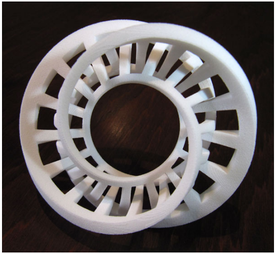
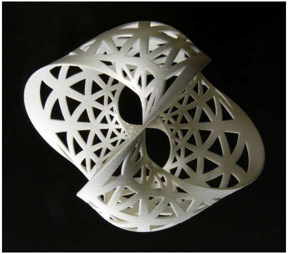
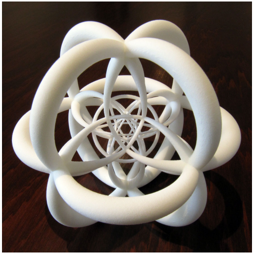
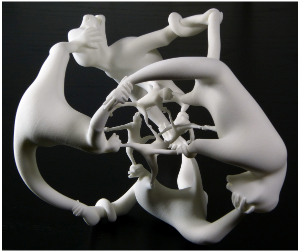
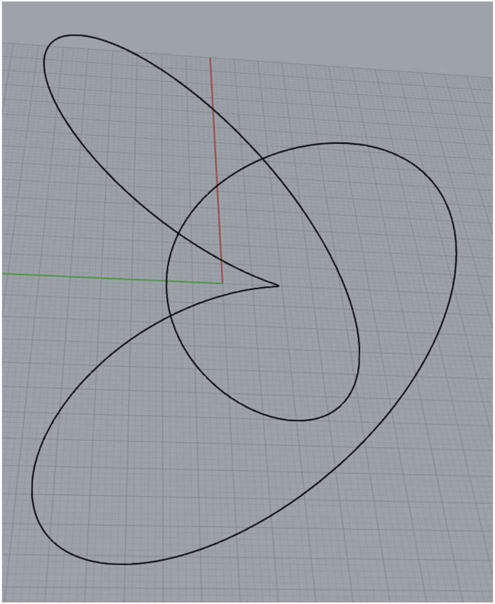




Henry Segerman
Oklahoma State University

Design of 3D printed mathematical art



Turning mathematics into sculpture: thickening



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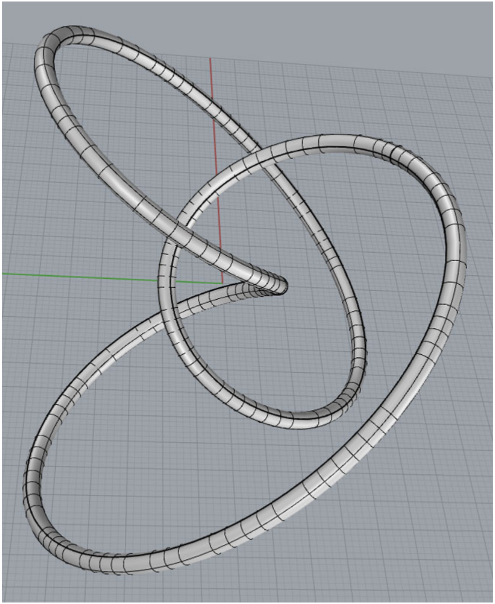

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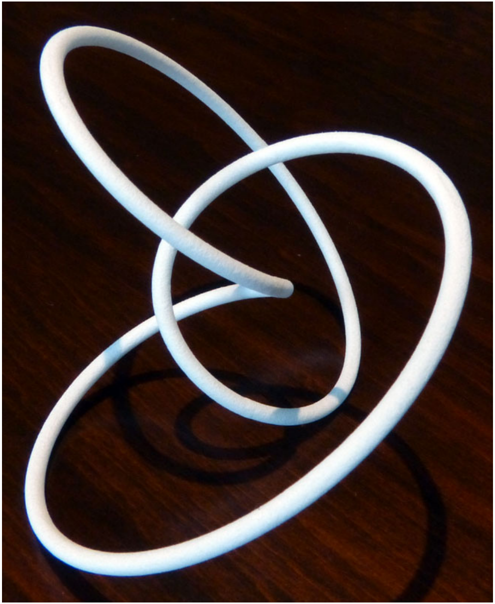

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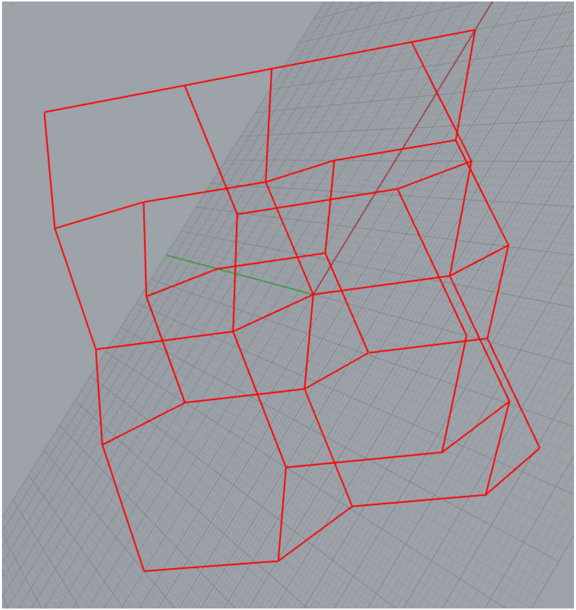

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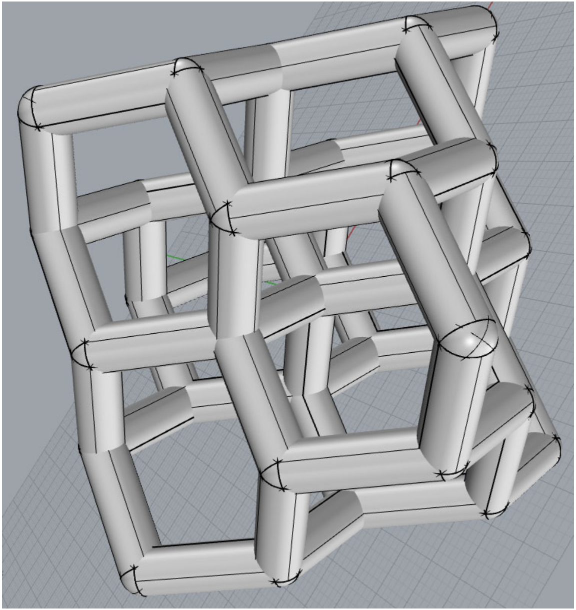

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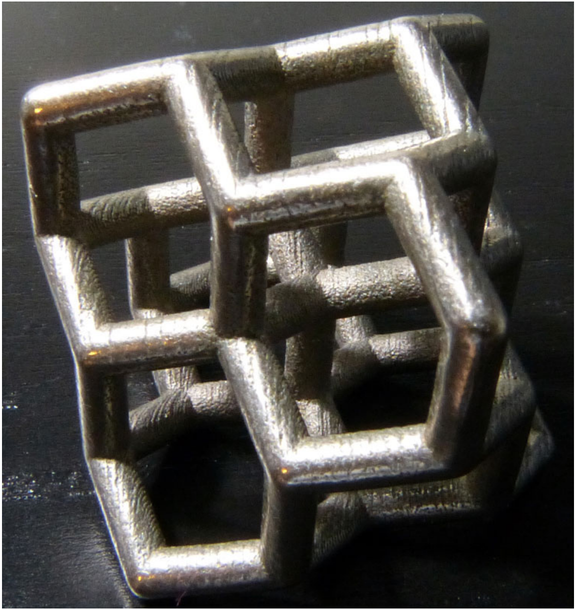

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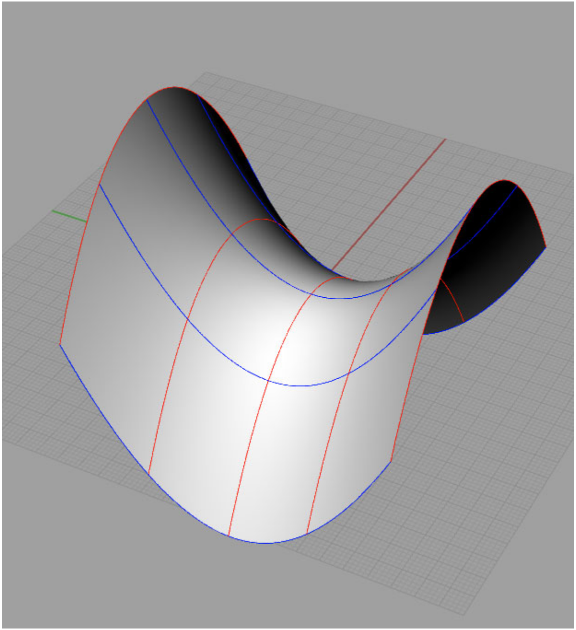

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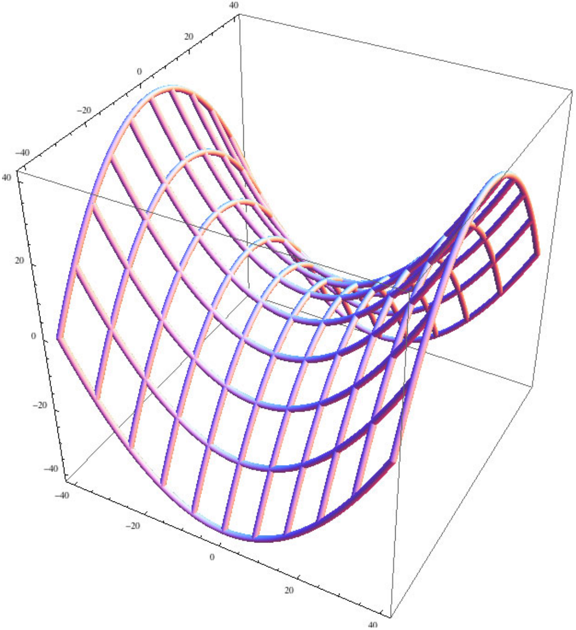

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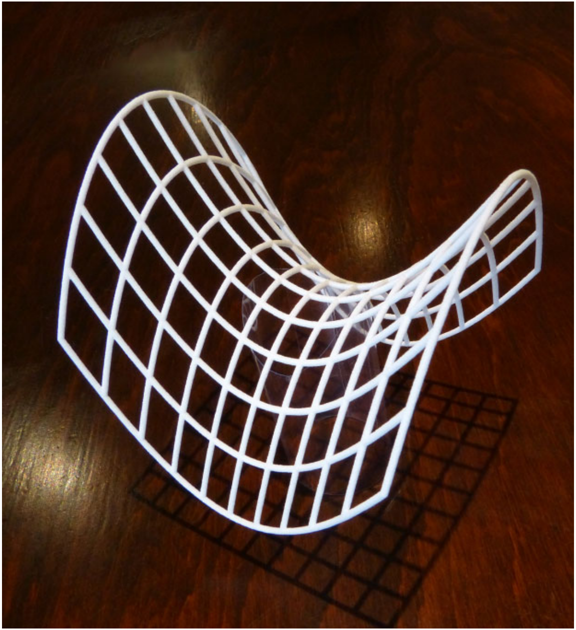
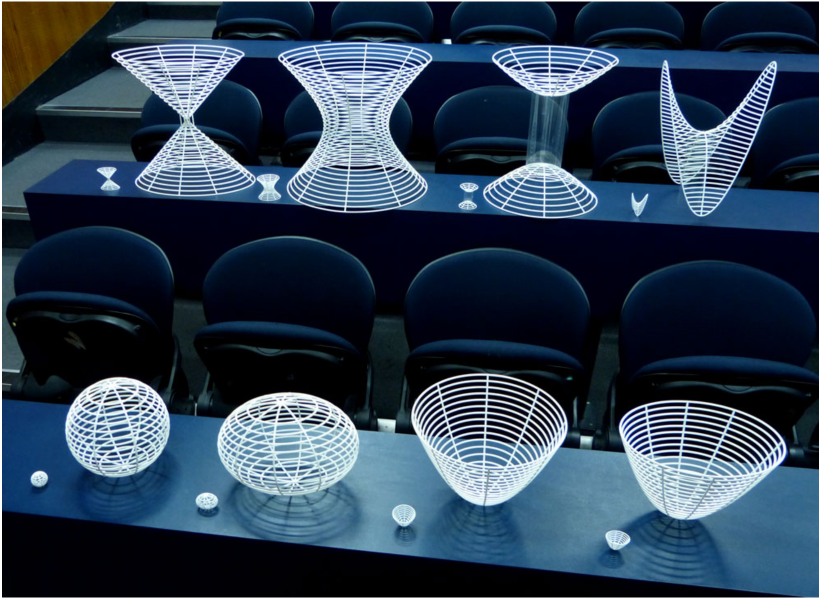


Quadric surfaces



Quadric surfaces

So far, we have looked at “algebraic” objects, where the geometry is precisely defined, and the only question is in thickening.

For topological objects, such as the knot, we also have to choose the geometry...

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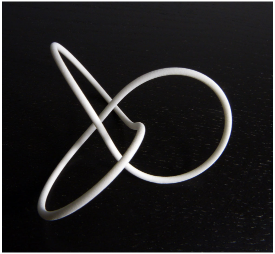
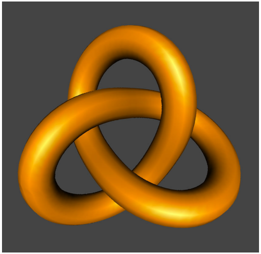
For topological objects, such as the knot, we also have to choose the geometry... (or not, if we print a flexible design!)

Strategies for choosing geometry

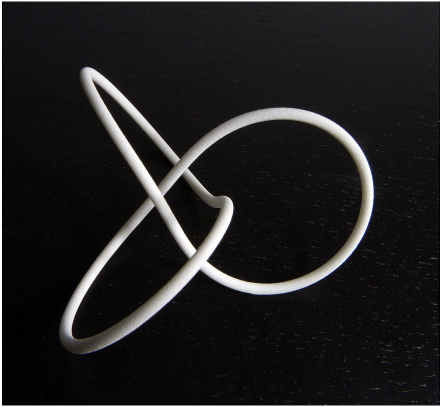
1. *Manual* - using whatever design software is available to build the object by hand.
2. *Parametric/implicit* - generating the desired geometry using a parametrisation or implicit description of the object.
3. *Iterative* - numerically solving an optimisation problem.

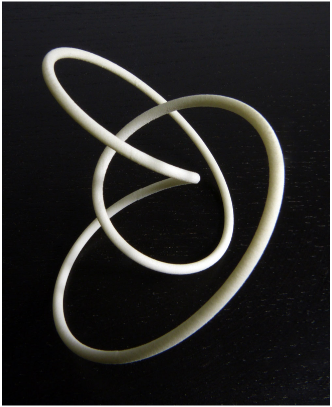




Manual trefoil

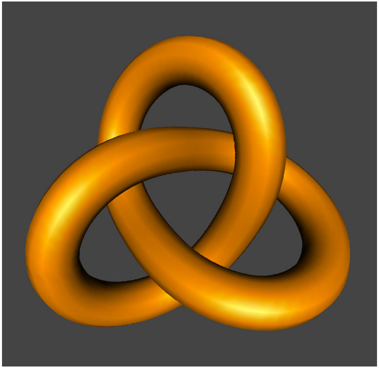
Cubic Trefoil Knot Pendant by Vertigo Polka

Parametric trefoils



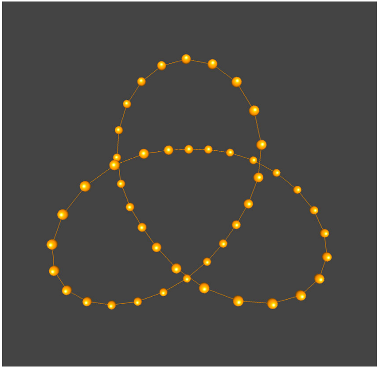


Iterative trefoils



KnotPlot by Robert Scharein

Iterative trefoils



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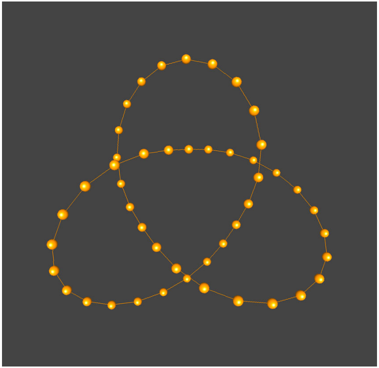
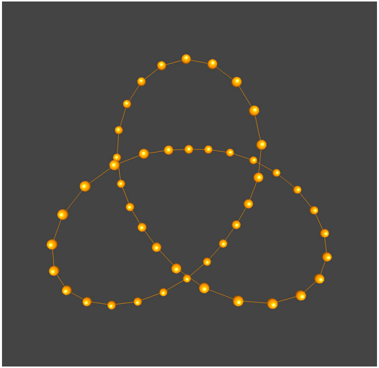


KnotPlot by Robert Scharein

Minimal rope length by Jason Cantarella

Iterative trefoils



KnotPlot by Robert Scharein

Minimal rope length by Jason Cantarella

Even more trefoils, by Laura Taalman

Aesthetic choices

How should we choose a geometrical representation of a topological object?

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2. Be as faithful as possible.

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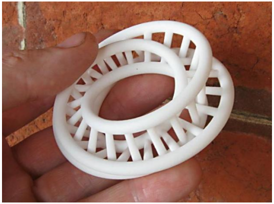
What we really want is a canonical geometric structure on a topological object.

$$\text{mathematics} \xrightarrow[\text{structure}]{\text{canonical}} \text{computer model} \xrightarrow[\text{printing}]{\text{3D}} \text{physical object}$$

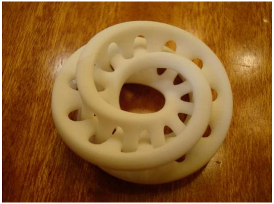
Example: Möbius ladders

Mobius Bangle by Denzyl Basterfield

Square Mobius Ribbed by Vertigo Polka



Double Trouble by Tones3-D



linked mobius by Zorink

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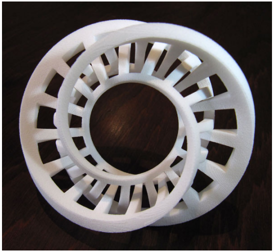

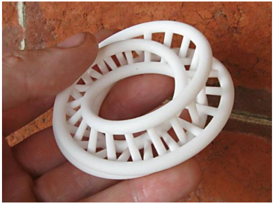

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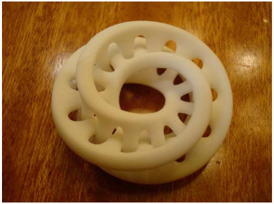

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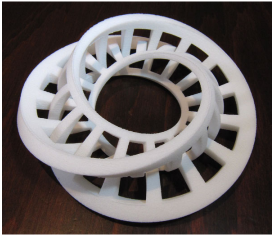
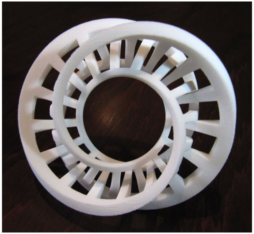
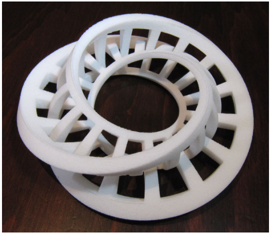
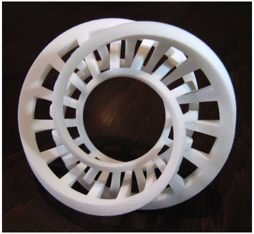
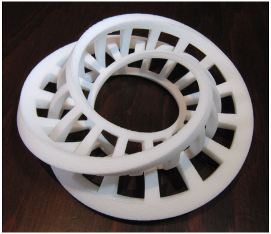


Interlocking Möbius Ladders by Schleimer and Segerman

In our version, the rungs meet the poles at right angles.

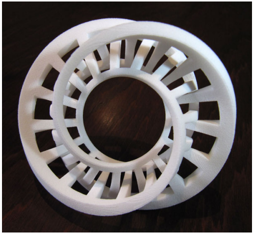
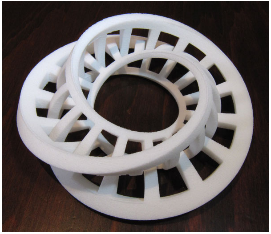



In our version, the rungs meet the poles at right angles.

It is parametrised in
$S^3 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 = 1\}$ by

$$f(\theta, \tau) = (\cos(\theta)\cos(\tau), \cos(\theta)\sin(\tau), \sin(\theta)\cos(\tau/2), \sin(\theta)\sin(\tau/2))$$

for θ in a small interval and $0 \leq \tau < 2\pi$.

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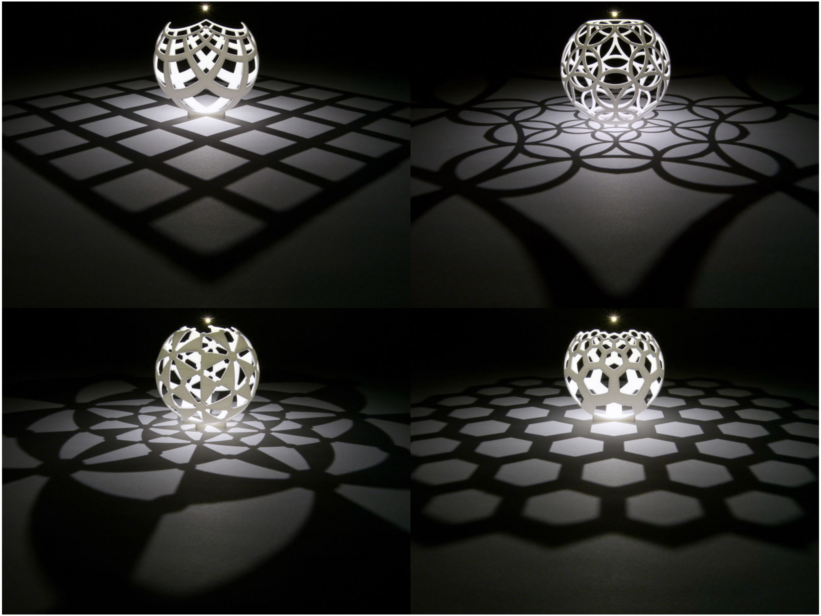
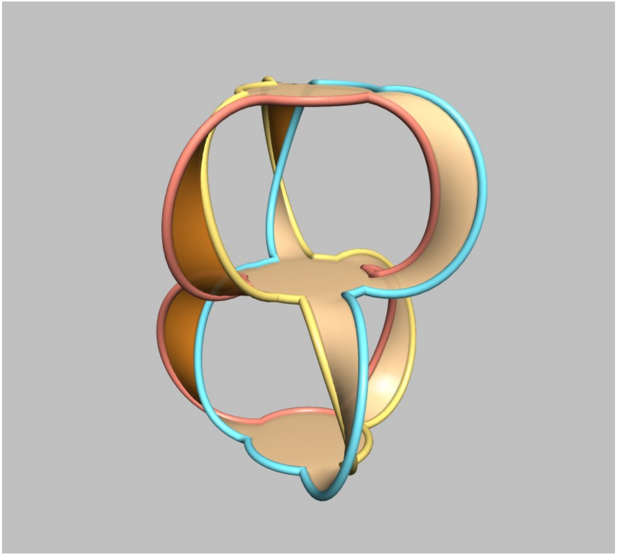
for θ in a small interval and $0 \leq \tau < 2\pi$.

Then $f_\theta \perp f_\tau$, and their images after stereographic projection from S^3 to $\mathbb{R}^3$ are also perpendicular, since stereographic projection is conformal.

Stereographic projection

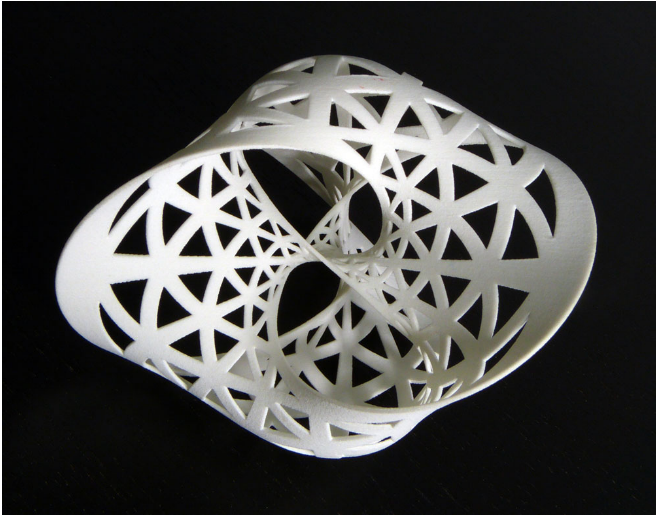


Iterative Seifert surfaces

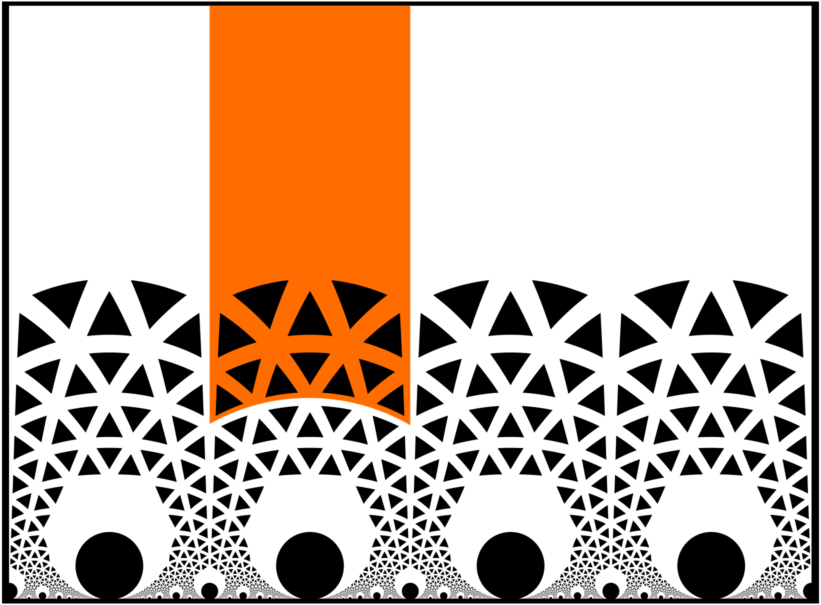


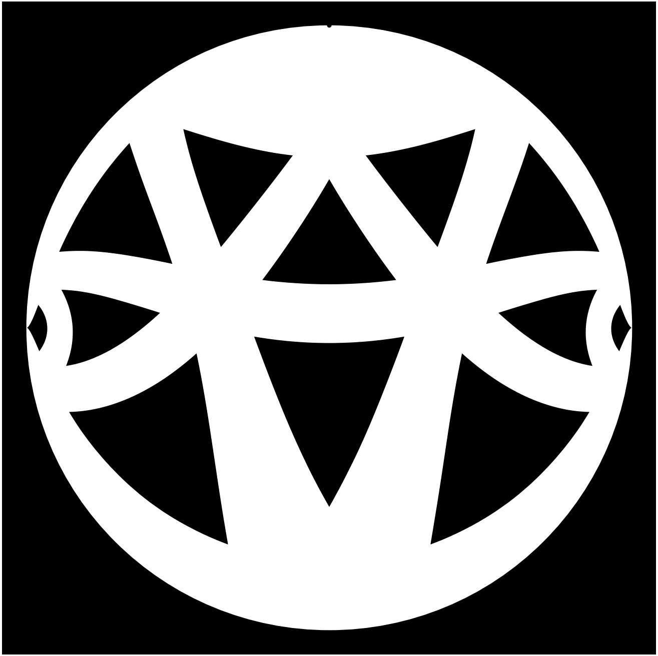
SeifertView, by Jarke J. van Wijk, `http://www.win.tue.nl/~vanwijk/seifertview/`.

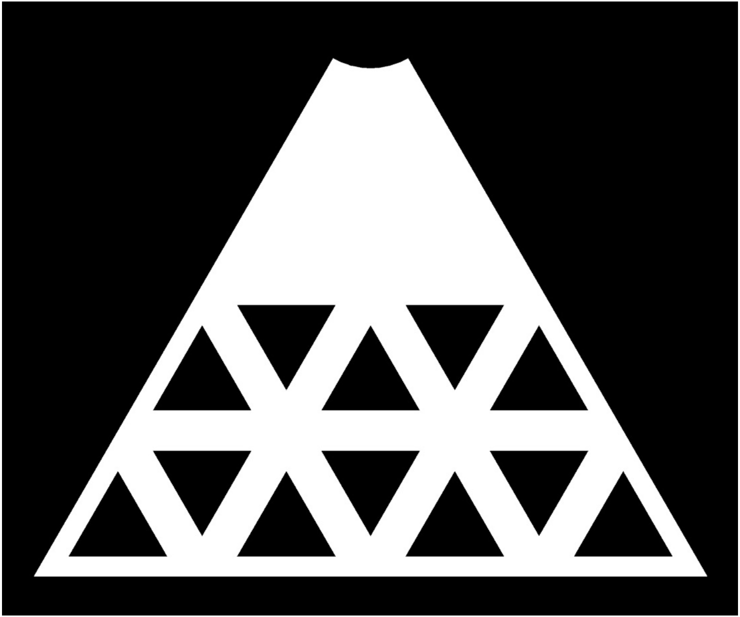
Parametric Seifert surfaces via Milnor fibers (joint work with Saul Schleimer)



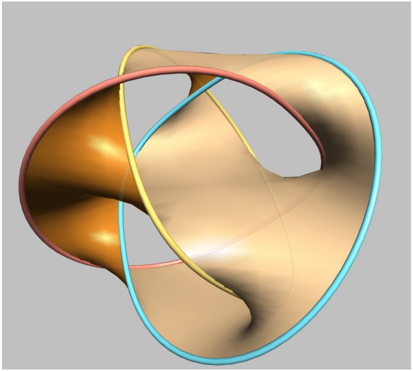
$w^3 + z^3 = 0$ $\qquad$ $\arg(w^3 + z^3) = 0$

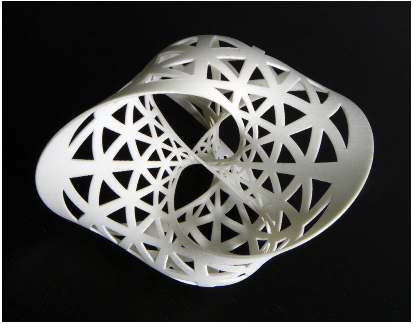






Comparison





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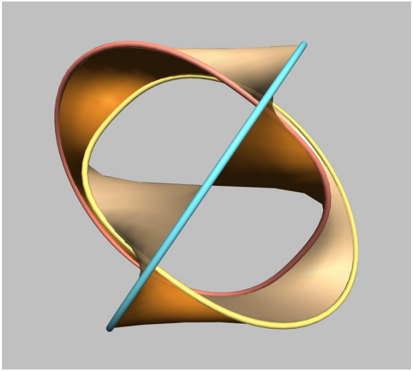

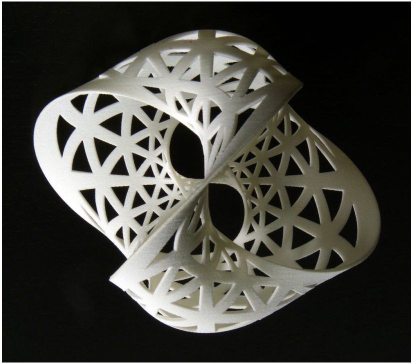

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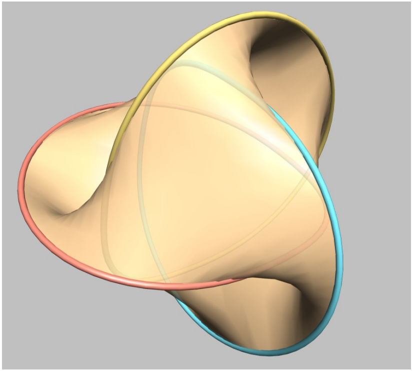

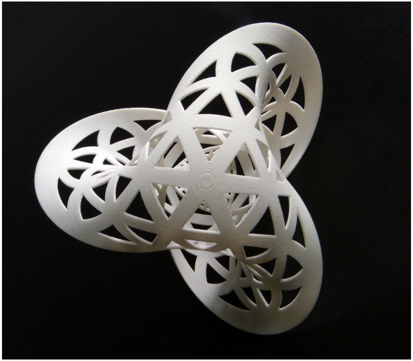

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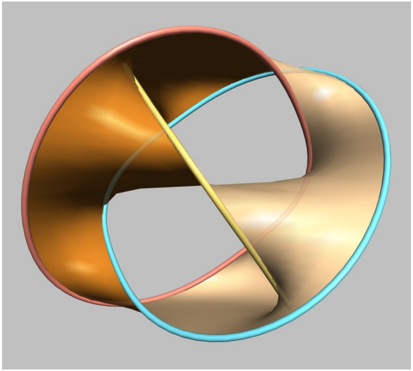

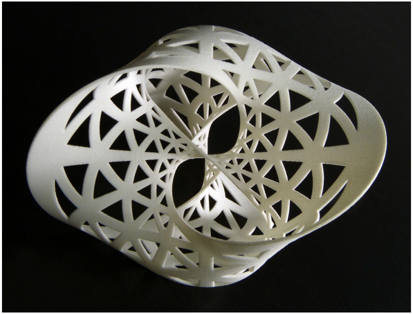

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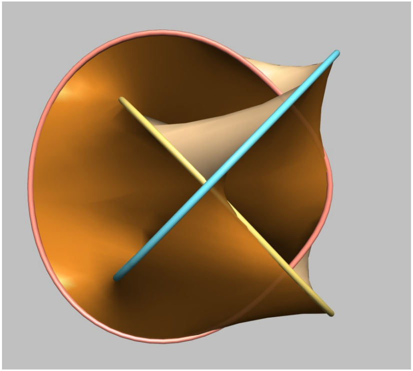
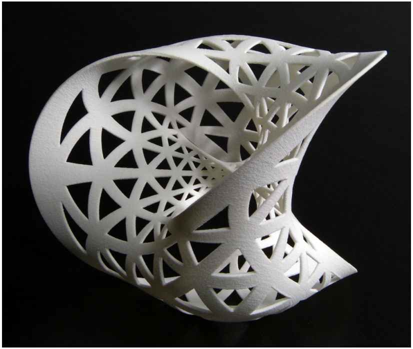
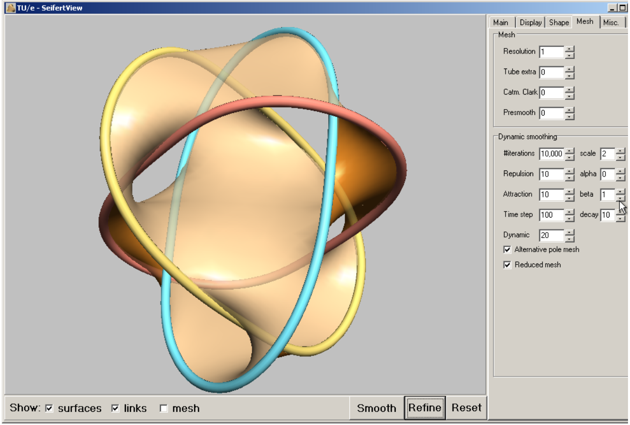




Borromean rings



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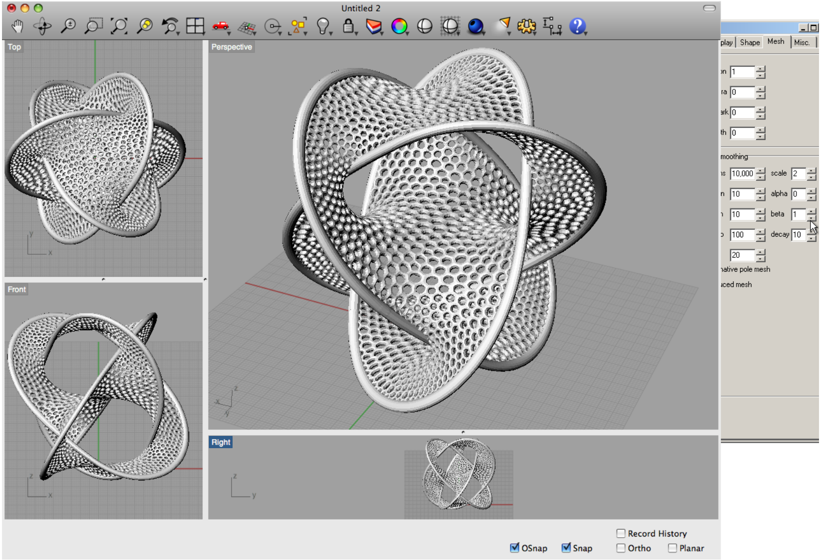

Borromean rings

Bathsheba Sculpture

Borromean Rings

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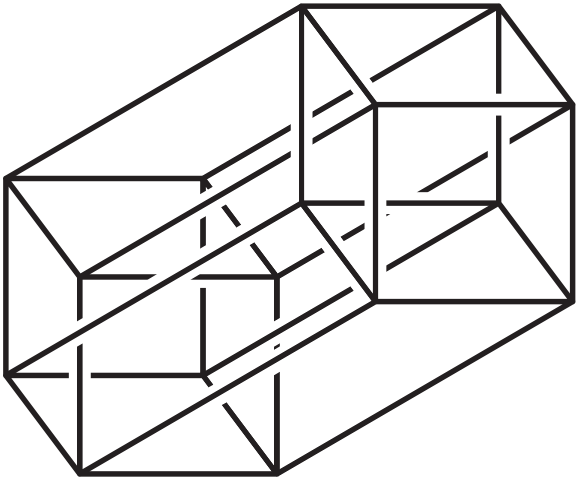
Click to rotate

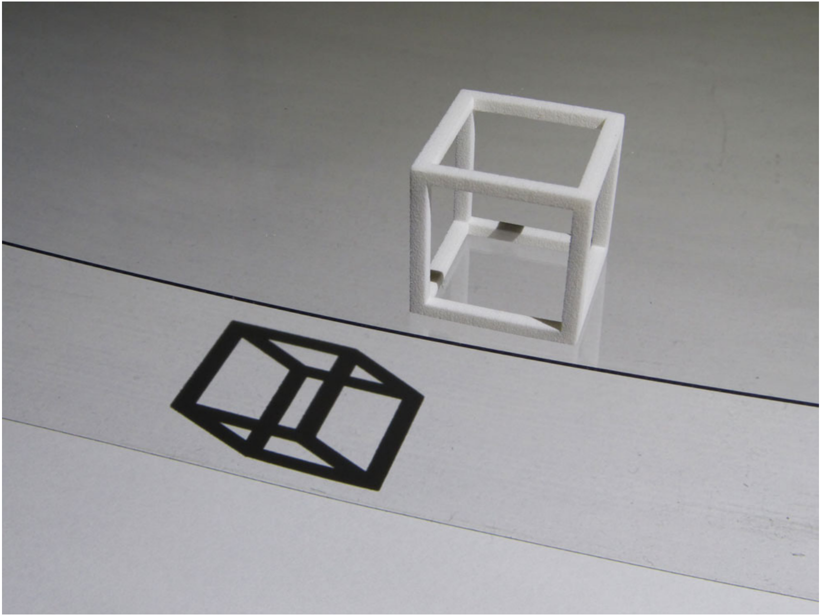
This is one of a delightful class of objects known as Seifert surfaces. Every knot and link (in matl
are closed loops, links are assemblages of knots) has a continuous surface which it is the edge of
introduction to these surfaces, along with free software to generate them, are at the SeifertView

These surfaces are often beautiful, especially for symmetrical knots and links, and here I've produ
sweeter ones. This surface has three edges, each a simple closed loop, which are locked togethe
form called the Borromean Rings. Named after its use in an Italian coat of arms, these three ring
together inextricably although no two of them are linked. Their Seifert surface twists through the

Example: Hypercube



Parallel projection of a cube



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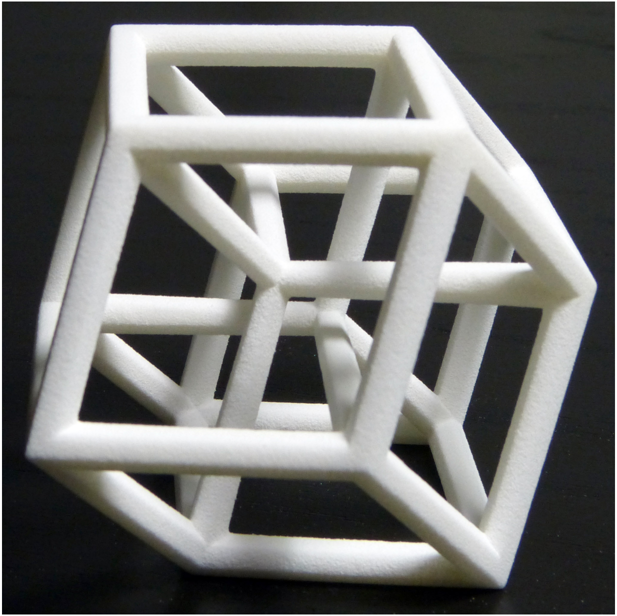
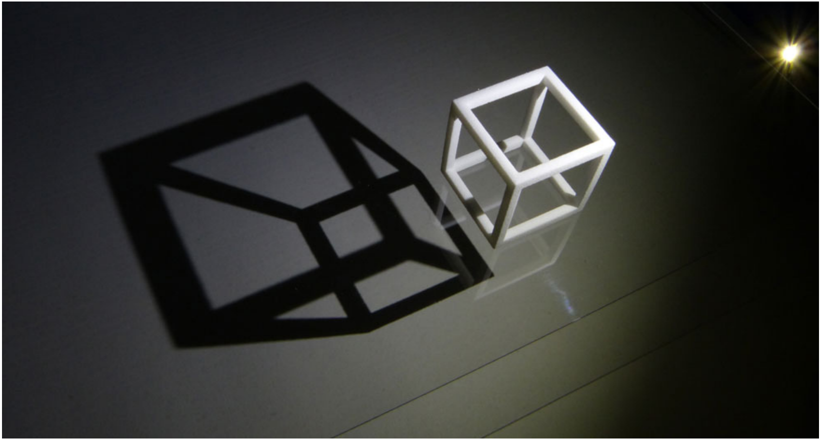
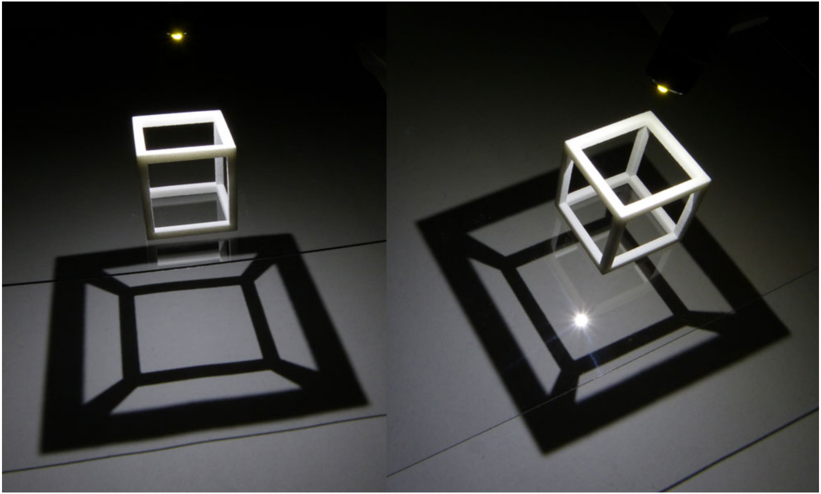


Hypercube B by Bathsheba Grossman.

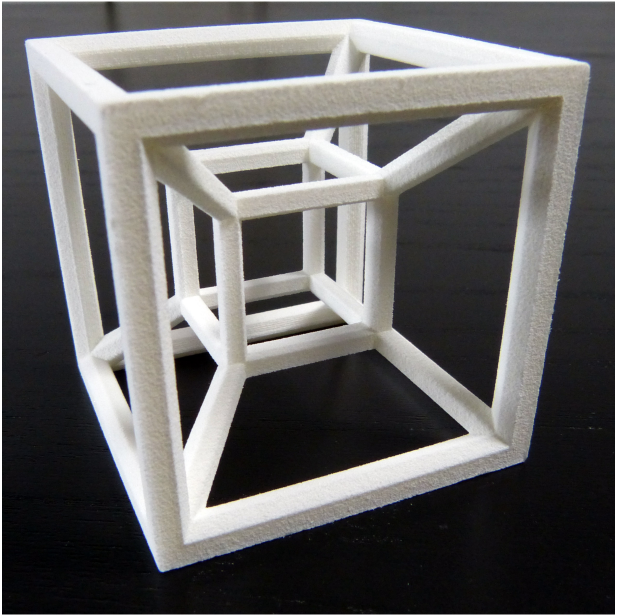
Perspective projection of a cube



Perspective projection of a cube

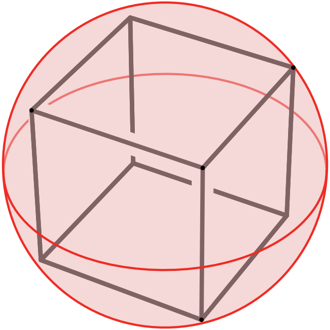


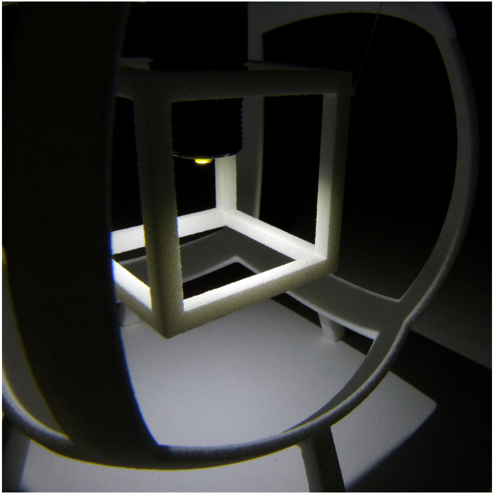
Perspective projection of a hypercube



Hypercube A by Bathsheba Grossman.

A better method: radially project the cube to the sphere...





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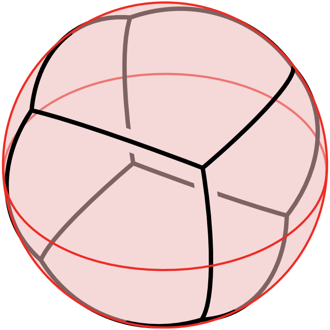
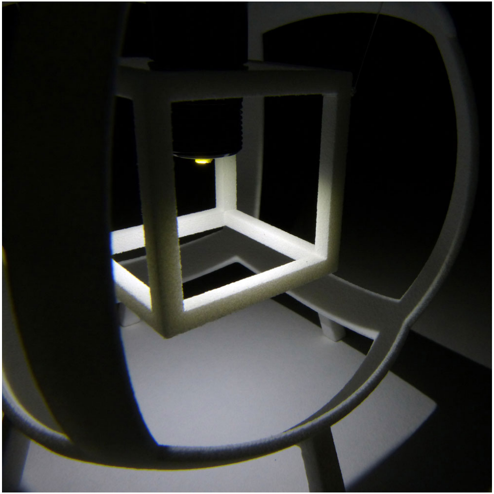
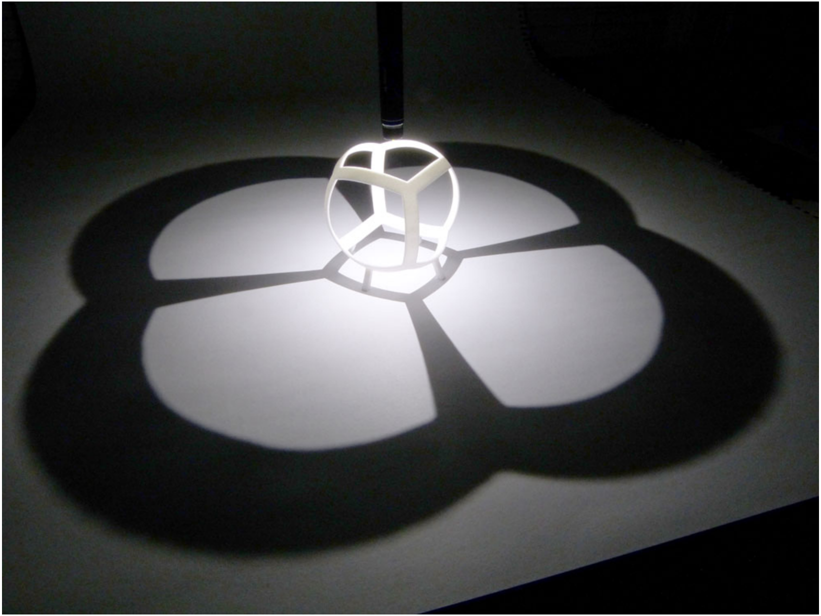
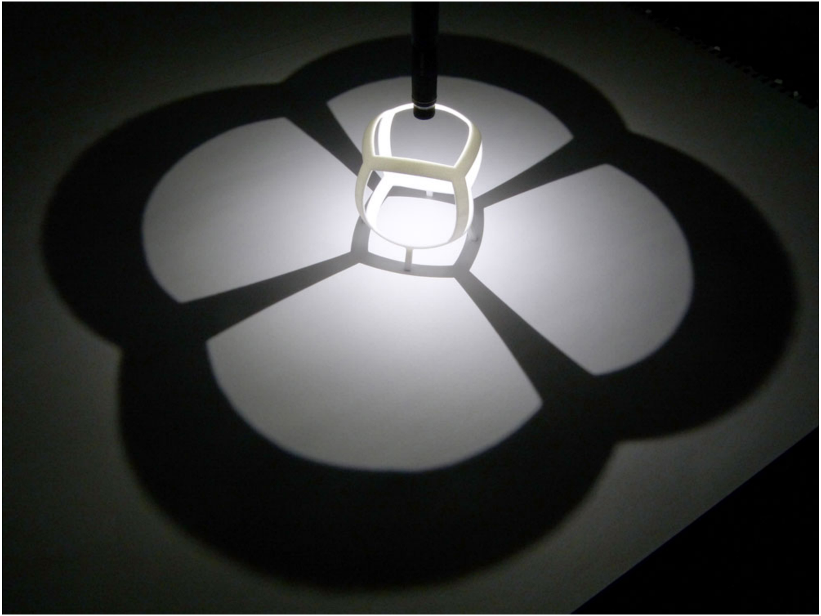




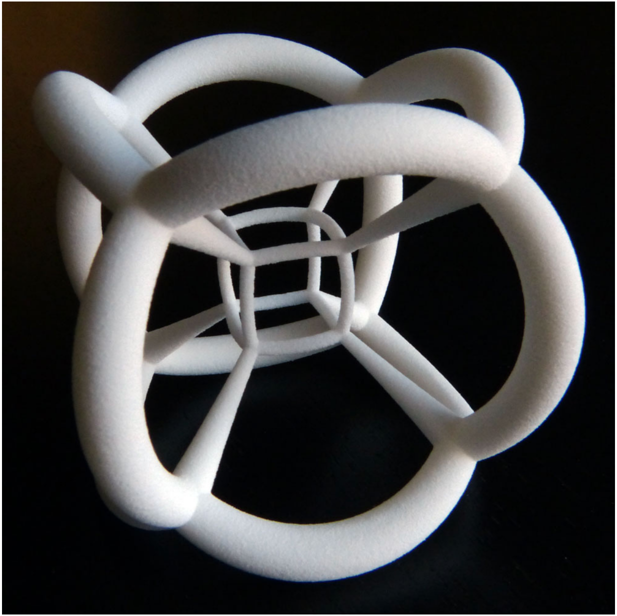
...then stereographically project to the plane



...then stereographically project to the plane



Do the same thing one dimension up for a hypercube



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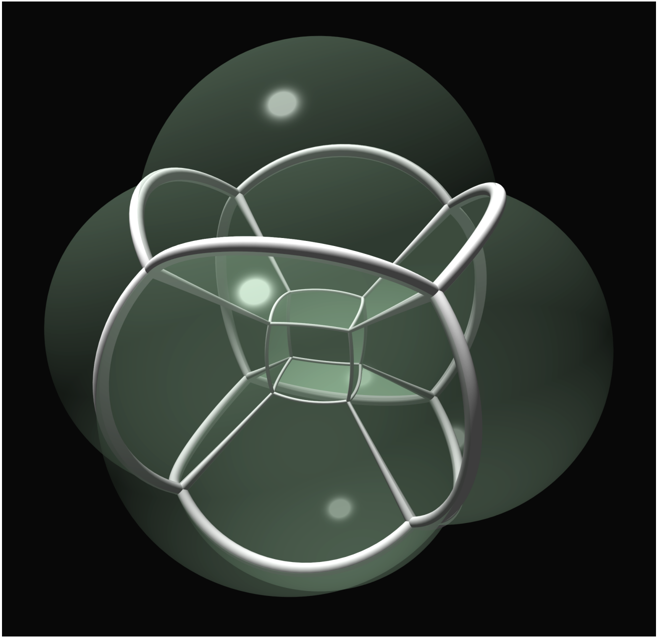
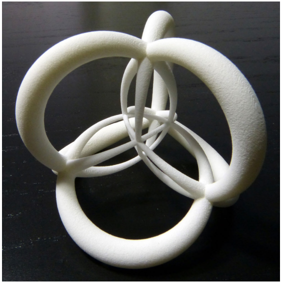
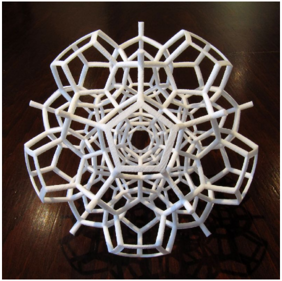


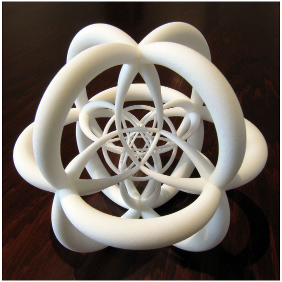
More regular 4-dimensional polytopes



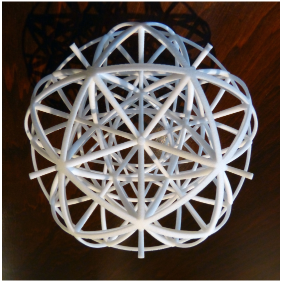
16-cell



Half of a 120-cell



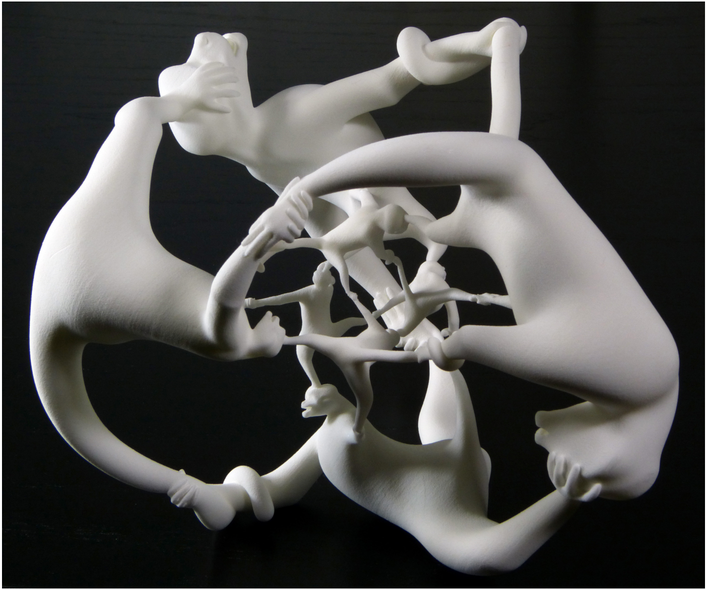
24-cell

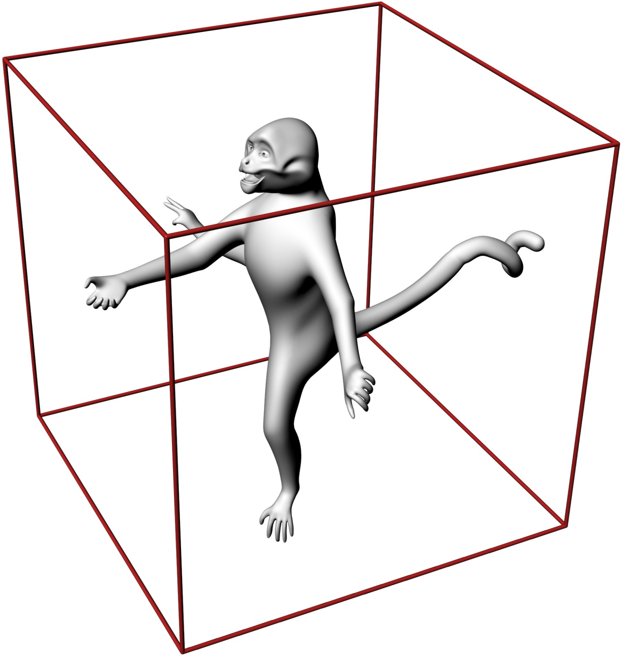


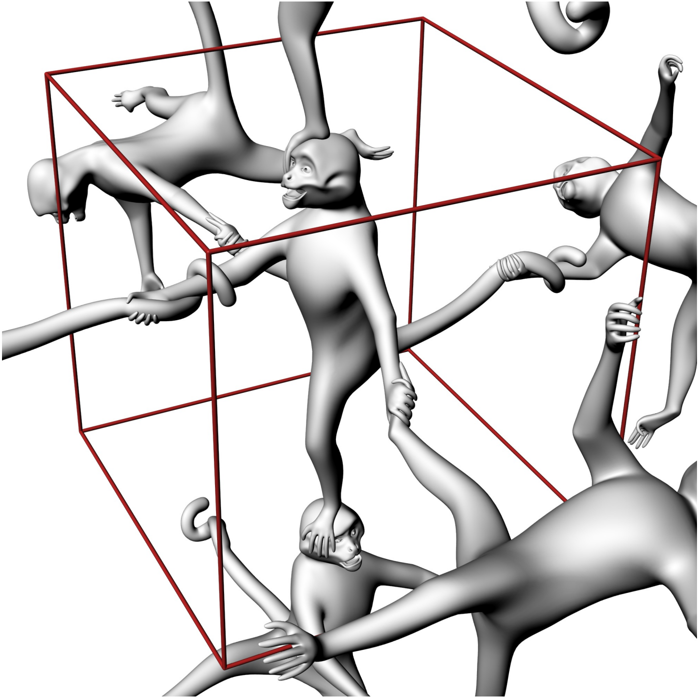
Half of a 600-cell

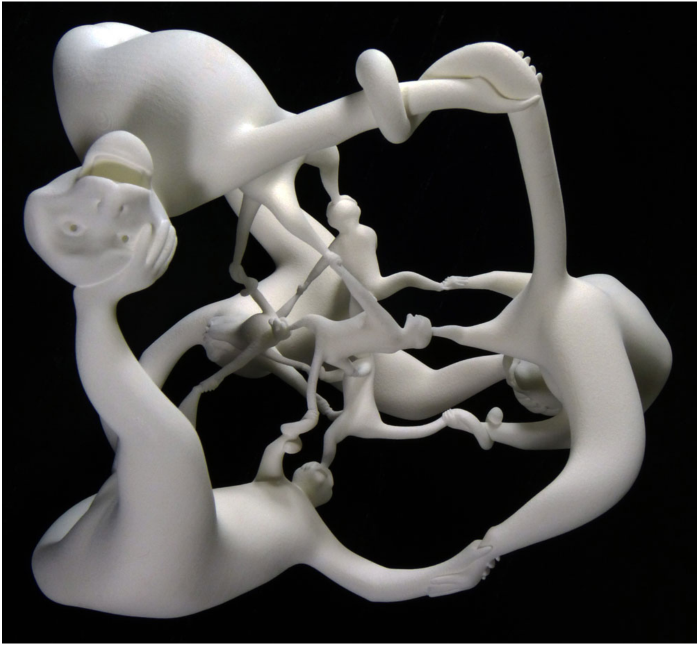
Quintessence (joint work with Saul Schleimer)

More fun than a hypercube of monkeys (joint work with Will Segerman)









http://monkeys.hypernom.com

http://monkeys.hypernom.com

http://monkeys.hypernom.com

Triple gear (joint work with Saul Schleimer)

https://skfb.ly/IuUI

Photo credit: Bill Beaty

Photo credit: meladramos of reddit.

Photo credit: Bill Beaty

Photo credit: meladramos of reddit.

Three pairwise meshing gears are usually frozen...

Photo credit: Bill Beaty

Photo credit: meladramos of reddit.

Three pairwise meshing gears are usually frozen...

A challenge: Find a triple of pairwise meshing gears that moves!

“Umbilic Rolling Link” by Helaman Ferguson.

“Knotted Gear” by Oskar van Deventer.

Our solution is inspired by these “linked” gears.

"Umbilic Rolling Link" by Helaman Ferguson.

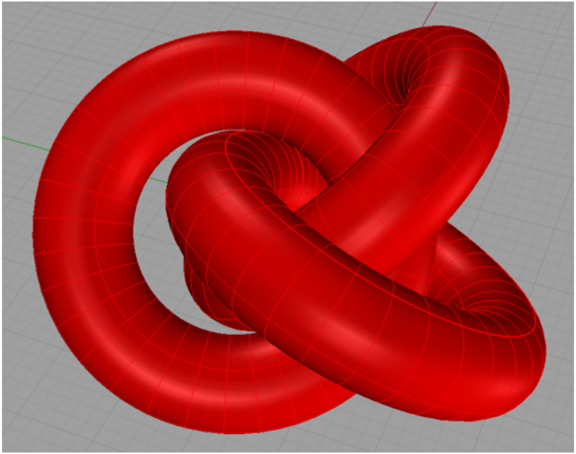
"Knotted Gear" by Oskar van Deventer.

Our solution is inspired by these "linked" gears.

They have two "gears"; we want to do the same with three.

We chose the three-component Hopf link as the basis of the design.

We gradually inflate the three rings, letting them bump against each other while preserving the 3-fold symmetry, until they reach maximum thickness.



We had hoped that these rings would only be able to rotate along their axes.

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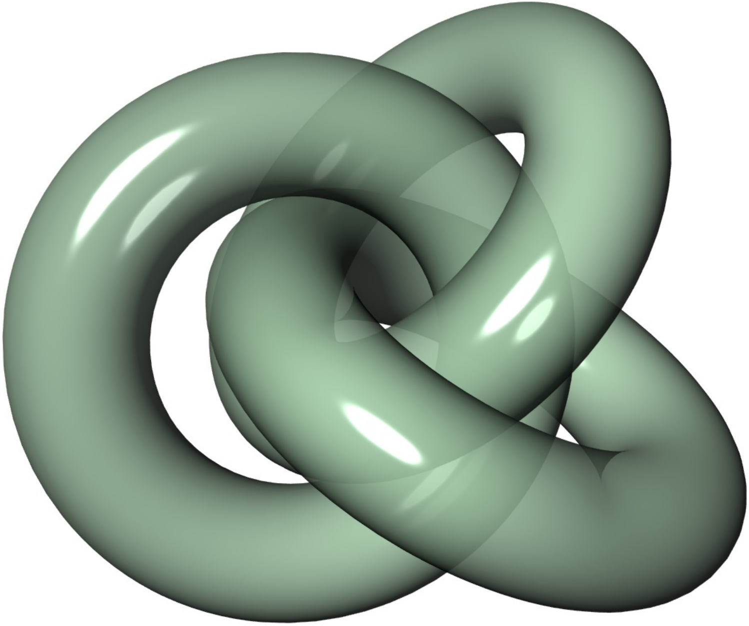
We had hoped that these rings would only be able to rotate along their axes. But unfortunately, they can move out of place, and then there is a little more room to inflate them further.

To stop them moving out of place, we design gear teeth.

To design the teeth, we investigate how the rings touch each other.



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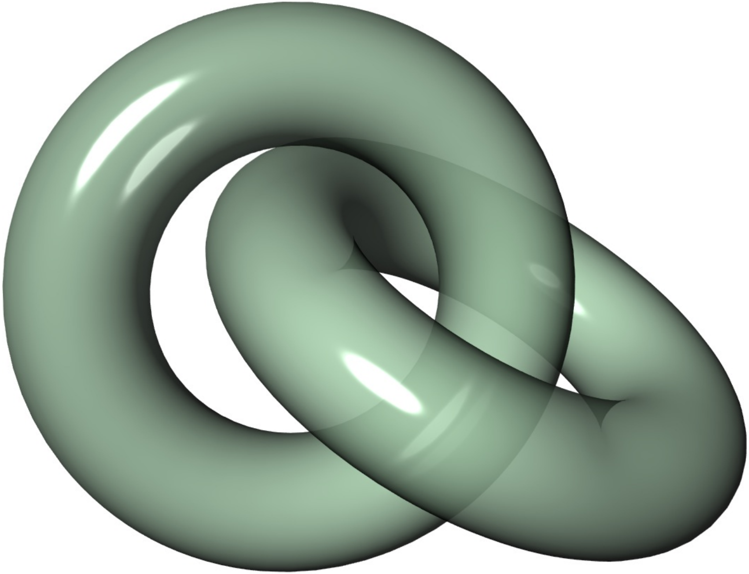

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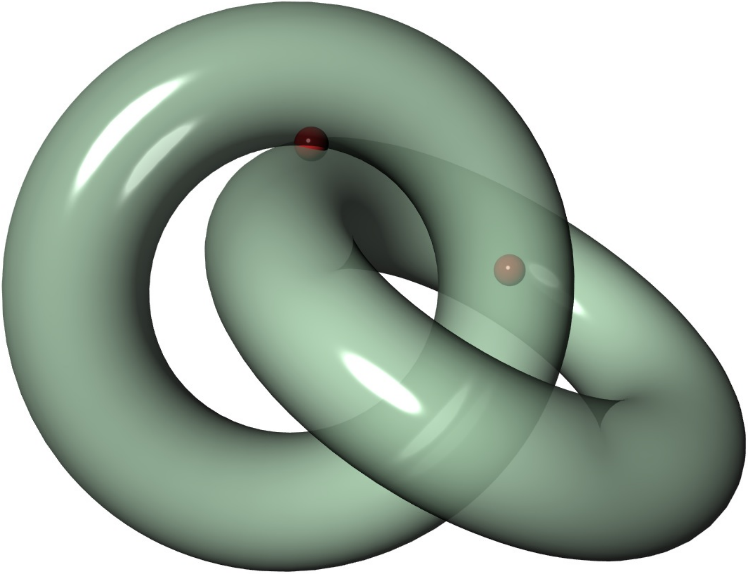

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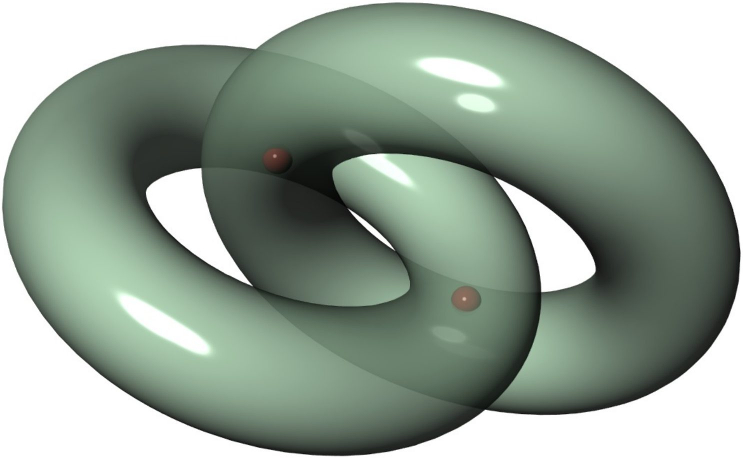

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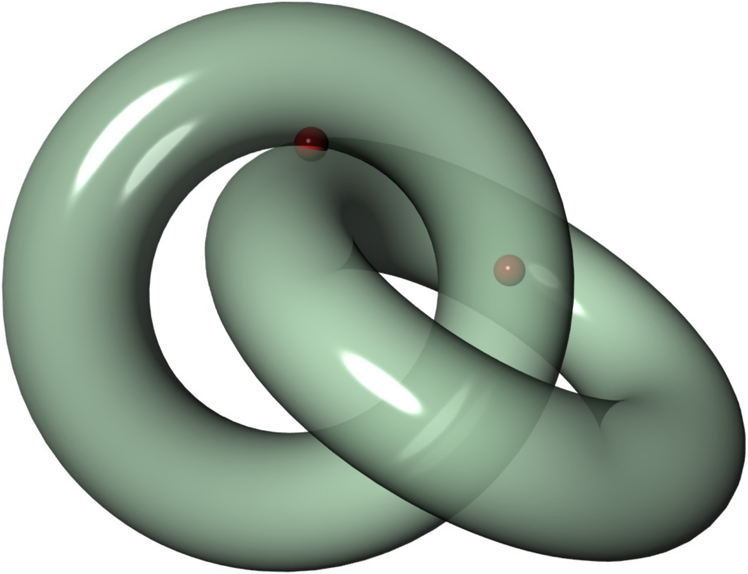

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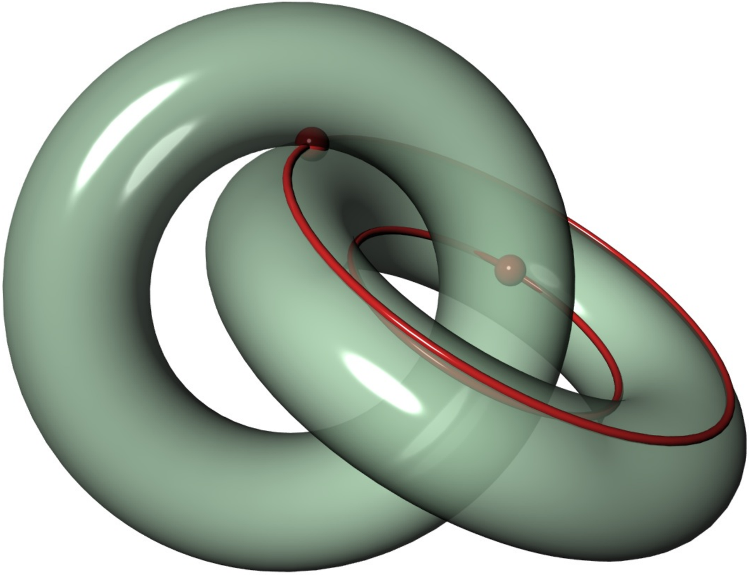

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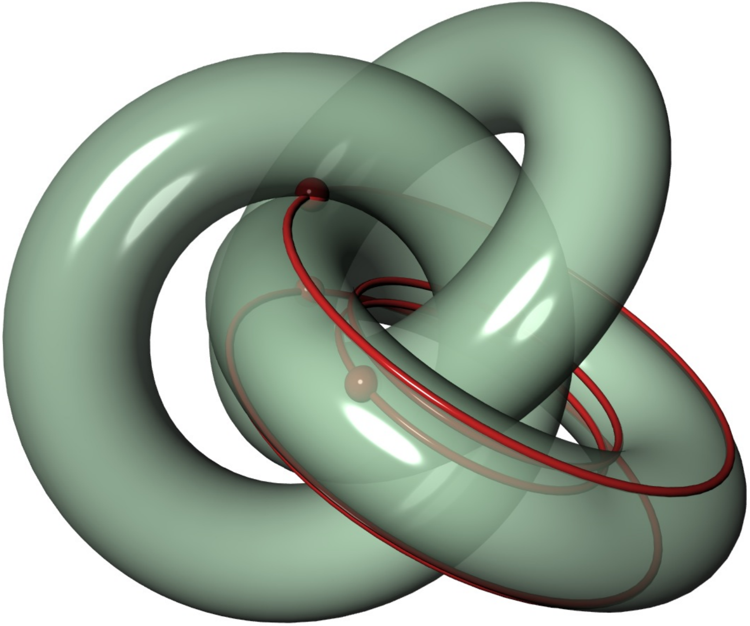

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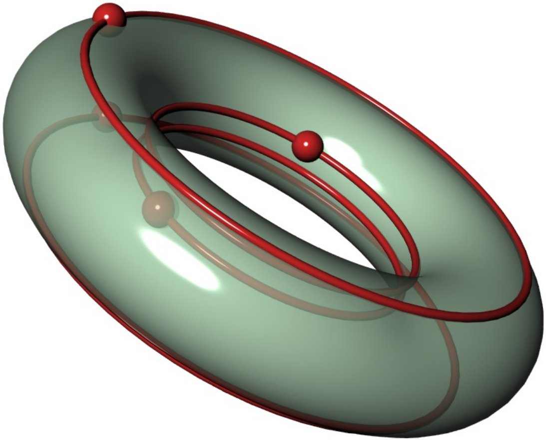

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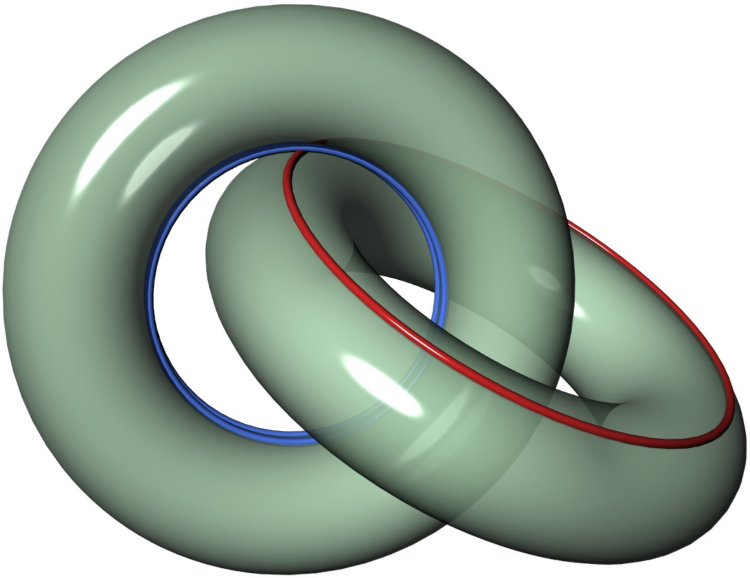
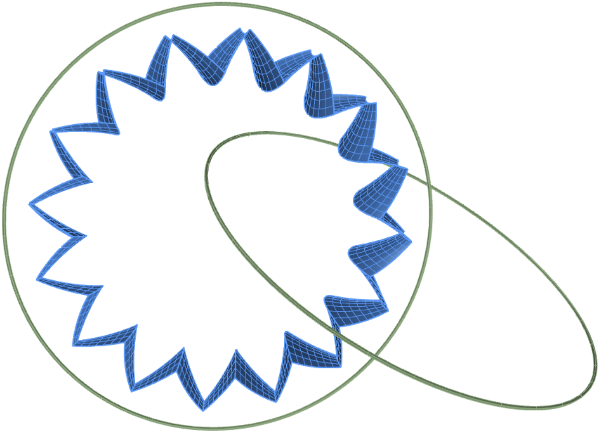
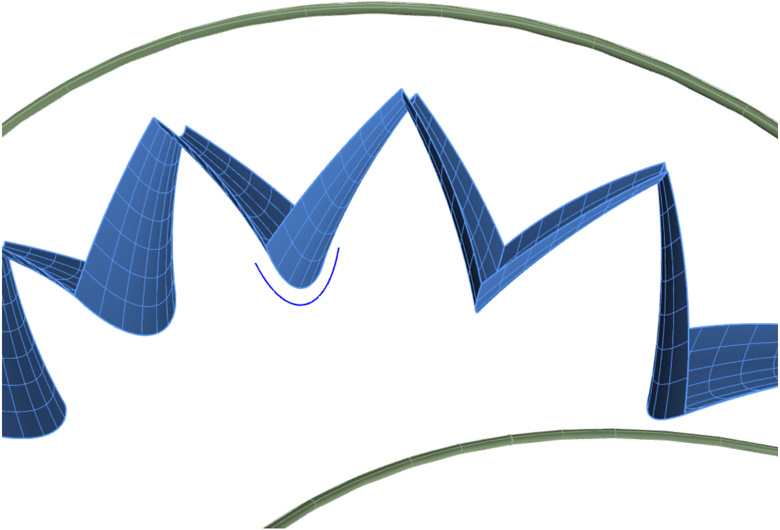


The "inner" teeth are the images of planes in toroidal coordinates.



The "inner" teeth are the images of planes in toroidal coordinates.
The "outer" teeth are determined by "carving".



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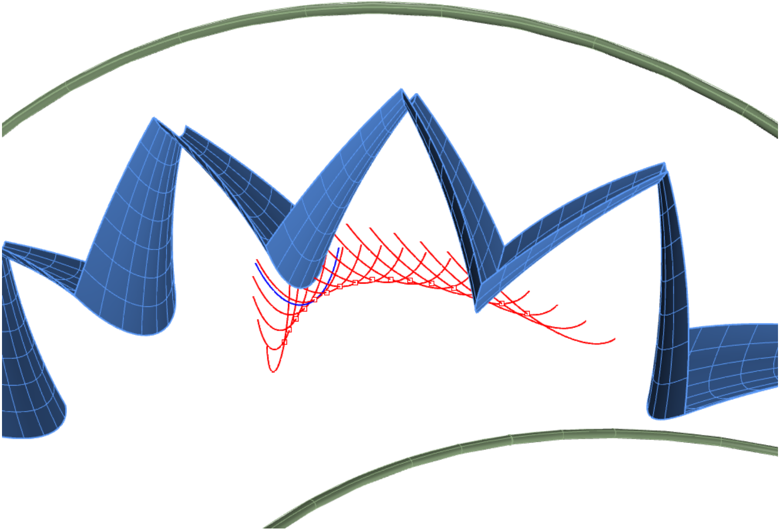

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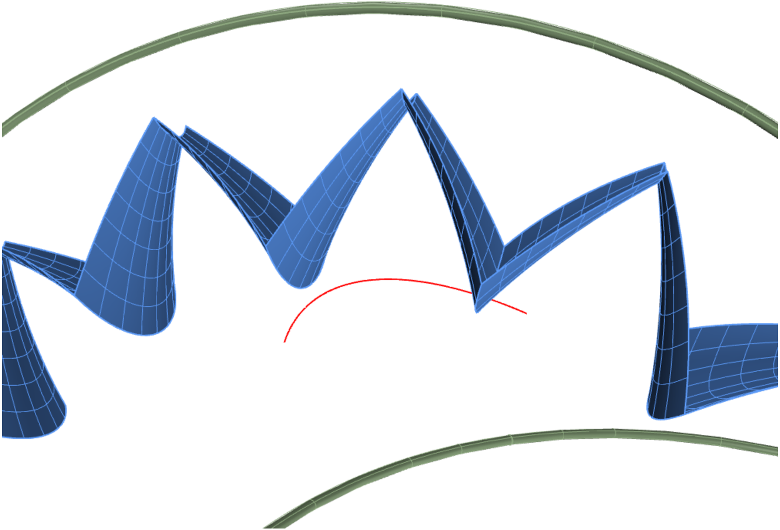

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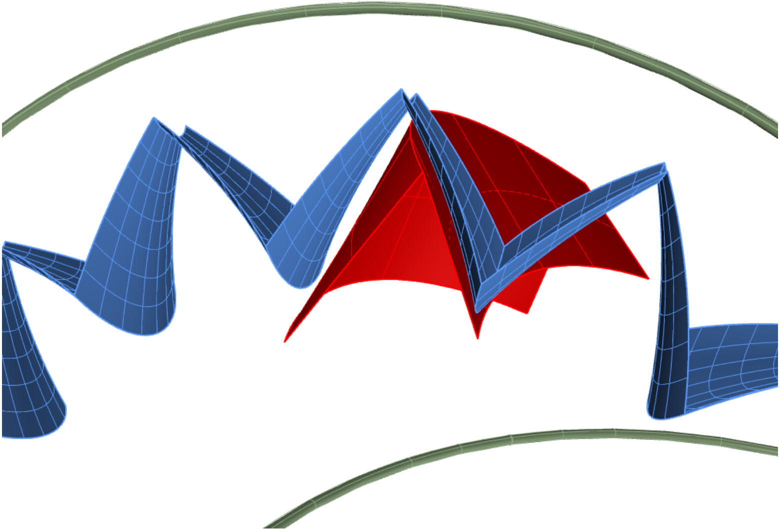

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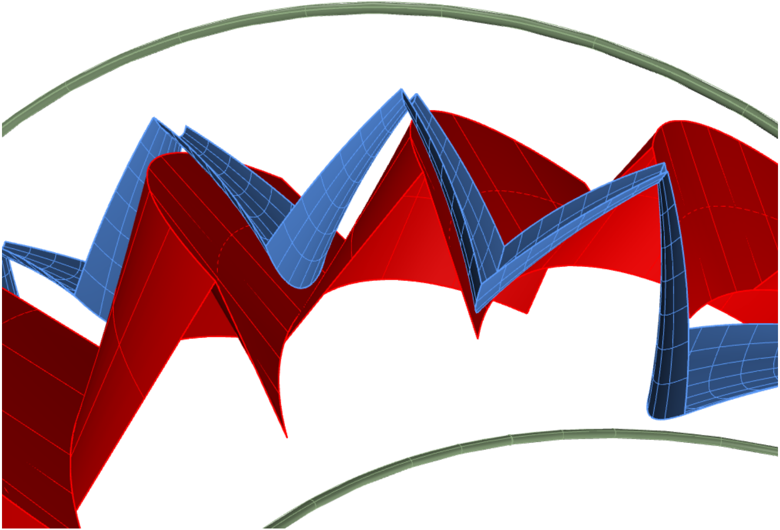

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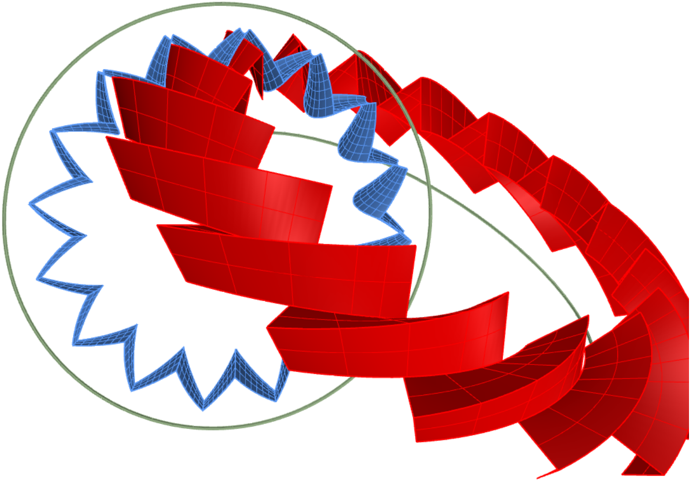
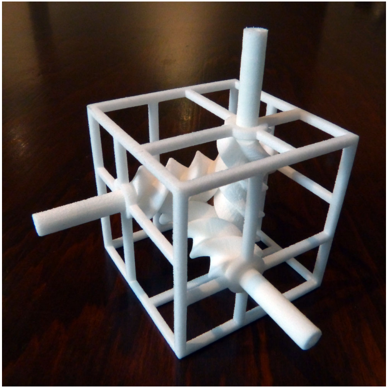
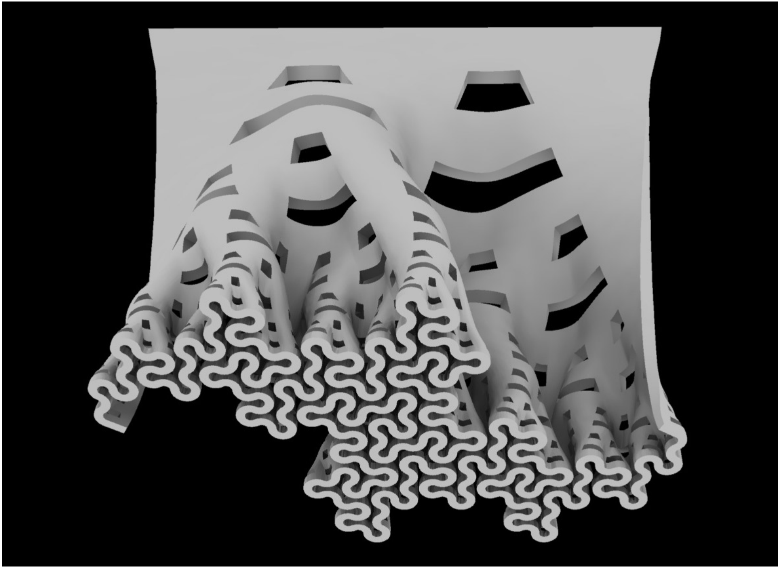
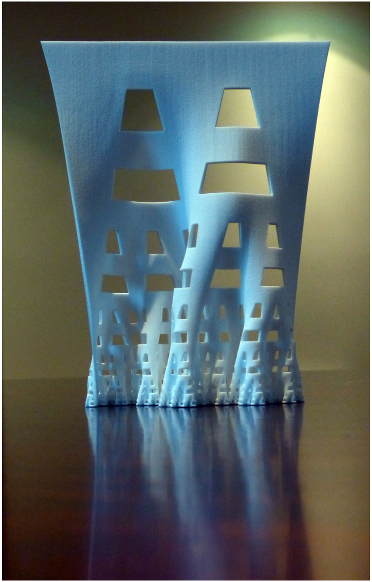
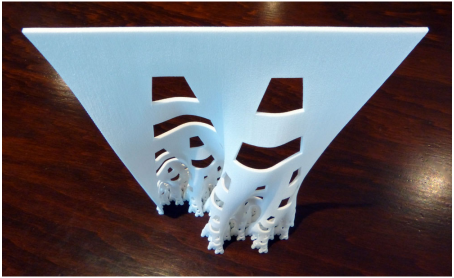
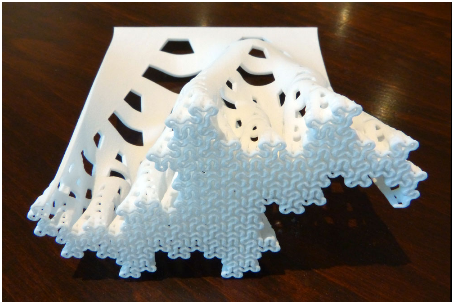


Alternative solutions

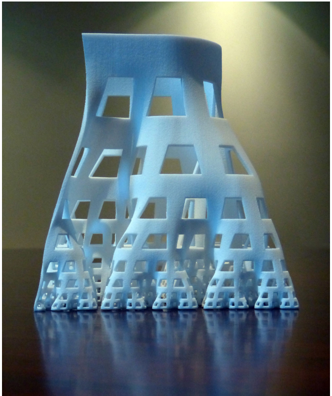
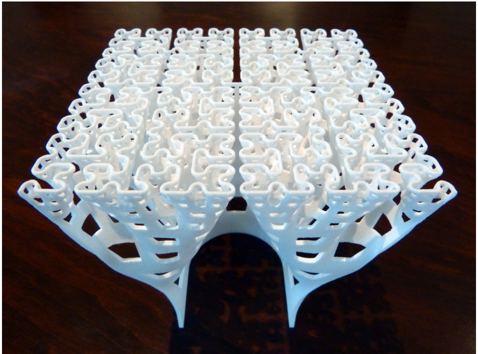


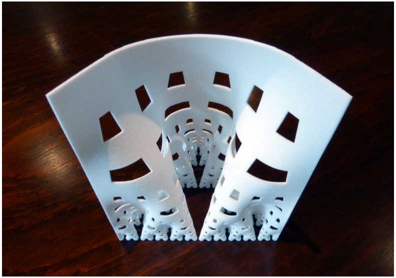
Developing fractal curves
(joint work with Geoffrey Irving)

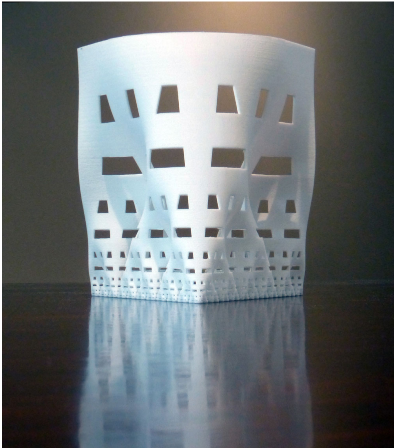


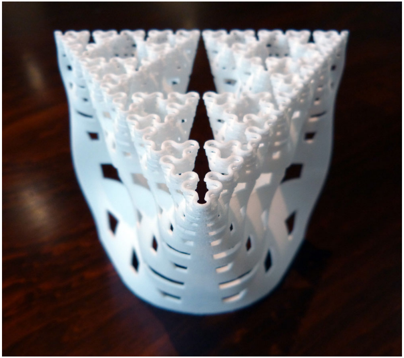




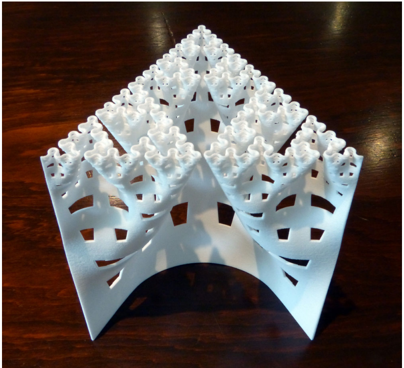


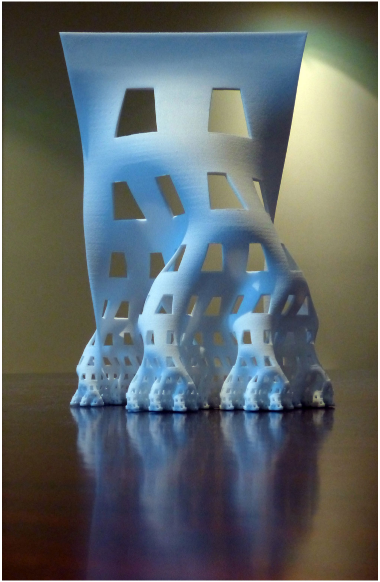



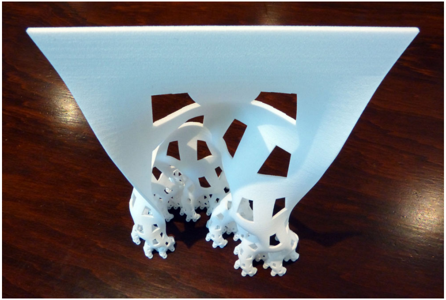


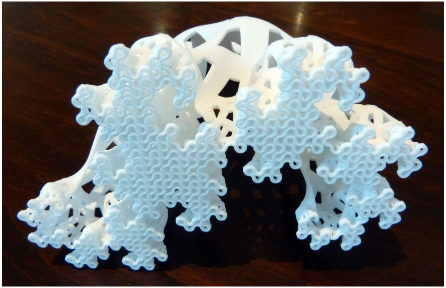


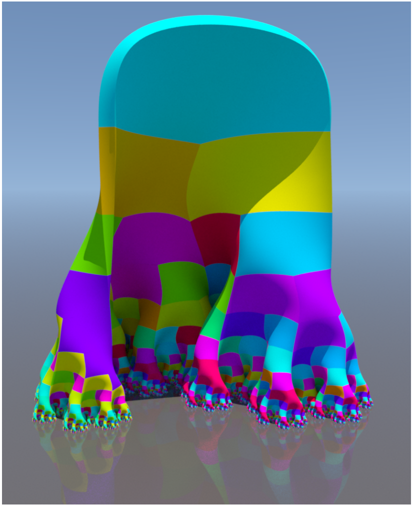
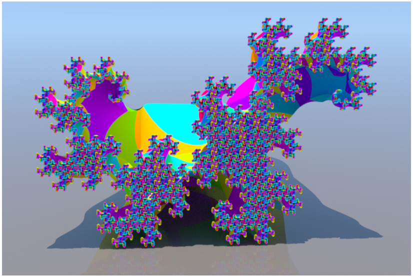
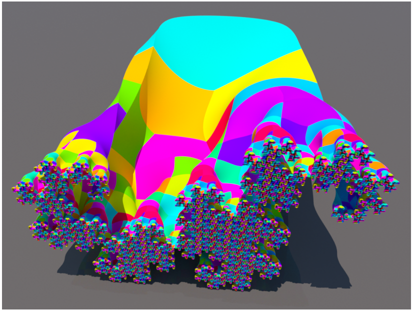




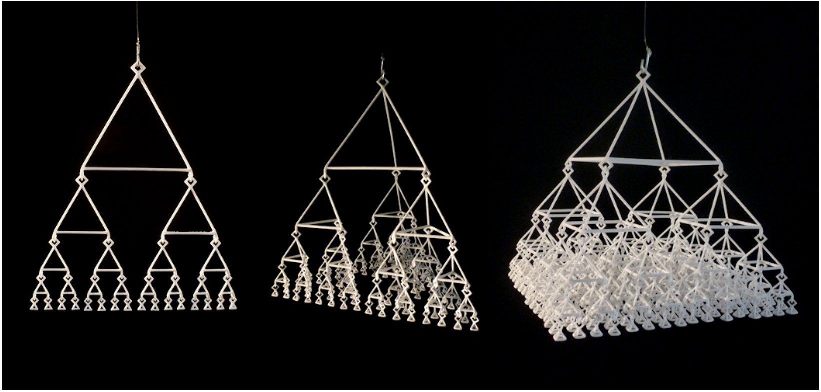






Mobiles (joint work with Marco Mahler)



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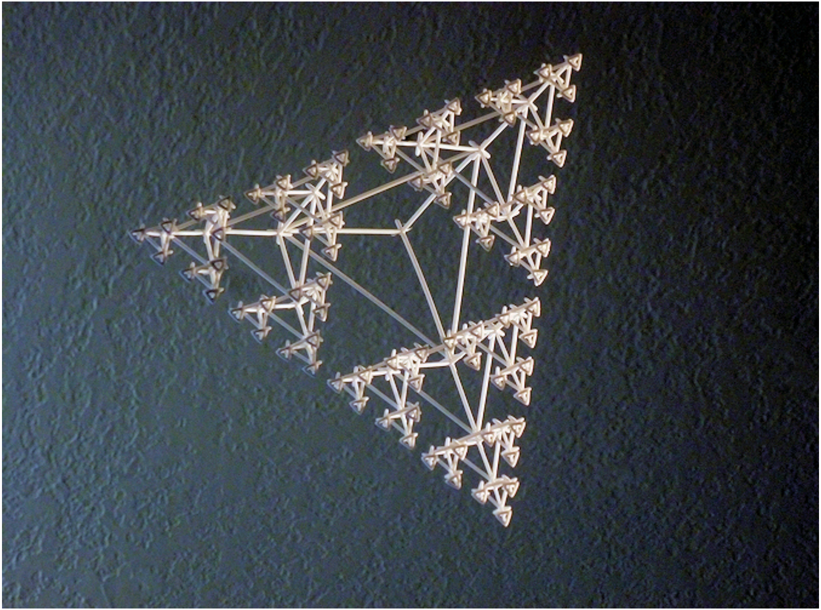
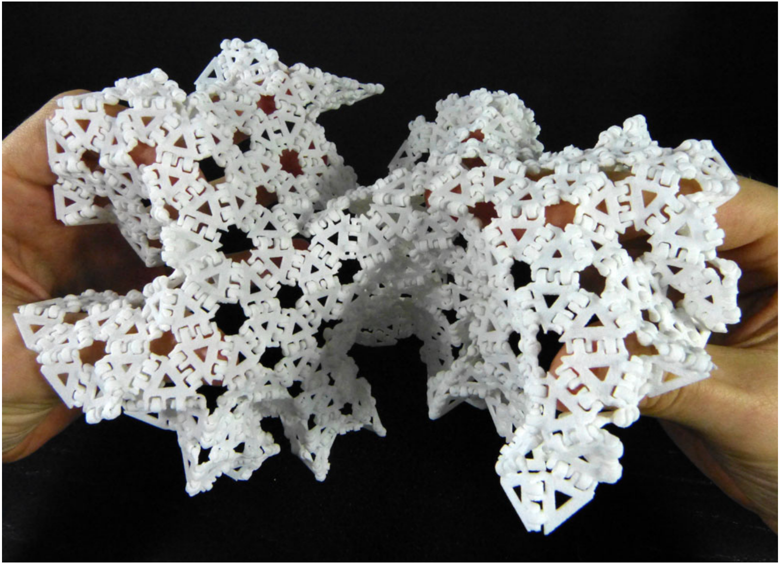
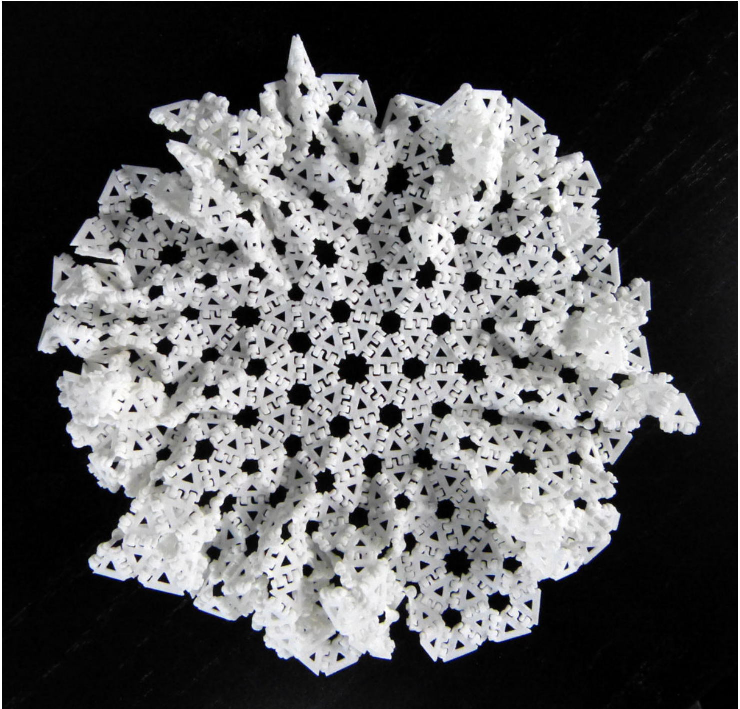
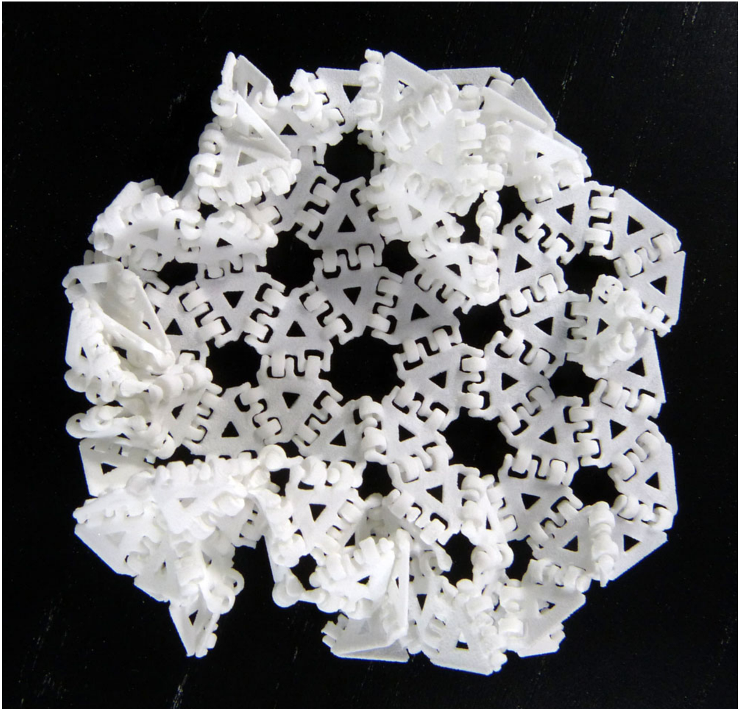
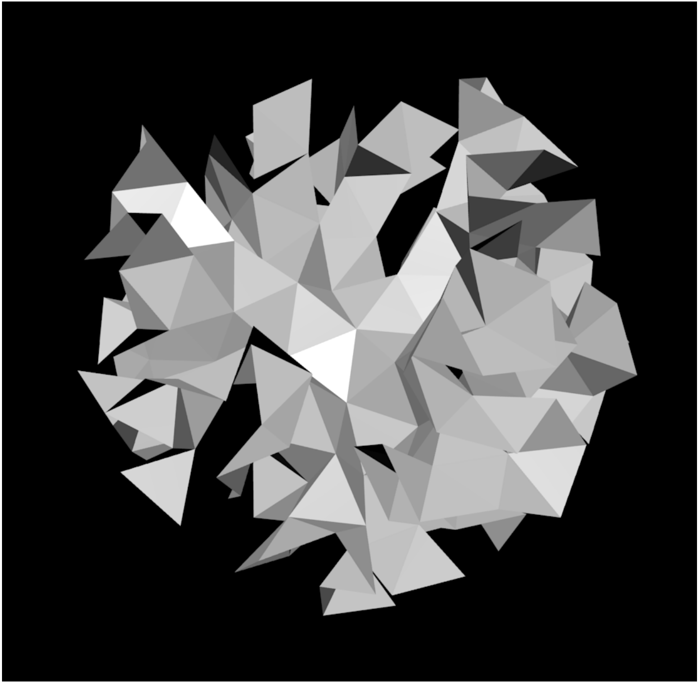


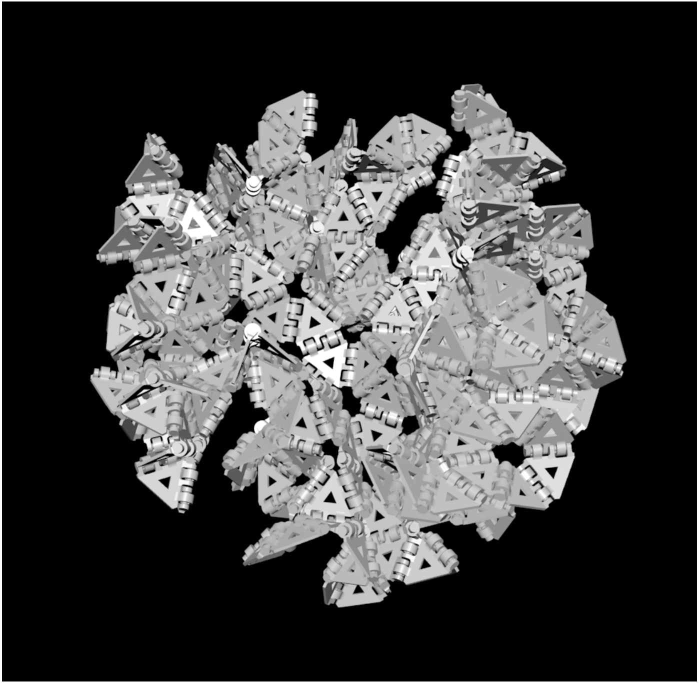
Hinged negatively curved surfaces
(joint work with Geoffrey Irving)



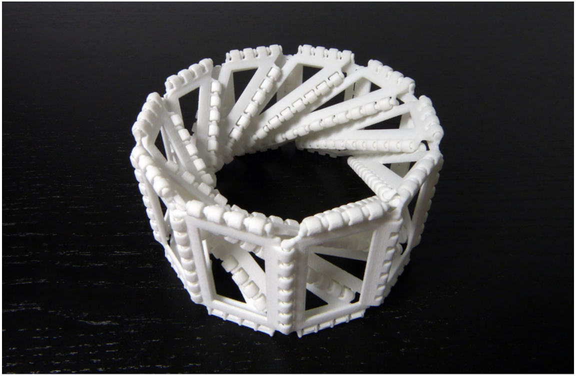


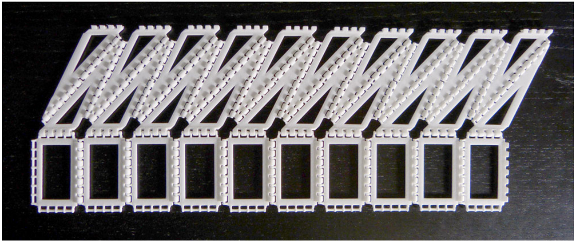






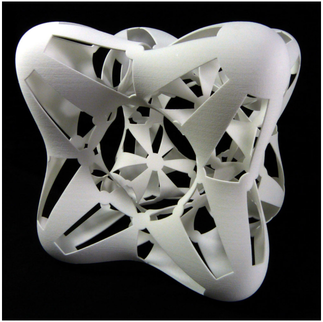
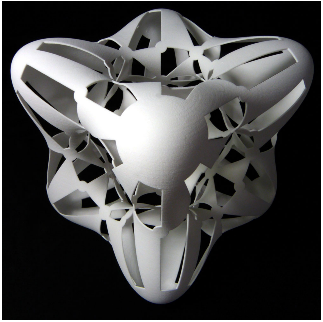
Hinged flat torus

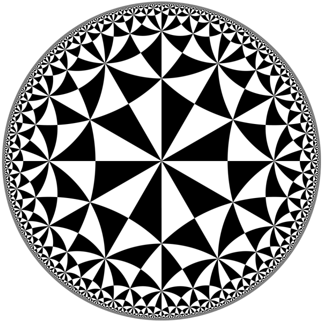




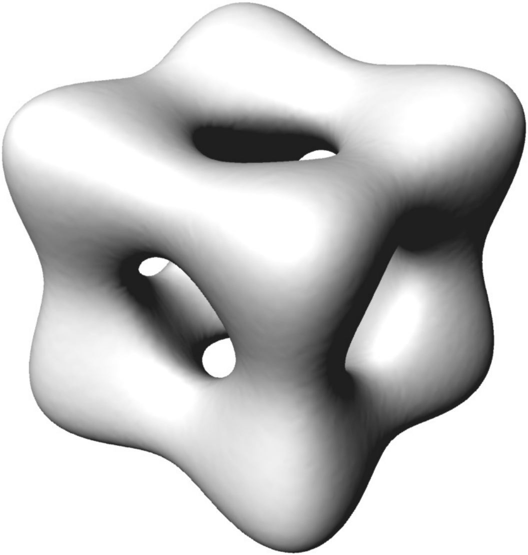
Topology Joke (joint work with Keenan Crane)

Conformal Chmutov (joint work with Saul Schleimer)

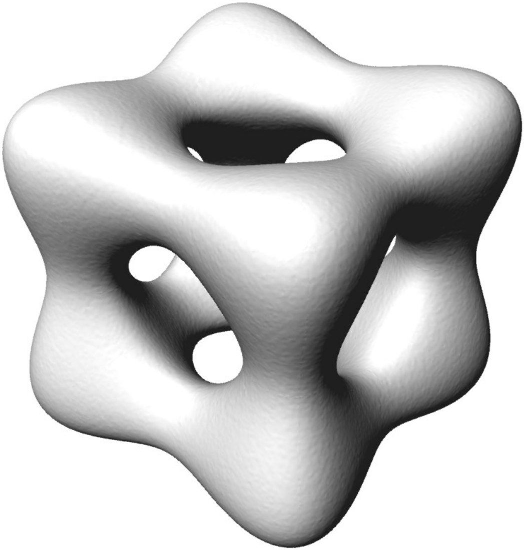


Conformal Chmutov (joint work with Saul Schleimer)



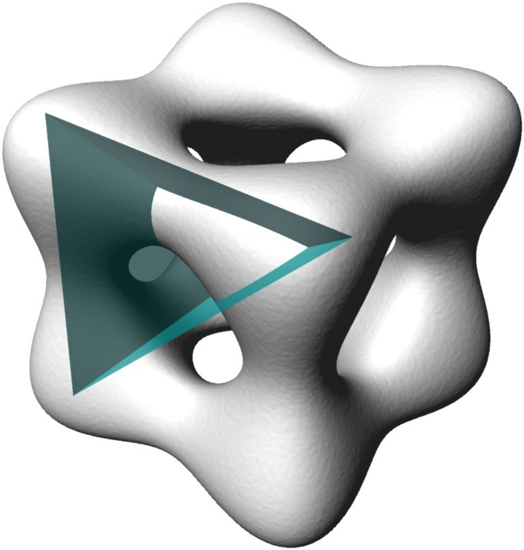
$$f(x, y, z) = -8(x^4 + y^4 + z^4) + 8(x^2 + y^2 + z^2) - 3 = 0.$$

Conformal Chmutov (joint work with Saul Schleimer)

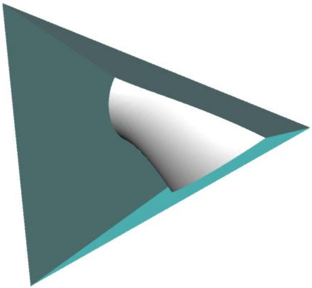


$$f(x, y, z) = -8(x^4 + y^4 + z^4) + 8(x^2 + y^2 + z^2) - 3 = 0.24114.$$

Conformal Chmutov (joint work with Saul Schleimer)



Conformal Chmutov (joint work with Saul Schleimer)



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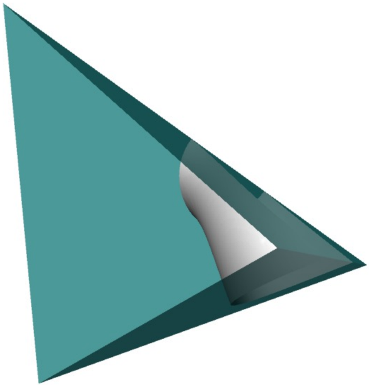
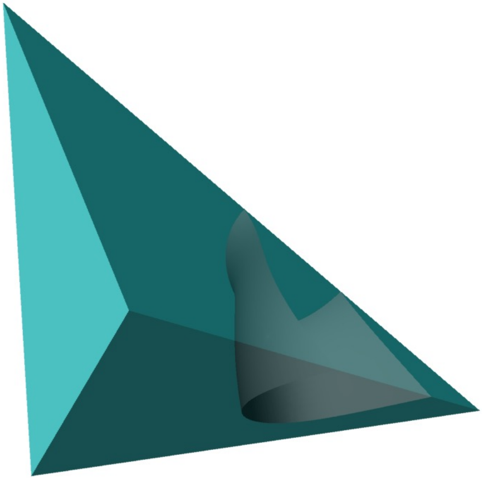
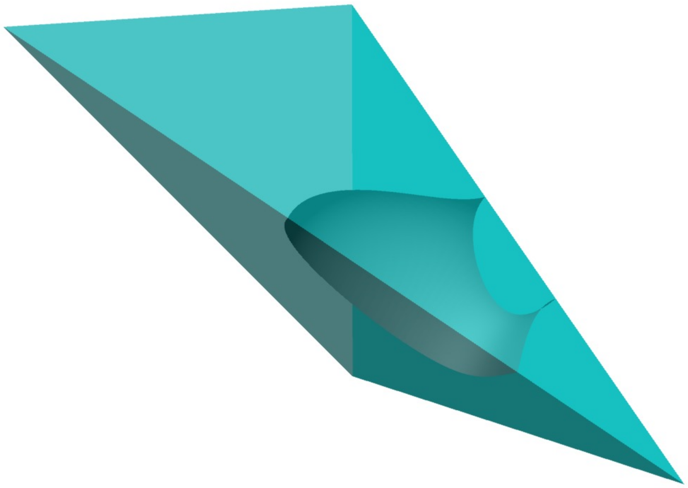


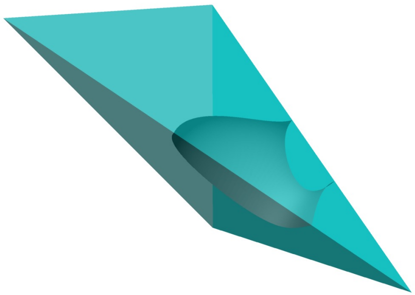
Conformal Chmutov (joint work with Saul Schleimer)

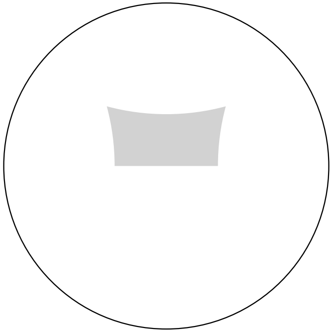


Conformal Chmutov (joint work with Saul Schleimer)

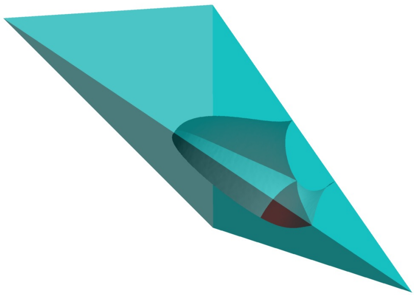


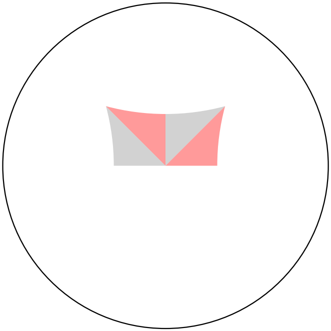
Conformal Chmutov (joint work with Saul Schleimer)





Conformal Chmutov (joint work with Saul Schleimer)





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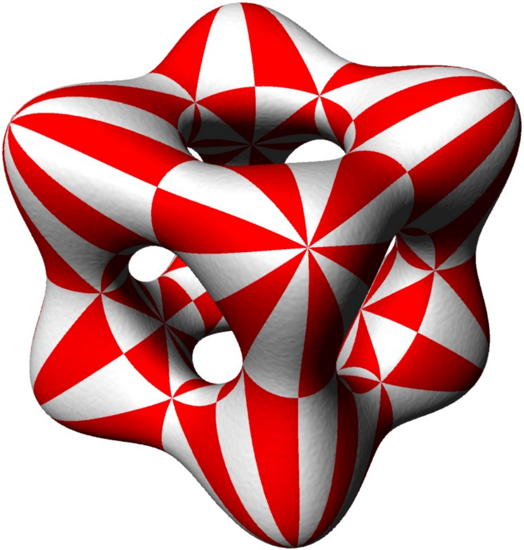
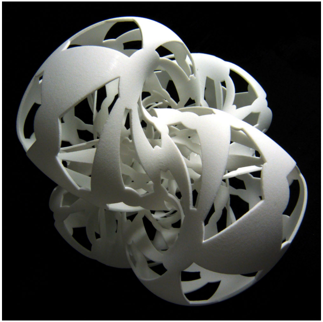
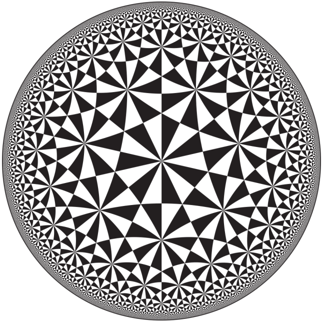


Klein quartic (joint work with Saul Schleimer)



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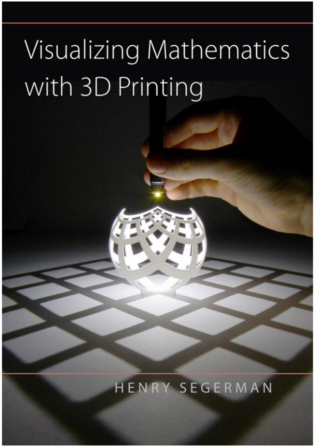

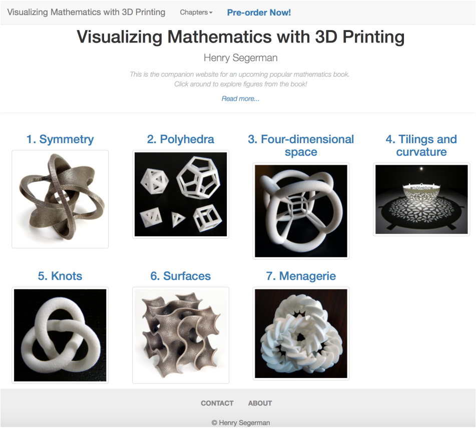

http://3dprintmath.com

Dice design (joint work with Robert Fathauer)

http://thedicelab.com

Skew dice (joint work with Robert Fathauer)

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d120 (joint work with Robert Fathauer and Bob Bosch)

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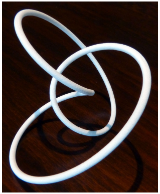
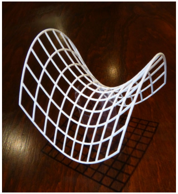
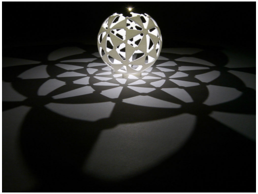
Thanks!



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