Math 109 – Homework Assignment 7
Due Friday, May 25

1. Exercise 11.4. Prove that, if a and b are non-zero integers with $\gcd(a, b) = d$, then the integers a/d and b/d are *coprime* (see definition below).

2. Exercise 13.1. Prove that there does not exist a rational number whose square is 3.

   Note: Simply mimic the proof of Theorem 13.2.1. You may use without proof the fact that for integers a, a^2 is divisible by 3 if and only if a is divisible by 3 (we'll prove this later).

3. Exercise 14.1. Prove that if A is finite and B is denumerable then $A \cup B$ is denumerable.

4. Exercise 14.2. Prove that if A and B are denumerable then $A \cup B$ is denumerable. Hence prove by contradiction that, if X is uncountable and a subset A of X is denumerable, then $X - A$ is uncountable.

5. Exercise 14.3. Prove that if $\{ A_n \mid n \in \mathbb{N} \}$ is a denumerable set of pairwise disjoint denumerable sets then the union

   $$\bigcup_{n \in \mathbb{N}} A_n \overset{\text{def}^{\text{n}}}{=} \{ x \mid x \in A_n \text{ for some } n \in \mathbb{N} \}$$

   is also denumerable.

   Note: In other words, you are asked to prove that *a denumerable set of denumerable sets is denumerable*. This is the main part of showing that *a countable union of countable sets is countable* (which is easier to say out loud).

6. Problem III.28. Prove that for each $n \in \mathbb{N}$ the number of polynomials of degree n with rational coefficients is denumerable. Deduce that the set of algebraic numbers (see definition below) is denumerable.

   Note: You may use the fact that a polynomial equation $x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$ with real coefficients has at most n real roots. Note that the result shows that the set of non-algebraic real numbers (*transcendental* numbers) is uncountable.

Definition: Two integers a and b, not both zero, are said to be *coprime* (or *relatively prime*) when $\gcd(a, b) = 1$, i.e. when their only common factors are 1 and -1.

Definition: A real number is called *algebraic* if it satisfies a polynomial equation

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$$

with the coefficients $a_1, \ldots, a_n$ all rational.
Note: Any rational number q is algebraic, since it is a solution of the degree 1 polynomial equation $x - q = 0$. The number $\sqrt{2}$ is also algebraic, since it is a solution to the equation $x^2 - 2 = 0$. It is possible to show that π and e are not algebraic numbers. Real numbers which are not algebraic are called *transcendental* numbers.