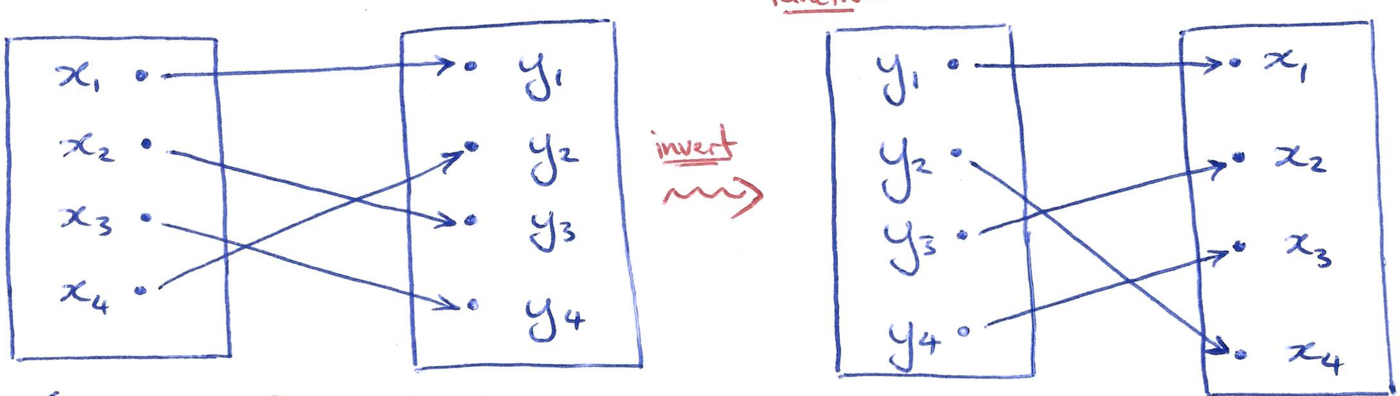
Chapter 9: Injections, Surjections and Bijections

Question: When does a function $f: X \to Y$ have an inverse?

meaning a function $g: Y \to X$ such that $g \circ f(x) = x \;\; \forall x \in X$ and $f \circ g(y) = y \;\; \forall y \in Y$.

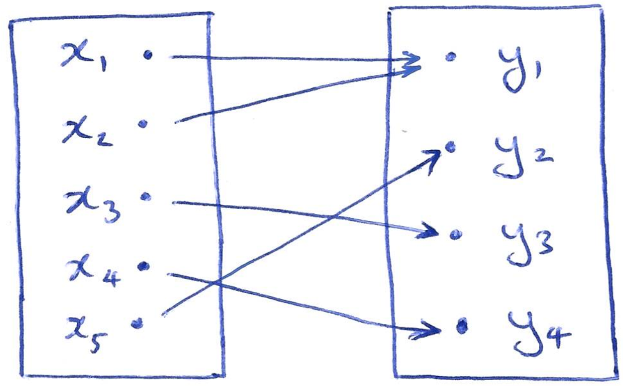
Let's consider some simple examples.

① $f: X \to Y$ inverse function $f^{-1}: Y \to X$



$X = \{x_1, x_2, x_3, x_4\}$ $Y = \{y_1, y_2, y_3, y_4\}$

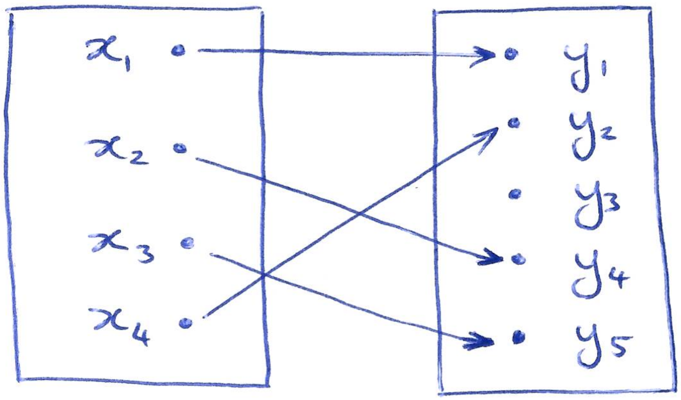
② $f: X \to Y$



- If we wanted to invert f, where would we send y_1?
- We say that f is not injective.

③

$f : X \to Y$



- If we wanted to invert f, where would we send y_3?
- We say that f is <u>not surjective</u>.

<u>Def^n</u>: Let $f : X \to Y$ be a function.

(i) We say that f is an <u>injection</u> (or is <u>injective</u>, or is <u>one-to-one</u>) if for any $y \in Y$ the equation $f(x) = y$ has <u>at most one solution</u>.

In other words, f is an injection if

$$\forall a, b \in X, \ (a \neq b \Rightarrow f(a) \neq f(b))$$

or, equivalently,

$$\forall a, b \in X, \ (f(a) = f(b) \Rightarrow a = b).$$

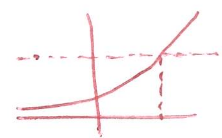
(ii) We say that f is a <u>surjection</u> (or is <u>surjective</u>, or is <u>onto</u>) if for any $y \in Y$ the equation $f(x) = y$ has <u>at least one solution</u>.

In other words, f is a surjection if $f(X) = Y$.

(iii) We say that f is a <u>bijection</u> if it is injective and surjective. (We also say f is <u>bijective</u> or <u>one-to-one and onto</u>.)

<u>Examples:</u>

- The function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=e^x$ is <u>injective</u>, but not surjective.
  ↑ passes the "horizontal line test"



- The function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=x+1$ is a bijection.

<u>Remark</u>: Whether f is an injection/surjection depends on the choice of domain & codomain.

- $f:\mathbb{R}\to\mathbb{R}^{\geq 0}$ given by $f(x)=x^2$ is surjective, but not injective.
- $f:\mathbb{R}^{\geq 0}\to\mathbb{R}$ given by $f(x)=x^2$ is injective, but not surjective.
- $f:\mathbb{R}^{\geq 0}\to\mathbb{R}^{\geq 0}$ given by $f(x)=x^2$ is a bijection.

Problem: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 4x^2 - 4x + 2$. Is f injective, surjective, or bijective?

Solution:

Let $b \in \mathbb{R}$, then $\forall x \in \mathbb{R}$,

$$4x^2 - 4x + 2 = b \iff 4x^2 - 4x + (2-b) = 0.$$

The discriminant of $4x^2 - 4x + (2-b)$ is $16 - 16(2-b) = 16(b-1)$. Thus

quadratic formula

(i) if $b > 1$, $f(x) = b$ has two solutions;

(ii) if $b = 1$, $f(x) = b$ has one solution;

(iii) if $b < 0$, $f(x) = b$ has no solutions $(x \in \mathbb{R})$.

So f is not injective and not surjective.

Def^n: Let $f : X \to Y$. We say f is invertible if there is a function $g : Y \to X$ such that

$$g \circ f(x) = x \quad \forall x \in X, \text{ and}$$
$$f \circ g(y) = y \quad \forall y \in Y.$$

In other words,

$$\underbrace{\forall x \in X, y \in Y}_{\forall x \in X,\ \forall y \in Y}, \quad \big(f(x) = y \iff x = g(y) \big).$$

Such a function $g : Y \to X$ is called an <u>inverse</u> of $f : X \to Y$.

<u>Proposition:</u> If $f : X \to Y$ is invertible, then f has a unique inverse $g : Y \to X$.

<u>Proof:</u>

Let $f : X \to Y$ be an invertible function.
Let $g_1, g_2 : Y \to X$ be two inverses of f (we do not assume they are different functions, indeed, we will prove they are the same).
Then $\forall y \in Y$

$$g_2(y) = \overbrace{g_1 \circ f}\circ g_2(y) = g_1(y).$$

Hence $g_1 = g_2$. $\square$

<u>Notation:</u> If $f : X \to Y$ is invertible, the inverse function is written $f^{-1} : Y \to X$.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x+1$, then $f^{-1}: \mathbb{R} \to \mathbb{R}$ is given by $\underbrace{f^{-1}(y) = \frac{y-1}{2}}$.

(often we would write $f^{-1}(x) = \frac{x-1}{2}$)

For $x, y \in \mathbb{R}$,
$y = 2x+1 \iff y-1 = 2x \iff x = \frac{y-1}{2}$.

Theorem: Let $f: X \to Y$.
The function f is invertible if and only if f is a bijection.

Proof:

("$\Leftarrow$") Reverse implication: Suppose f is a bijection. Then for each $y \in Y$ there is a unique element $x \in X$, which we may denote by $g(y)$, such that $f(x) = y$ ($f(g(y)) = y$). We define $g: Y \to X$ by $y \mapsto g(y)$, with $g(y)$ the unique element of X s.t. $f(g(y)) = y$. By the definition of g, $\forall x \in X, \forall y \in Y$,

$$f(x) = y \iff x = g(y),$$

so g is the inverse of f.

("$\Rightarrow$") Forward implication: Suppose now instead that f is invertible. To see that f is surjective take any $b \in Y$, then $f(x) = b$ where $x = g(b)$. So f is indeed surjective. To see that f is injective, let $x_1, x_2 \in X$, then

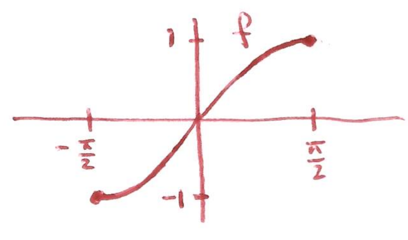
$$f(x_1) = f(x_2) \implies f^{-1}(f(x_1)) = f^{-1}(f(x_2))$$
$$\implies x_1 = x_2 .$$

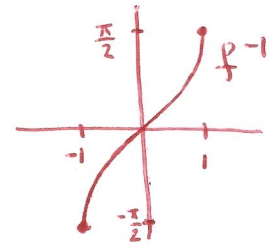
So f is injective. Hence f is a bijection. $\square$

<u>Examples</u> (Trigonometric Functions):

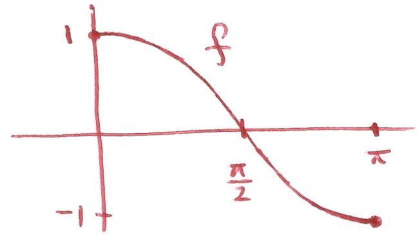
- Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin(x)$. Is f invertible? No.
  - We need to restrict the codomain to be the image $[-1, 1] = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$.
  - We also need to restrict the domain so that the function becomes a bijection (currently it is <u>not injective</u>).

- Let $f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ be given by $f(x) = \sin(x)$. Then f is invertible. The inverse is written as $f^{-1} = \sin^{-1}$ or as $f^{-1} = \arcsin$.

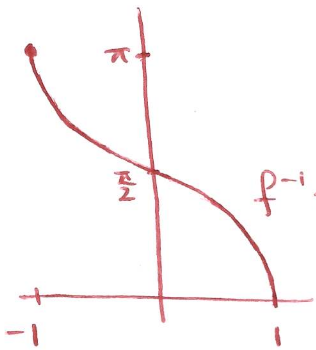




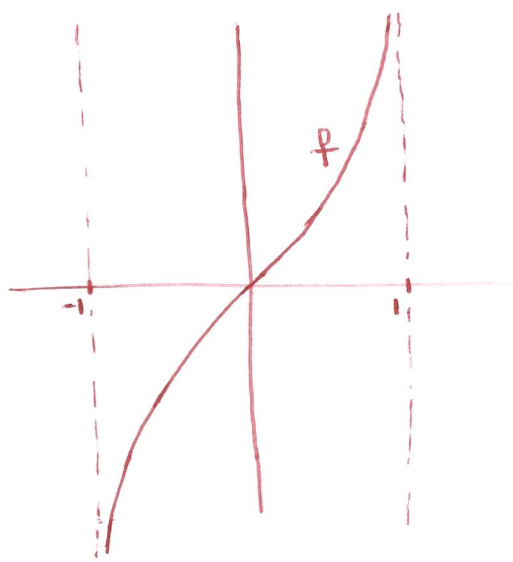
- The function $f : [0, \pi] \to [-1, 1]$ given by $f(x) = \cos(x)$ is a bijection.



The inverse function is written as $\cos^{-1}$ or arccos ($\cos^{-1} : [-1, 1] \to [0, \pi]$).



- The function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by $f(x) = \tan(x)$ is a bijection.



<u>Remark:</u> Note that f puts $(-\frac{\pi}{2}, \frac{\pi}{2})$ in *one-to-one correspondence* with $\mathbb{R}$. We say that f puts these two sets in bijection.

The inverse function is written as $\tan^{-1}$ or arctan ($\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$).

Question: Let $n \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^n$. Is f invertible?

Answer:

- If n is even, $(-1)^n = 1 = 1^n$, so f is not injective.
- If n is odd then f is a bijection with inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = x^{\frac{1}{n}}$.

Exercise: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x^3 - 1}{2}$. Is f invertible? If so, find the inverse.

Images & Preimages:

Images: Let $f : X \to Y$ be a function. We already defined the image $f(X)$ of f. If A is a subset of X then we similarly define

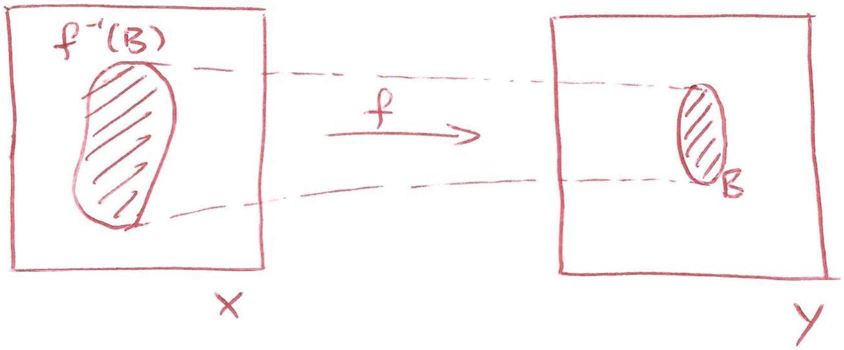
$$\underline{f(A)} = \{ f(a) \mid a \in A \}.$$

the image of A under f.

Remark: The textbook uses the notation $\vec{f}(A)$ (which is not used by anyone else!).

Preimages: Let $f: X \to Y$ be a function.
Given $B \subseteq Y$ we define the preimage of B under f, denoted $f^{-1}(B)$, by
or "inverse image"

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \}.$$



Remarks:

- The function f need not be invertible for $f^{-1}(B)$ to be defined. If f is invertible then the preimage of B under f is the same as the image of B under f^{-1}, so the notation $f^{-1}(B)$ is unambiguous.

- The textbook uses the notation $\overleftarrow{f}(B)$ rather than $f^{-1}(B)$. (But it is usual to write $f^{-1}(B)$.)

- $f^{-1}(Y) = f^{-1}(f(X)) = X$

Exercise: Let $f: X \to Y$ be a function, $A \subseteq X$ and $B \subseteq Y$. Prove that $A \subseteq f^{-1}(f(A))$ and $f(f^{-1}(B)) \subseteq B$. Can these inclusions be proper?

__Example:__ Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then

- $f(\underbrace{[2, \infty)}_{\{x \in \mathbb{R} \mid x \geq 2\}}) = \underbrace{[4, \infty)}_{\{x \in \mathbb{R} \mid x \geq 4\}}$;

- $f^{-1}([2, \infty)) = \underbrace{(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)}_{\{x \in \mathbb{R} \mid x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}\}}$;

- $f^{-1}(\{1\}) = \{-1, 1\}$.