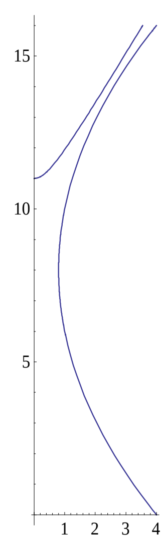
Voluminous Vessel

By Rob Grondahl

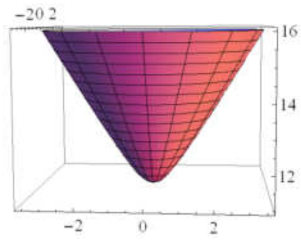
The goal of this project is to design a drinking vessel described by the volume of revolution of at least two curves. One or more curves will represent the outside of the vessel and one or more curves will represent the inside of the vessel. The region between the two curves or the outer curve and the axis of revolution makes up the material of our vessel. We are limited in several ways by our design constraints. The vessel must be between 10 cm. and 30 cm. tall. The base of the vessel must have a diameter between 5 cm. and 10 cm. The opening at the top must have a diameter between 7 cm. and 10 cm. The thickness of the rim of the vessel must be positive, but be less than 0.5 cm. Lastly, the functions must have non-constant derivative.

For my design, I decided to create a modernist take on a medieval chalice. I had the following image in mind as a typical chalice. In keeping with modernist design, I wanted remove any design flourishes while maintaining the basic geometry.

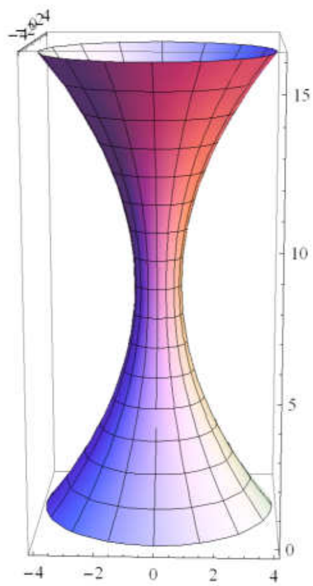
As such I wanted the liquid to be contained in the upper portion of the vessel with the surface of the vessel being described by a smooth curve. I therefore used a hyperbola for the inner curve that describes the portion holding the liquid, $I(y) = 0.6\sqrt{(y-10)^2 - 1}$, $11 \le y \le 16$ and a parabola for the outer curve that describes the exterior of the vessel, $O(y) = 0.2(0.5y-4)^2 + 0.8$, $0 \le y \le 16$. In both cases x and y are measured in cm. The following image shows these two curves.



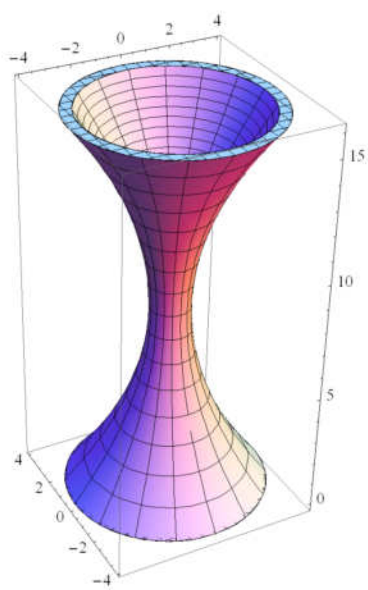
These curves were then revolved about the y-axis in order to create the vessel. The inner curve revolved about the y-axis looks like,



The outer curve revolved about the y-axis looks like,



Together the vessel looks like,



Visually the design accomplished my goal of a modernist chalice, but how about the design constraints? The vessel needed to be between 10 cm. and 30. cm tall. The outside curve of my chalice runs the full height of the vessel and it runs from $y=0$ to $y=16$, so my vessel is 16 cm. tall. The vessel needed to have a base with diameter between 5 cm. and 10 cm. The radius of the base of my vessel is determined by,

$$O(0)=0.2(0-4)^2+0.8=4$$

so the diameter of the base is 8 cm. The opening of the top of the vessel needed to have a diameter of between 7 cm and 10 cm. The radius of my opening is determined by,

$$I(16)=0.6\sqrt{(16-10)^2-1}\approx 3.55$$

so the diameter of the opening is approximately twice that or 7.10 cm. The thickness of the rim of the vessel had to be between 0 cm. and 0.5 cm. The thickness at the top of the vessel is determined by,

$$0(16)-I(16)=0.2(0.5\cdot 16-4)^2+0.8-0.6\sqrt{(16-10)^2-1}\approx 4-3.55=0.45\text{ cm}$$

Lastly, the curves had to have non-constant derivative,

$$I'(y)=\frac{0.6(y-10)}{\sqrt{(y-10)^2-1}}\neq C$$

and

$$O'(y)=0.2(0.5y-4)\neq C.$$

Now that the design requirements have been met, let's look at some of the resulting properties of our vessel. First of all, how much will it hold? Since the inside function describes the region that holds the liquid, we need to find the volume of revolution of that curve. Using the disk method of finding the volume of revolution, we have the following,

$$V_{\text{Liquid}}=\int_a^b \pi r^2 dy=\int_{11}^{16}\pi\left(0.6\sqrt{(y-10)^2-1}\right)^2 dy\approx 75.3982\text{ cm}^3\approx 2.5\text{ U.S. fl. oz.}$$

$$\left(\text{Calculation done by Mathematica: NIntegrate}[\pi*(0.6*((y-10)\wedge 2-1)\wedge .5)\wedge 2,\{y,11,16\}]\right)$$

Next we would like to know how much material it would take to make this vessel. Using the disk method, we calculated the volume generated by revolving $O(x)$ about the y-axis and then subtracted the volume generated by revolving $I(x)$ about the y-axis. This difference gave us the volume of material constituting the vessel.

$$V_{\text{Material}}=\int_a^b \pi r^2 dy=\int_0^{16}\pi\left(0.2(0.5y-4)^2+0.8\right)^2 dy-\int_{11}^{16}\pi\left(0.6\sqrt{(y-10)^2-1}\right)^2 dy$$
$$\approx 220.9-75.3982=145.5018\text{ cm}^3$$

$$\left(\begin{array}{l}\text{Calculation done by Mathematica:}\\ \text{NIntegrate}[\pi*(0.2(0.5y-4)\wedge 2+0.8)\wedge 2,\{y,0,16\}]-\text{NIntegrate}[\pi*(0.6*((y-10)\wedge 2-1)\wedge .5)\wedge 2,\{y,11,16\}]\end{array}\right)$$

Lastly, we would like to know the surface area of the outside of the vessel. This just requires us to find the surface area of the revolution of the curve $O(x)$

$$S = \int_0^{16} \left(0.2(0.5y-4)^2 + 0.8\right)\sqrt{1+(0.1y-.8)^2}\,dy \approx 34.0972 \text{ cm}^2$$

$$\left(\begin{array}{l}\text{Calculation done by Mathematica:} \\ \text{NIntegrate}[(0.2(0.5y-4)^\wedge 2+0.8)(1+(0.1y-0.8)^\wedge 2)^\wedge 0.5,\{y,0,16\}]\end{array}\right)$$

The finished design for the chalice seems to fit all of the design requirements and visually is comparable with what I set out to accomplish. I think the narrow stem with its gradually changing diameter will make it easy to hold. The parabolic inside may make it prone to spills over the side. Next time, I might try using inverse secant to get sides that are more vertical. In retrospect, I would have liked it to be a bit larger. I went for the low end of most of the design requirements. Next time I would have aimed for the high end. The fact that it only holds 2.5 oz. of liquid makes the vessel of limited use. I guess it could work as a futuristic take on a martini glass, but next time I would start with a target volume for the liquid and build the vessel around that.