Math 109 – Homework Assignment 3
Due Friday, April 27

Notation: (cf. Exercise 6.1.) Let a, b be real numbers with $a < b$. Then the *open interval* from a to b, denoted (a, b), is defined by

$$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}.$$

1. Exercise 6.2. Prove that
   (i) $\{ x \in \mathbb{R} \mid x^2 + x - 2 = 0 \} = \{1, -2\}$,
   (ii) $\{ x \in \mathbb{R} \mid x^2 + x - 2 < 0 \} = (-2, 1)$,
   (iii) $\{ x \in \mathbb{R} \mid x^2 + x - 2 > 0 \} = \{ x \in \mathbb{R} \mid x < -2 \} \cup \{ x \in \mathbb{R} \mid x > 1 \}$.
2. Exercise 6.4. By using a truth table prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Draw a Venn diagram to illustrate the proof.
3. Exercise 6.5. Prove that
   (i) $A \subseteq B \;\Leftrightarrow\; A \cup B = B$,
   (ii) $A \subseteq B \;\Leftrightarrow\; A \cap B = A$.
4. Exercise 6.6. Prove by contradiction that, if $A \cap B \subseteq C$ and $x \in B$, then $x \notin A - C$. [Work from the definitions of '$A \cap B$', '$A - C$', and '$\subseteq$'.]
5. Exercise 6.7. Using the fact that an implication is equivalent to its contrapositive, prove that, for subsets of a universal set U, $A \subseteq B$ if and only if $B^c \subseteq A^c$.
6. Problem II.4. Prove by contradiction or otherwise that for sets A, B, C,

$$A \cap B = A \cap C \text{ and } A \cup B = A \cup C \quad \Leftrightarrow \quad B = C.$$

The following problems are not assigned for this week, but **will be on Homework Assignment 4**. You may want to start thinking about them this week.

- Problem II.6. Use the distributivity law to prove that

  $$(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A).$$

  [Hint: There is a similar problem worked out in the book. See the proof Proposition 6.3.5 on page 72.]

- Problem II.7. For subsets of a universal set U prove that $B \subseteq A^c$ if and only if $A \cap B = \emptyset$ By taking complements deduce that $A^c \subseteq B$ if and only if $A \cup B = U$. Deduce that $B = A^c$ if and only if $A \cap B = \emptyset$ and $A \cup B = U$.

- Problem II.8. Let X be a set. Given sets $A, B \in \mathcal{P}(X)$, their *symmetric difference* $A \Delta B$ is defined by

  $$A \Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

  Prove that

  (i) the symmetric difference is associative, $(A \Delta B) \Delta C = A \Delta (B \Delta C)$ for all $A, B, C \in \mathcal{P}(X)$,

  (ii) there exists a unique set $N \in \mathcal{P}(X)$ such that $A \Delta N = A$ for all $A \in \mathcal{P}(X)$ [Hint: Guess what N is!],

  (iii) for each $A \in \mathcal{P}(X)$, there exists a unique $A' \in \mathcal{P}(X)$ such that $A \Delta A' = N$,

  (iv) for each $A, B \in \mathcal{P}(X)$, there exists a unique set C such that $A \Delta C = B$.

- Problem II.9. Using the notation in the previous problem, prove that for sets $A, B, C, D \in \mathcal{P}(X)$

  $$A \Delta B = C \Delta D \;\Leftrightarrow\; A \Delta C = B \Delta D.$$