Name: ____________________

Math 109
Final Examination
June 11, 2007

Turn off and put away your cell phone.
You may use a calculator; but no other electronic devices are allowed during this exam.
You may use one page of notes, but no books or other assistance during this exam.
Read each question carefully, answer each question completely, and show all of your work.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

#	Points	Score
1	6	
2	6	
3	6	
4	6	
5	8	
6	6	
7	6	
8	6	
Σ	50	

1. (a) (3 points) A function $f : A \to B$ is injective if and only if for every pair of elements a_1, a_2 in A such that $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$. Write what it means for a function to not be injective.

(b) (3 points) A function $f : A \to B$ is surjective if and only if for every element b in B, there exists an element a in A such that $f(a) = b$. Write what it means for a function to not be surjective.

2. (6 points) Let a and b be positive integers such that $\gcd(a, b) = 1$.

(a) Prove that if m is a positive integer such that a divides m and b divides m, then ab divides m.

(b) Exhibit an example that shows that $\gcd(a, b) = 1$ is necessary. That is, find a pair of positive integers a and b that are not coprime and an integer m such that a divides m and b divides m, but ab does *not* divide m.

3. (6 points) Let $u_1, u_2, u_3, \ldots$ be the Fibonacci numbers, defined by the recurrence $u_1 = 1$, $u_2 = 1$, and $u_{n+2} = u_n + u_{n+1}$ for each n in $\mathbb{N}$. Prove that

$$u_1^2 + \cdots + u_n^2 = u_n u_{n+1}$$

for every natural number n.

4. (a) (6 points) Use the Euclidean Algorithm to find $\gcd(230381, 222503)$.

   (b) Use the results found above to write $\gcd(230381, 222503)$ as a linear combination of 230381 and 222503 with integer coefficients.

5. Let $\mathcal{S} = \mathbb{R}$. For $x, y \in S$, let $x \simeq y$ if and only if $y - x \in \mathbb{Z}$.

(a) (6 points) Prove that $\simeq$ is an equivalence relation on $\mathcal{S}$.

(b) (2 points) Characterize $[0]$, the equivalence class of 0 in $\mathcal{S}$.

6. (6 points) Let m be a positive integer. Recall that $\mathbb{Z}_m$ is the set of congruence classes modulo m.

(a) Show that the function $f : \mathbb{Z}_m \to \mathbb{Z}_m$ given by $f([a]) = [6a]$ is well-defined. That is, show that if $[a] = [b]$, then $f([a]) = f([b])$.

(b) Show that f is injective if and only if $\gcd(6, m) = 1$.

7. (6 points) Prove that a positive integer is divisible by 9 if and only if the sum of its decimal digits is divisible by 9. That is, prove that if $0 \le a_k \le 9$ for $0 \le k \le n$, then $\sum_{k=0}^{n} a_k \, 10^k$ is divisible by 9 if and only if $\sum_{k=0}^{n} a_k$ is divisible by 9.

8. (6 points) In a group of 26 people, 15 speak English, 13 speak Spanish and 8 speak Chinese. Of these, 5 speak both English and Spanish, 3 speak both English and Chinese, and 3 speak both Spanish and Chinese.

   (a) How many people in the group speak all three languages?

   (b) How many people in the group speak only Chinese?