# Recent 3D Printed Sculptures

Henry Segerman

November 13, 2011

## 1 Introduction

I am a mathematician and a mathematical artist, currently a research fellow in the Department of Mathematics and Statistics at the University of Melbourne, Australia. My mathematical research is in 3-dimensional geometry and topology, and concepts from those areas often appear in my work. Other artistic interests involve procedural generation, self reference, ambigrams and puzzles.

All of the sculptures in this article were fabricated by the 3D printing service Shapeways. The material used is nylon plastic (PA 2200, Selective-Laser-Sintered) for all of the sculptures apart from "Knotted Cog", which is made from stainless steel infused with bronze. I design my sculptures in Rhinoceros, which is a NURBS based modelling tool, often using the python scripting interface. In designing sculptures using scripting and then producing them using 3D printing, it is possible to get very close to mathematically precise geometry, which is often difficult to achieve by other means.

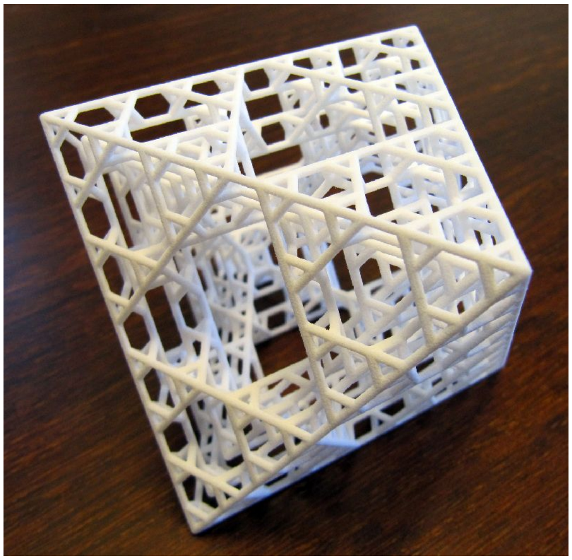
Sections 2, 3 and 4 of this article are on the themes of fractal graphs, surfaces native to the 3-dimensional sphere, and 4-dimensional polytopes. Finally, section 5 is a miscellany of other designs.

## 2 Fractal graphs

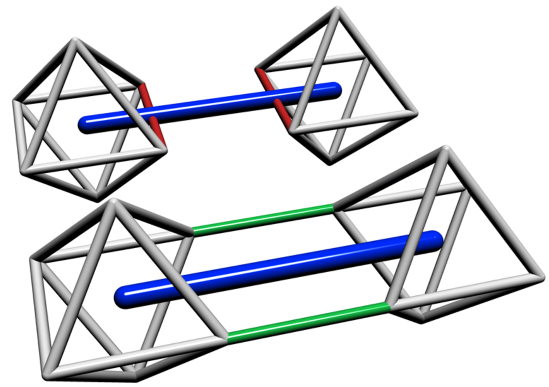
These are part of a series of sculptures exploring graphs embedded in $\mathbb{R}^3$ with a fractal structure analogous to that found in constructions of space filling curves. Starting from a simple initial graph, we repeatedly apply a substitution move. The substitution move replaces each vertex of the graph from step $i$ with a small graph in step $i+1$, and it replaces each edge of the graph from step $i$ with some number of parallel edges connecting the small graphs in step $i+1$ corresponding to the endpoints of the vertices in step $i$. This construction is explored in detail in [1].

**Octahedron Fractal Graph** This is a graph embedded in $\mathbb{R}^3$ as a subset of an "octahedral lattice", which is related to the tessellation of space using octahedra and tetrahedra. See Figure 1b for the construction rule: Each vertex at each step of the construction is degree 4, and is replaced at the next step by a small octahedron. For each edge at the previous step (shown in blue) we remove the two red edges that intersect it, and add in the two green edges parallel to it. We begin the construction with the first step being the edges of an octahedron. After the first substitution, we get an outer shell around a small octahedron. We discard this small octahedron (otherwise the object would not be connected) and continue. This is the result at the fourth step.

**Fractal Graph 3** This is a graph embedded in 3-dimensional space as a subset of the cubic lattice. Each vertex at each step of the construction is degree 3, and has the incident edges arranged either in a 'T' formation, or like a corner of a cube. The vertex is replaced at the next step by a subgraph of a $3 \times 3 \times 3$ cubical grid, the choice determined by whether the edges meeting at the vertex are in the 'T' or 'corner'



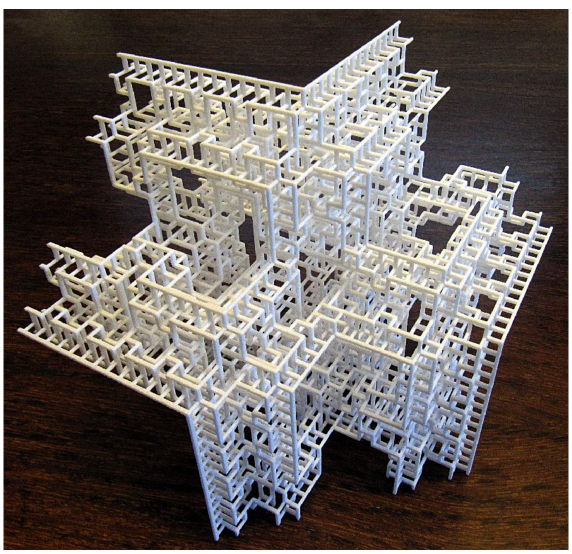
(a) The sculpture.



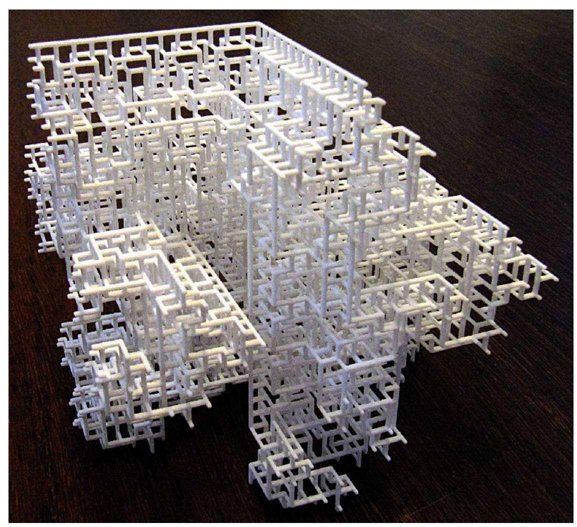
(b) The substitution move.

**Figure 1**: Octahedron Fractal Graph, 2010, $10.3 \times 10.3 \times 10.3$ cm.

shape. Each edge is replaced at the next step by four parallel edges, joining to the midpoints of the sides of each $3 \times 3$ face of the $3 \times 3 \times 3$ cubical grid. We begin the construction with the first step being three edges meeting in a corner formation, and this is the result at the fourth step.



(a) A view from 'outside the corner'.



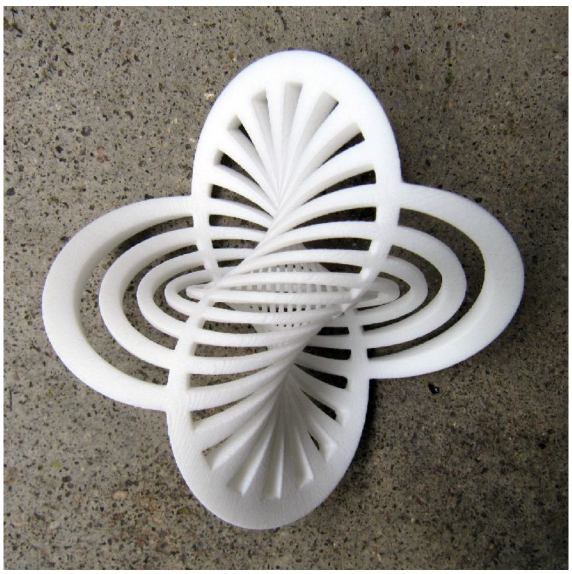
(b) Reverse view.

**Figure 2**: Fractal Graph 3, 2011, $10.6 \times 10.6 \times 10.6$ cm.

## 3 Surfaces in $S^3$

These designs are based on various parametric surfaces, given by trigonometric functions in the 3-sphere, $S^3$, viewed as the set of points at distance 1 from the origin in $\mathbb{R}^4$. We use stereographic projection from $S^3$ to $\mathbb{R}^3$ to get an object that can be printed in our universe. When the surface goes through the point from which we stereographically project, the image would go off to infinity in $\mathbb{R}^3$. To save on costs, I have removed the parts that would require an infinite amount of plastic to print.

**Hopf fibration** The surface here is a torus (in fact, the image of a Clifford torus in $S^3$), with a disk removed around infinity. Although it is possible to choose a projection that does not go through infinity, the result wouldn't have the same symmetries in $\mathbb{R}^3$. In particular, the torus cuts $\mathbb{R}^3$ into two pieces, and these can only be congruent if the torus goes through infinity. I came to this design through trying to represent the Hopf fibration. The Hopf fibration describes $S^3$ as being a union of fibres, each fibre being a circle, and with one fibre for each point of the 2-sphere. Each of the corner curves of the square cross-section tubes is a circular fibre of the fibration. The pair of tubes going in the transverse direction to all of the others come from the mirror image fibration, and are necessary to keep the sculpture connected.



(a) A 2-fold symmetry axis.

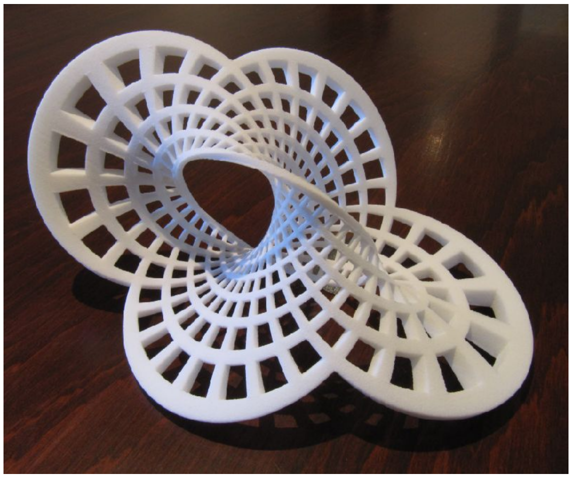
(b) A generic viewpoint.

**Figure 3**: Hopf Fibration, 2010, $10.8 \times 10.8 \times 3.4$ cm.

**Round Möbius Strip** The usual version of a Möbius strip has as its single boundary curve an unknotted loop. This unknotted loop can be deformed into a round circle, with the strip deformed along with it. This shows one possible way to do this (sometimes called the "Sudanese Möbius strip"). The boundary of the strip is the circle in the middle. The other boundary of the design is the cut out around infinity, so strictly speaking this is a punctured Sudanese Möbius strip. Again, the choice of a projection that goes through infinity makes the sculpture particularly symmetric. This was designed with the assistance of Saul Schleimer, who is a mathematician at the University of Warwick.

**Round Klein Bottle** This is made by gluing two copies of the Round Möbius Strip along their boundaries, and so this is actually a doubly punctured Klein bottle. A Klein bottle in 3-dimensional space must intersect itself, and in this case it intersects along a straight line.

**Knotted cog** The parametrisation used here would make a Möbius strip with 3 half-twists, if the cog teeth on the inside linked up across the gap. Instead, the design is a trefoil knot, with near-meshing cogs. In order for the teeth to mesh rather than collide, there has to be an odd number of teeth around the strip, and in order



(a) A generic viewpoint.

(b) A 2-fold symmetry axis.

**Figure 4**: Round Möbius Strip, 2011, $15.2 \times 10.9 \times 6.2$ cm.

to preserve the 3-fold symmetry, the number must also be a multiple of three. This restricted the possibilities to the extent that it was easy to choose 33 on aesthetic grounds.

## 4 4-dimensional polytopes

These are representations of the edges of regular 4-dimensional polytopes, also known as polychora. Just as 3-dimensional polyhedra are named for the number of 2-dimensional faces they have, the 4-dimensional polychora are named for the number of 3-dimensional cells they have. The polytopes are native to $\mathbb{R}^4$, but can be radially projected onto the unit $S^3$ in $\mathbb{R}^4$, where we fatten the edges into tubes of constant radius. From here, we stereographically project to $\mathbb{R}^3$ to produce our sculptures, as with the surfaces in $S^3$.

**24-Cell** This is the 24-cell, which has 24 octahedral cells. The point from which we stereographically project is at the centre of one of the octahedra, so the edges do not pass through infinity, and we need not cut out a region to get a finite sculpture.

**Half of a 120-Cell** The 120-cell has 120 dodecahedral cells. The dodecahedra are considerably smaller in $S^3$ than the octahedra of the 24-cell, and so even with the stereographic projection point at the centre of a dodecahedron, the edges nearest to this point would be very large after projection. The smallest edges are limited by the resolution of the 3D printer to a minimum diameter of around 1mm, and so unless we altered the way in which we project, printing the whole object would be prohibitively expensive. The solution, due to Saul Schleimer, is to print only half of the object, sliced along the equatorial $S^2$ in $S^3$ dual to the projection point. Reflecting the sculpture across this sphere would recover the whole object.

## 5 Other works

**Archimedean Spire** This is a ruled surface, which is a surface that can be made up out of straight line segments. The grid structure of the sculpture shows the ruling, and is more visually interesting than the filled in surface would be. Here, the straight line segments are determined by the paths that their endpoints follow. In this case, one end of the line segment follows one turn of an Archimedean spiral in the x-y plane, while the other moves up the z-axis at a constant speed.

(a) A generic viewpoint.

(b) The 4-fold symmetry axis.

(c) One of the 2-fold symmetry axes.

(d) The other 2-fold symmetry axis.

**Figure 5**: Round Klein Bottle, 2011, $15.2 \times 15.2 \times 10.9$ cm.

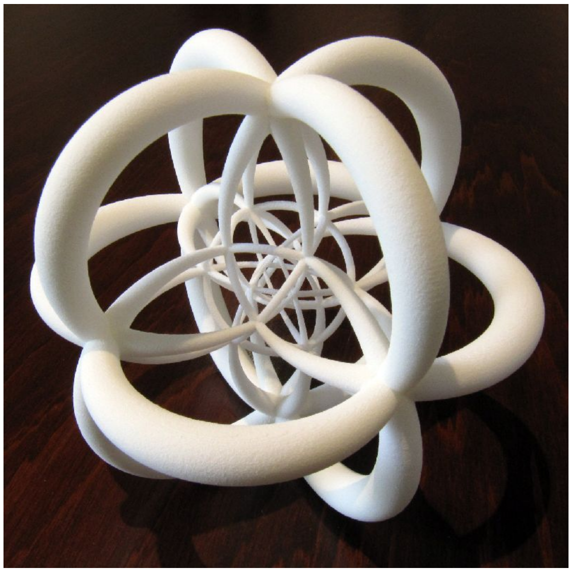
**Sphere Autologlyph** An *autologlyph* is a word written or represented in a way which is described by the word itself. Here, 20 copies of the word "SPHERE" are tessellated to from the surface of a sphere. The sculpture has a 10-fold rotational axis, and ten 2-fold axes perpendicular to it, so it has the symmetries of a regular 10-gon. The form of the sculpture is in some ways forced by the limits of technology: If it had been easy to (2-dimensionally) print onto the surface of a sphere, then I would have done so, and presumably a design like that could be realised using the machinery that makes geographic globes. Within the medium of 3D printing, one could print a solid sphere, with grooves or ridges used to form the letters. This would be very expensive however. Drawing the letters as a network of tubes solves this problem, although it does mean that the holes in the "P" and "R" have to be inferred.

**Juggling Club Motion** This shows (a somewhat idealised version of) the path of a juggling club as it is thrown from the right hand to the left, making a single spin. The club is shown in a "multiple exposure" style as it follows a parabolic path while rotating at constant speed.

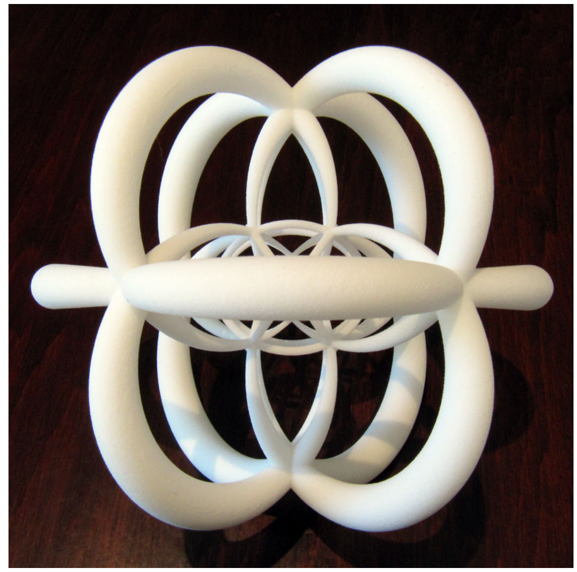
**Figure 6**: Knotted Cog, 2011, $3.8 \times 3.4 \times 1.3$ cm.

## References

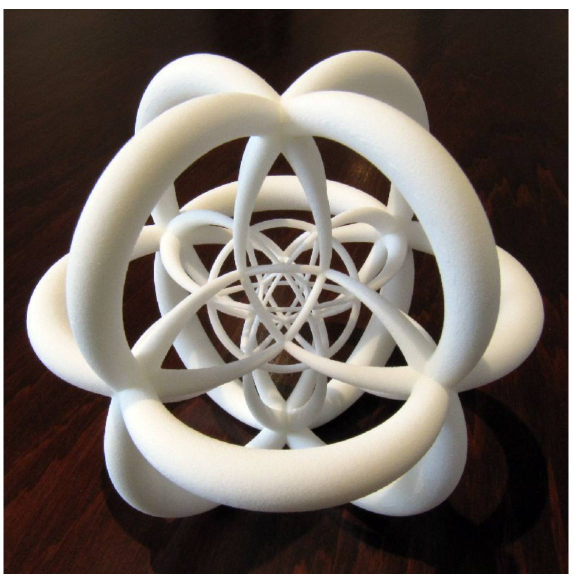
[1] Henry Segerman, *Fractal graphs by iterated substitution*, *Journal of Mathematics and the Arts*, ©Taylor and Francis, Volume 5, Issue 2, 2011, pp. 51–70.



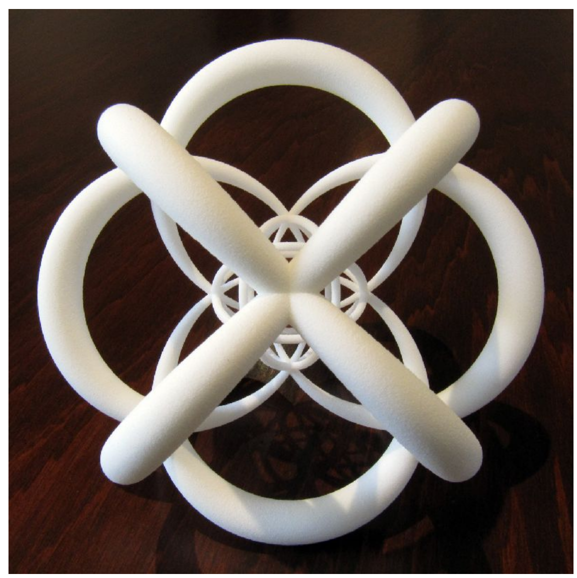
(a) A generic viewpoint.



(b) A 2-fold symmetry axis.

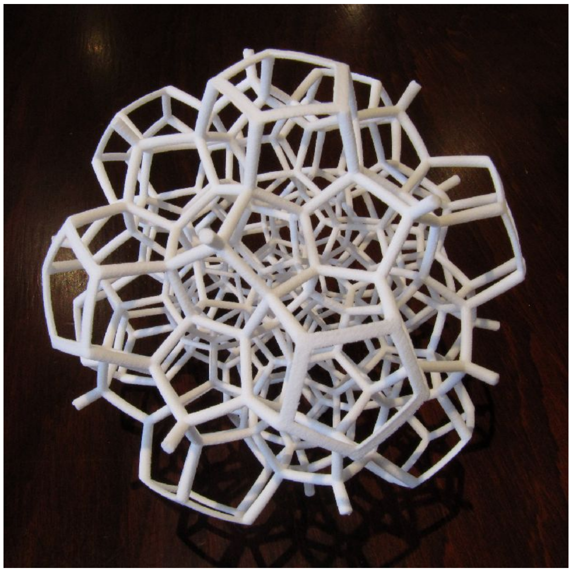


(c) A 3-fold symmetry axis.

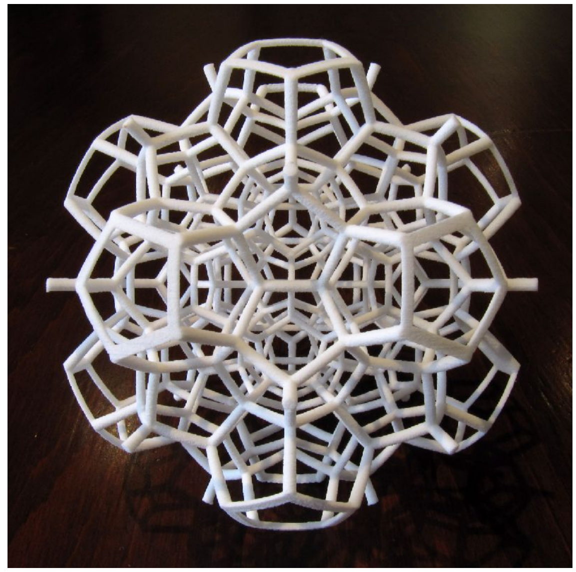


(d) A 4-fold symmetry axis.

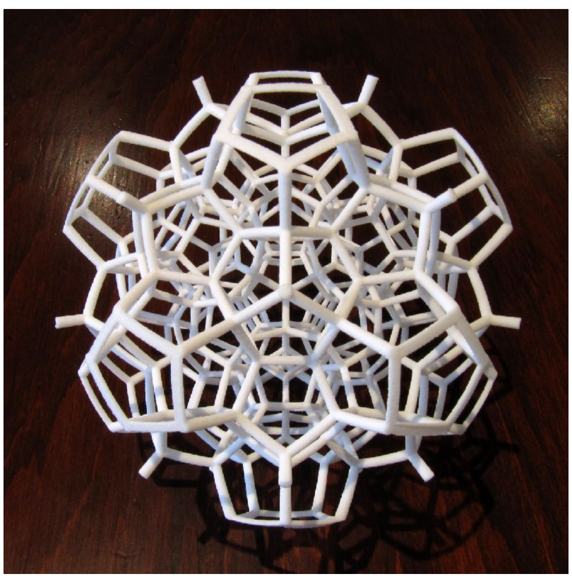
**Figure 7**: 24-Cell, 2011, $9.0 \times 9.0 \times 9.0$ cm.



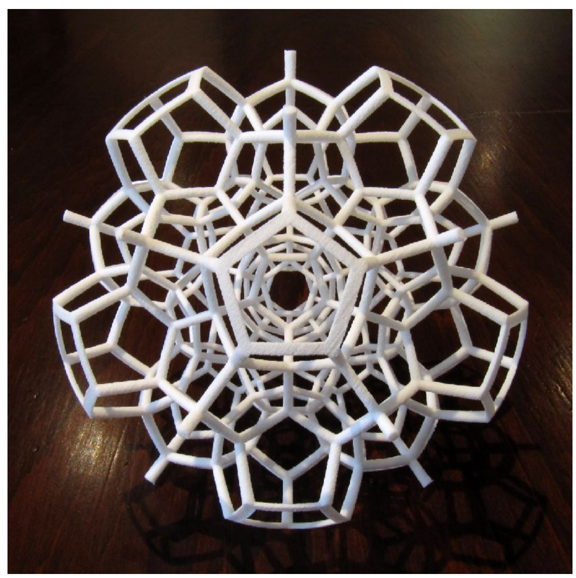
(a) A generic viewpoint.



(b) A 2-fold symmetry axis.



(c) A 3-fold symmetry axis.

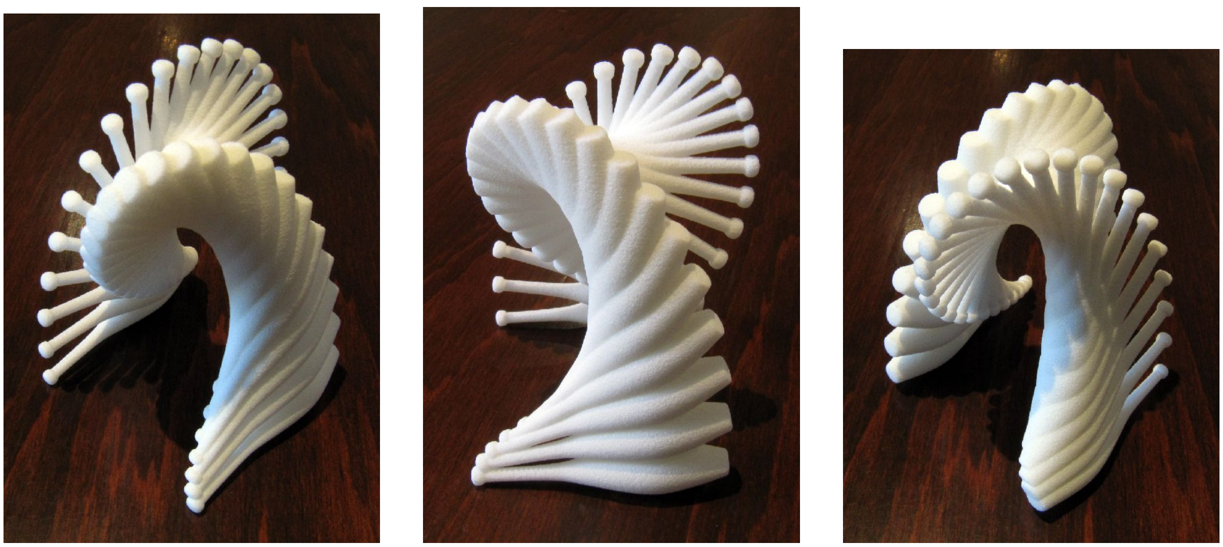


(d) A 5-fold symmetry axis.

**Figure 8**: Half of a 120-Cell, 2011, $9.9 \times 9.9 \times 9.9$ cm.

**Figure 9**: Archimedean Spire, 2009, 2011, 7.7 × 11.0 × 7.3 cm.

**Figure 10**: Sphere Autologlyph, 2009, 10.4 × 10.4 × 10.4 cm.



(a) A view from behind the juggler. (b) A view from the side. (c) A view from in front of the juggler.

**Figure 11**: Juggling Club Motion, 2011, $4.5 \times 4.1 \times 6.1$ cm.