Math 109 – Homework Assignment 2
Due Friday, April 20

1. Excercise 4.1. Prove by contradiction that there do not exist integers m and n such that $14m + 21n = 100$.

2. (cf. Exercise 4.4)

   (a) Let P, Q, R be three statements. Draw a truth table for $P \Rightarrow (Q \vee R)$ and also for $(P \wedge (\neg Q)) \Rightarrow R$. Conclude that

   $$[P \Rightarrow (Q \vee R)] \;\Leftrightarrow\; [(P \wedge (\neg Q)) \Rightarrow R]\,.$$

   (b) Use the above observation to prove that if a is a real number and $a^2 \geq 7a$, then $a \leq 0$ or $a \geq 7$. In other words, prove the desired implication by showing that if a is a real number with $a^2 \geq 7a$ and $a > 0$, then $a \geq 7$.

3. Exercise 4.7. Prove that, for real numbers a and b,

   $$|a + b| \leq |a| + |b|.$$

   Give a necessary and sufficient condition for equality.

4. Exercise 5.1. Prove by induction that, for positive integers n, $n^3 - n$ is divisible by 3.

5. Exercise 5.6. For non-negative integers n define the number u_n inductively as follows. Let

   $$\begin{aligned} u_0 &= 0, \\ u_{k+1} &= 3u_k + 3^k \quad \text{for } k \geq 0. \end{aligned}$$

   Prove that $u_n = n3^{n-1}$ for all non-negative integers n.

6. Problem I.8. Prove the following statements concerning a real number x.

   (i) $x^2 - x - 2 = 0 \;\Leftrightarrow\; x = -1$ or $x = 2$.

   (ii) $x^2 - x - 2 > 0 \;\Leftrightarrow\; x < -1$ or $x > 2$.

7. Problem I.11. Prove by contradiction that there does not exist a smallest positive real number.

8. Problem I.12. Prove by induction on n that, for all positive integers n, 3 divides $4^n + 5$.

9. Problem I.19. Prove that

   $$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

   for integers $n \geq 2$. *Note:* Given a natural number n and real numbers $a_1, a_2, \ldots a_n$, the product $a_1 a_2 \cdots a_n$ is denoted $\prod_{i=1}^{n} a_i$. If $n \geq 2$, then $\prod_{i=2}^{n} a_i$ simply means the product $a_2 \cdots a_n$.

10. Problem I.25. Let F_n denote the n^{th} Fibonacci number (Definition 5.4.2). Prove, by induction on n (without using the Binet formula), that

$$F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$$

for all natural numbers m and n (take $F_0 = 0$).
Deduce, again using induction on n, that for natural numbers m and n, F_m divides F_{nm}.