Math 109 – Homework Assignment 4
Due Friday, May 4

1. Problem II.6. Use the distributivity law to prove that
$$(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A).$$
[Hint: There is a similar problem worked out in the book. See the proof Proposition 6.3.5 on page 72.]

2. Problem II.7. For subsets of a universal set U prove that $B \subseteq A^c$ if and only if $A \cap B = \emptyset$. By taking complements deduce that $A^c \subseteq B$ if and only if $A \cup B = U$. Deduce that $B = A^c$ if and only if $A \cap B = \emptyset$ and $A \cup B = U$.

3. Problem II.8. Let X be a set. Given sets $A, B \in \mathcal{P}(X)$, their *symmetric difference* $A \Delta B$ is defined by
$$A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$
Prove that
   (i) the symmetric difference is associative, $(A \Delta B) \Delta C = A \Delta (B \Delta C)$ for all $A, B, C \in \mathcal{P}(X)$,
   (ii) there exists a unique set $N \in \mathcal{P}(X)$ such that $A \Delta N = A$ for all $A \in \mathcal{P}(X)$ [Hint: Guess what N is!],
   (iii) for each $A \in \mathcal{P}(X)$, there exists a unique $A' \in \mathcal{P}(X)$ such that $A \Delta A' = N$,
   (iv) for each $A, B \in \mathcal{P}(X)$, there exists a unique set C such that $A \Delta C = B$.

4. Problem II.9. Using the notation in the previous problem, prove that for sets $A, B, C, D \in \mathcal{P}(X)$
$$A \Delta B = C \Delta D \;\Leftrightarrow\; A \Delta C = B \Delta D.$$

5. Exercise 7.4. Prove or disprove the following statements.
   (i) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0.$
   (ii) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 0.$
   (iii) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 0.$
   (iv) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy = 0.$
   (v) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1.$
   (vi) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy = 1.$
   (vii) $\forall n \in \mathbb{N}$, (n is even or n is odd).
   (viii) ($\forall n \in \mathbb{N}$, n is even) or ($\forall n \in \mathbb{N}$, n is odd).

   Comment: It is a good idea to make sure you can also do Problem II.11., but Problem II.11. is not assigned as homework to be turned in.

6. Exercise 8.2. Define functions $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$ and $g(x) = 1 - x$. Find the functions (i) $f \circ f$, (ii) $f \circ g$, (iii) $g \circ f$, (iv) $g \circ g$.

   List the elements of the set $\{ x \in \mathbb{R} \mid f \circ g(x) = g \circ f(x) \}$.

7. Exercise 8.3. Find functions $f_i : \mathbb{R} \to \mathbb{R}$ with images as follows:

   (i) $\mathrm{Im} f_1 = \mathbb{R}$;

   (ii) $\mathrm{Im} f_2 = \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the positive real numbers;

   (iii) $\mathrm{Im} f_3 = \mathbb{R} - \mathbb{Z}$;

   (iv) $\mathrm{Im} f_4 = \mathbb{Z}$.

   Notation: Here $\mathrm{Im} f_i = f_i(\mathbb{R})$.

8. Problem II.15. Let X be a set. Given $A \in \mathcal{P}(X)$ define the *characteristic function* $\chi_A : X \to \{0, 1\}$ by

$$\chi_A(x) = \begin{cases} 0, & \text{if } x \in X - A \\ 1, & \text{if } x \in A. \end{cases}$$

   Suppose that A and B are subsets of X.

   (i) Prove that the function $x \mapsto \chi_A(x)\chi_B(x)$ (multiplication of integers) is the characteristic function of the intersection $A \cap B$.

   (ii) Find the subset C whose characteristic function is given by

$$\chi_C(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x).$$