Math 109 – Homework Assignment 5
Due Friday, May 11

1. Exercise 9.5. Suppose that $f : X \to Y$ and $g : Y \to Z$ are bijections of sets. Prove that the composite $g \circ f : X \to Z$ is also a bijection and that
$$(g \circ f)^{-1} = f^{-1} \circ g^{-1} : Z \to X.$$

2. Exercise 9.7. Let $f : X \to Y$ be a function and $B_1, B_2 \in \mathcal{P}(Y)$. Prove that
   (i) $B_1 \subseteq B_2 \;\Rightarrow\; f^{-1}(B_1) \subseteq f^{-1}(B_2)$,
   (ii) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$,
   (iii) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

   Prove that the converse of the first statement is not universally true (by constructing a counterexample).

   Note: The textbook uses the notation $\overleftarrow{f}(B)$ for the preimage of B under f, i.e. $f^{-1}(B)$. But this arrow notation is not used by anyone else.

3. Problem II.19. Let $f : X \to Y$ be a function. Prove that there exists a function $g : Y \to X$ such that $f \circ g = I_Y$ if and only if f is a surjection. [Such a function g is called a *right inverse* of f.]
   Notation: Here $I_Y : Y \to Y$ is the *identity function* on Y, defined by $I_Y(y) = y$ for all $y \in Y$.

4. Exercise 10.4. Each of a collection of 144 tiles is either triangular or square, either red or blue, and either wooden or plastic. Given that there are 68 wooden tiles, 69 red tiles, 75 triangular tiles, 36 red wooden tiles, 40 triangular wooden tiles, 38 red triangular tiles, and 23 red wooden triangular tiles, how many blue plastic square tiles are there?

5. Exercise 11.6. Prove by induction on n that if A is a set of positive integers without a least element then $\mathbb{N}_n \subseteq \mathbb{N} - A$ for every n so that A is the empty set.

   Deduce the well-ordering principle: every non-empty set of positive integers has a least element.

6. Problem III.3. At an international conference of 100 people, 75 speak English, 60 speak Spanish and 45 speak Swahili (and everyone present speaks at least one of these languages).
   (i) What is the maximum possible number of these people who can speak only one language? In this case how many people speak only English, how many speak only Spanish, how many speak only Swahili, and how many speak all three?
   (ii) What is the maximum number of people who can speak only English? In this case what can be said about the number who speak only Spanish and the number who speak only Swahili?
   (iii) Prove that the greater the number of people who speak all three languages, the greater the number of people who speak only one language.