Math 109 – Homework Assignment 8
Due Friday, June 1

1. Problem IV.6. Use the Euclidean algorithm to find the greatest common divisors of (i) 252 and 165, (ii) 4284 and 3480.

2. Problem IV.10. In each case of Problem IV.6. write the greatest common divisor as an integral linear combination of the two numbers.

3. Problem IV.14. Let a and b be positive integers. By the well-ordering principle the non-empty set of positive integers
$$\{\, am + bn \mid m, n \in \mathbb{Z} \text{ and } am + bn > 0 \,\}$$
has a minimum element c. Prove by contradiction that c is a common divisor of a and b and hence give an alternative proof of Theorem 17.1.1.

   Hint: Look at the appropriate remainder when dividing by c.

4. Problem IV.7. Let u_n be the nth Fibonacci number (for the definition see Definition 5.4.2). Prove that if $n \geq 2$ the Euclidean algorithm takes precisely $n - 1$ steps to prove that $\gcd(u_{n+1}, u_n) = 1$. Here we mean that the Euclidean algorithm, as written in class or in the book, takes $n - 1$ steps to complete.

The next exercise is for your interest, and does not need to be turned in. The problem is to prove Lamé's theorem which states that the Euclidean algorithm never takes more steps than 5 times the number of decimal digits in the smaller of the two positive integers you start with. Even just reading the statement and thinking about it is worthwhile.

- Problem IV.8. Suppose that a and b are two positive integers with $a \geq b$. Let $a_0, a_1, \ldots, a_n$ be the sequence of integers generated by the Euclidean algorithm so that $a_n = \gcd(a, b)$. Prove by induction on k that $a_{n-k} \geq u_{k+2}$ where u_m is the mth Fibonacci number. Deduce that $b \geq u_{n+1}$.

  Hence prove Lamé's theorem: if b has r decimal digits then $n \leq 5r$.

  [Prove that $\log_{10}\alpha > 1/5$ where $\alpha = (1 + \sqrt{5})/2$.]