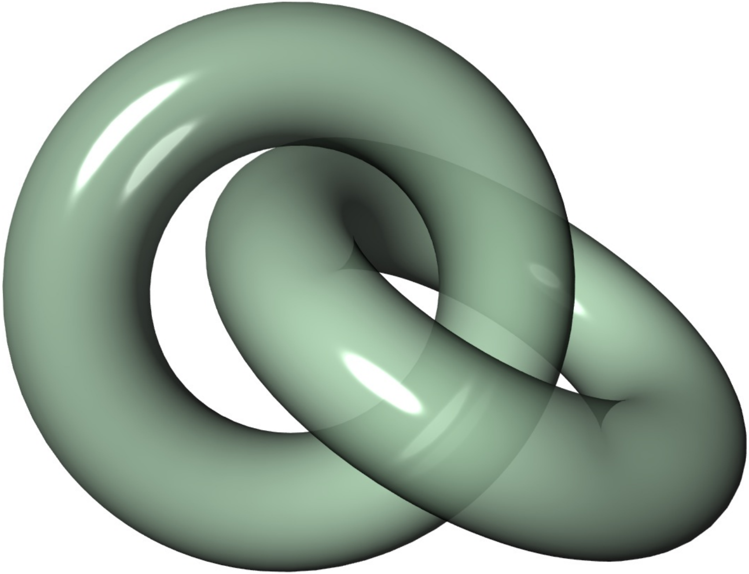

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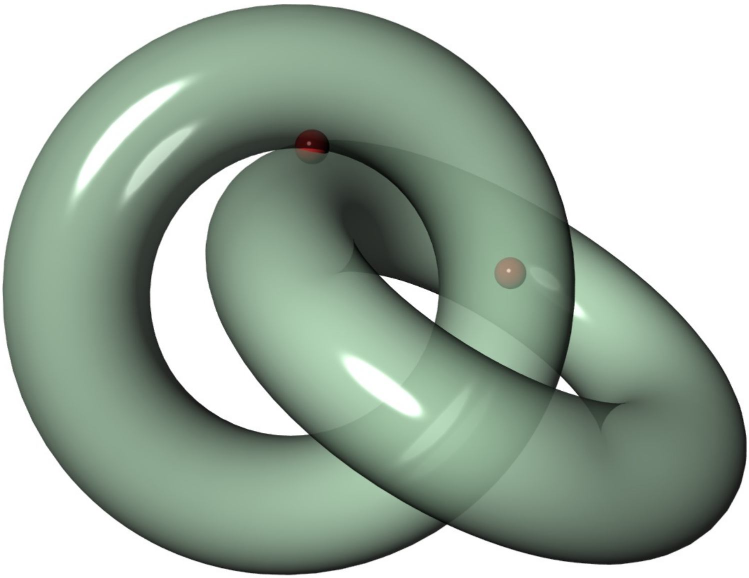

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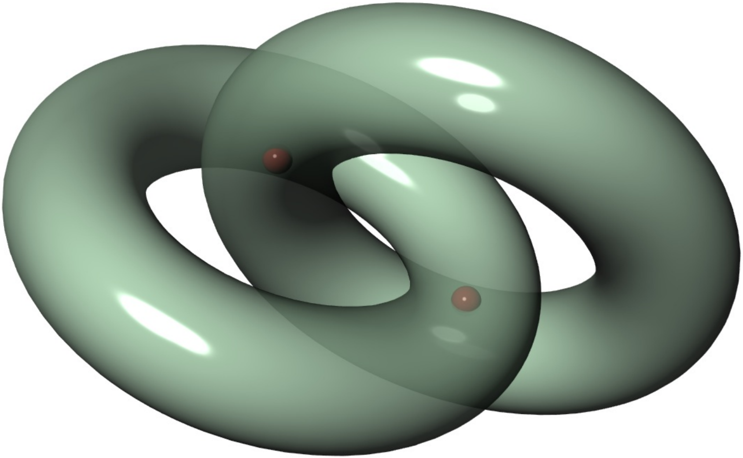

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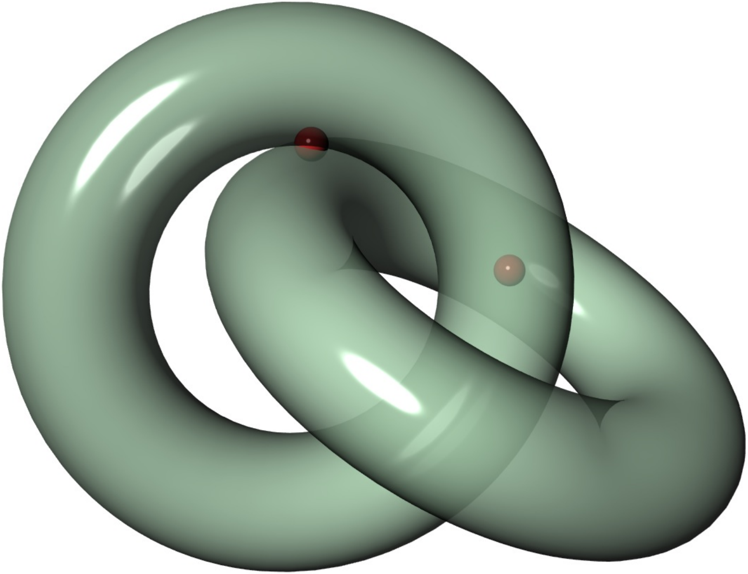

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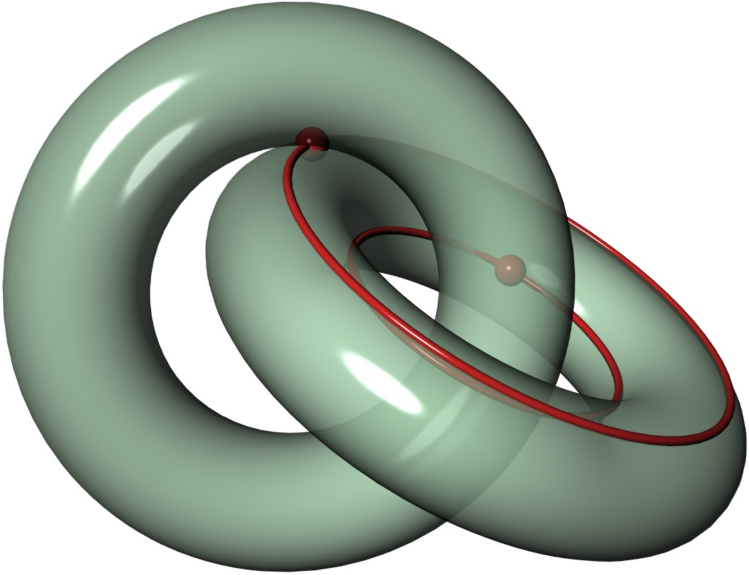

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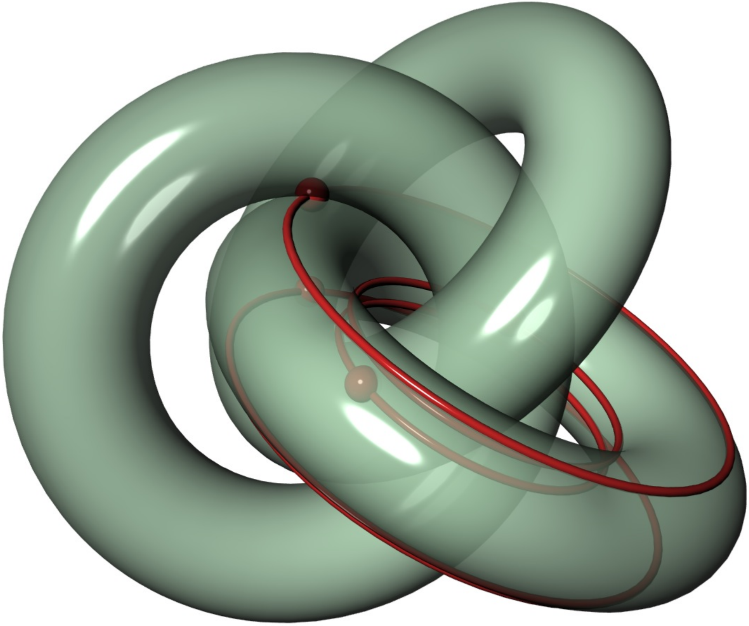

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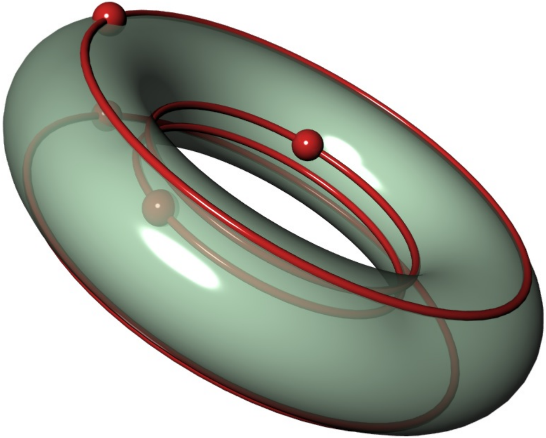

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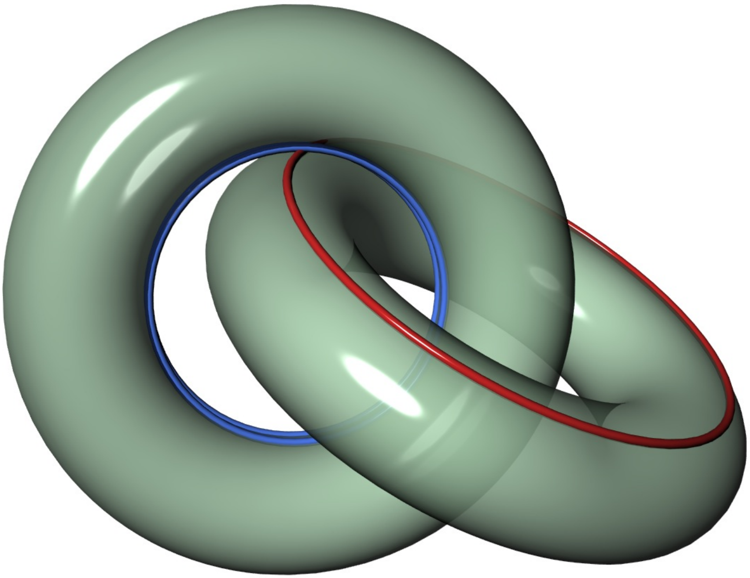
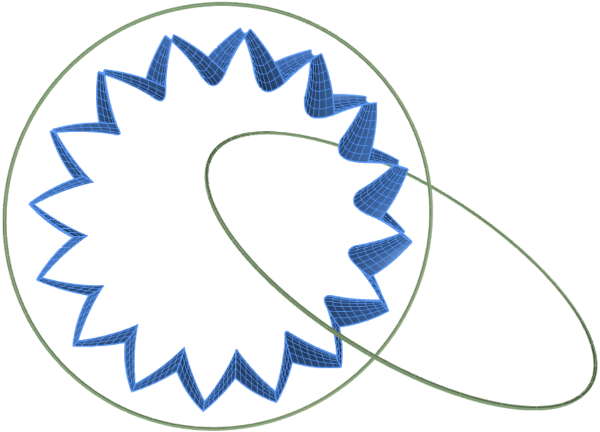
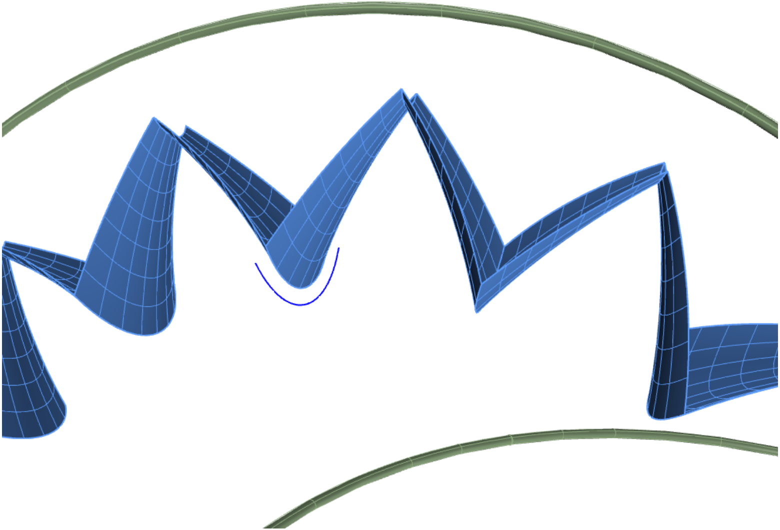


The “inner” teeth are the images of planes in toroidal coordinates.



The “inner” teeth are the images of planes in toroidal coordinates. The “outer” teeth are determined by “carving”.



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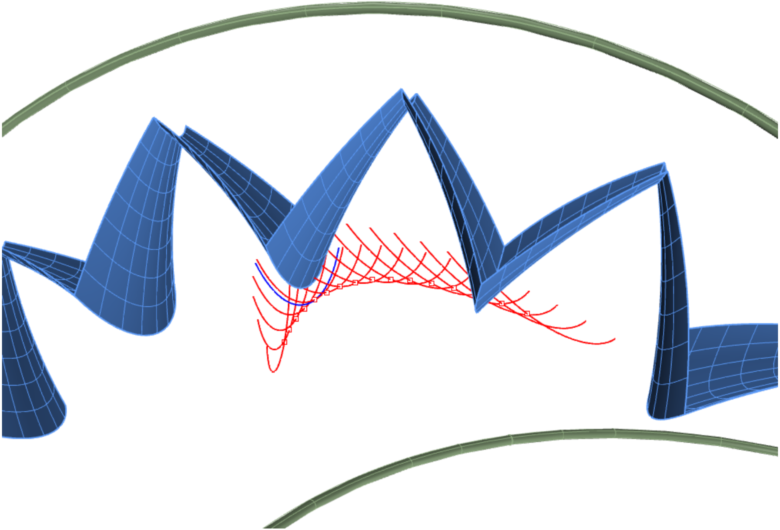

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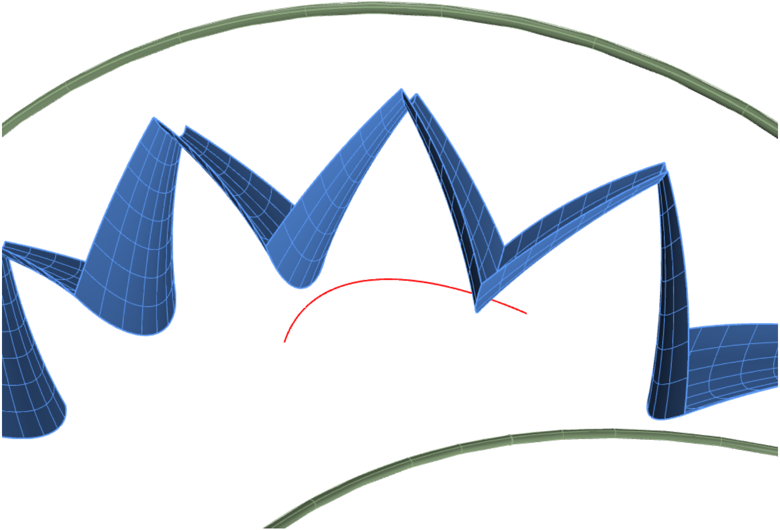

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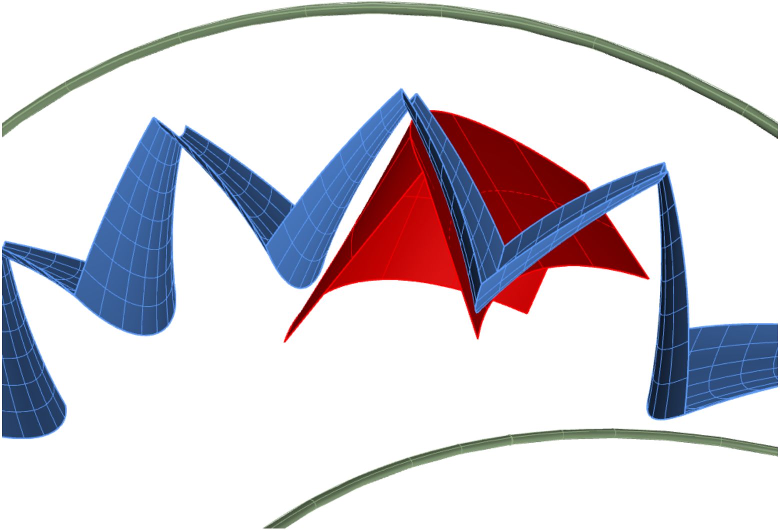

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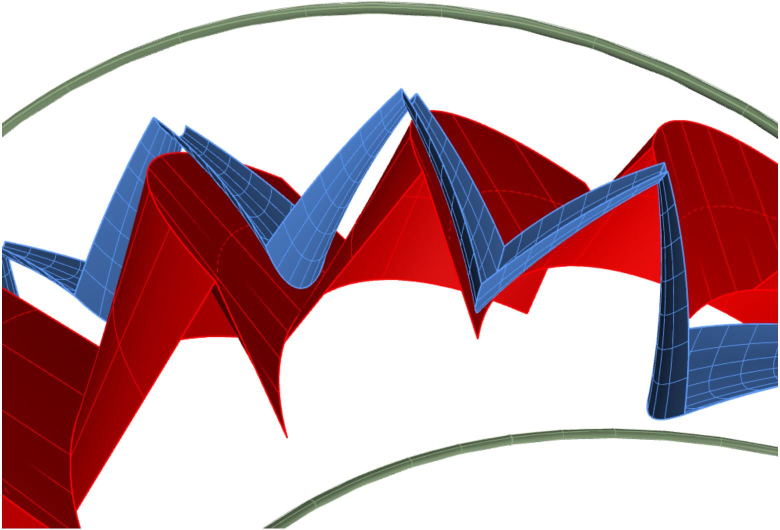

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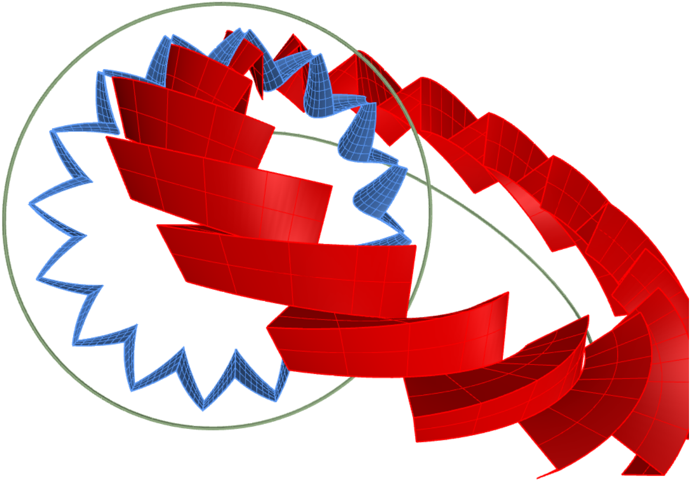
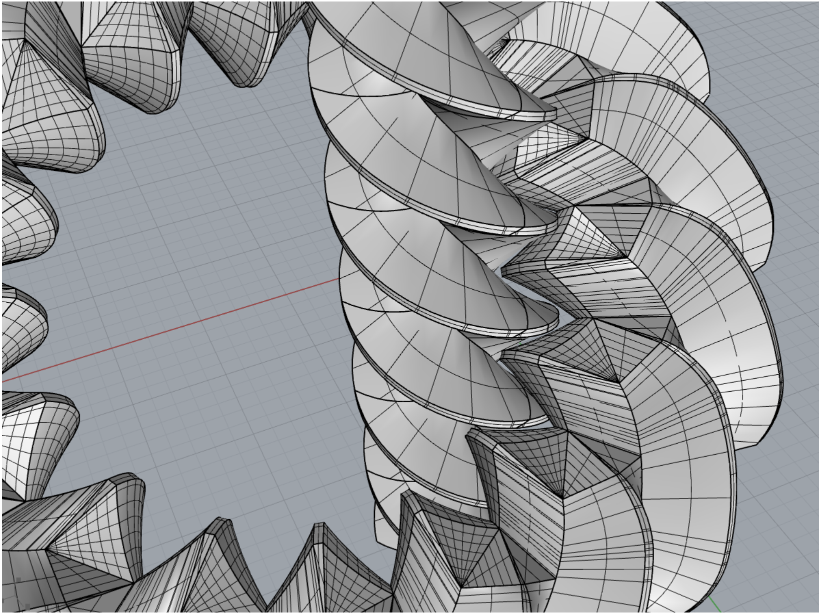
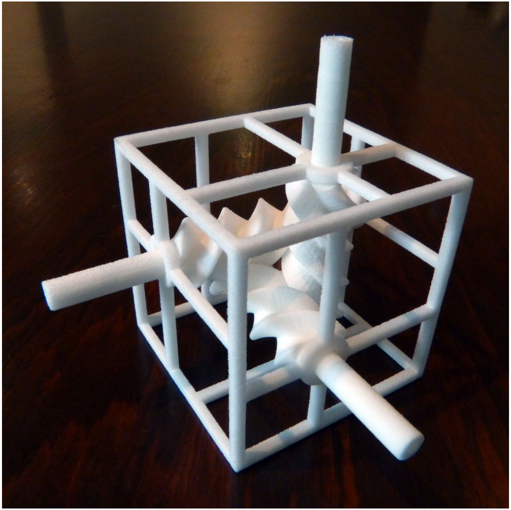


The gears can be powered by a central helical axle.



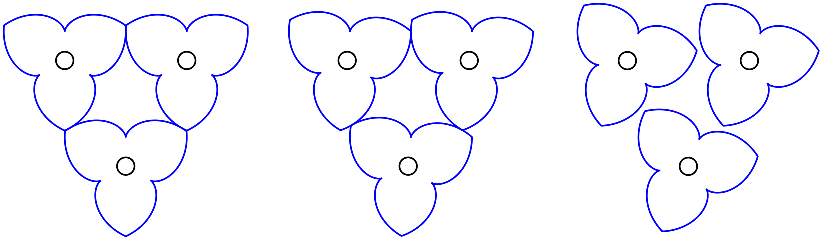
The axle is connected to a motor in the base. Thanks to Adrian Goldwaser for initial prototyping, and to Stuart Young for much more prototyping and construction of the base.

Alternate non-frozen arrangements of three gears



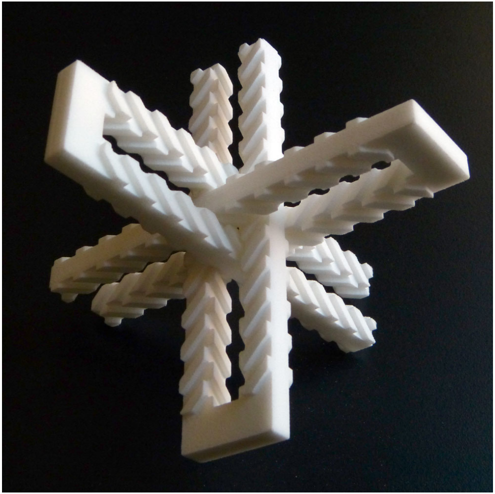
Three helical gears can also pairwise mesh, and they can all move.

Alternate non-frozen arrangements of three gears



It can even be done with gears with parallel axes!

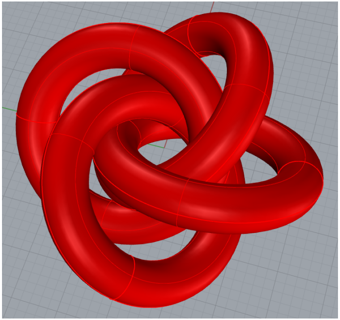
Alternate non-frozen arrangements of three gears



A similar mechanism with three “racks” - objects with gear teeth that move linearly rather than by rotating.

Future directions

- Do the same with the 4-component Hopf link.
- Other configurations of rings?



More generally, we are exploring mechanisms that move in unusual ways.

Thanks!

segerman.org

math.okstate.edu/∼segerman/

youtube.com/user/henryseg

shapeways.com/shops/henryseg