Chapter 8: Functions

Def^n: Let X, Y be sets. A function (or map) f from X to Y is the assignment to each $x \in X$ a single element $y \in Y$, denoted $y = f(x)$.

the image of x under f
(or the value of f at x)

Notation: If f is a function from X to Y, we write $f: X \to Y$.

domain of f — codomain of f

It is usual to always specify the domain and the codomain of f in this way.

Examples:

- The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

- The function $f: \mathbb{R} \to \mathbb{R}$ given by

absolute value function

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

- The function $f: \mathbb{N} \to \mathbb{N}$ given by $n \mapsto f(n)$ where $f(n)$ is the number of primes less than n.

  "maps to"

  Not all functions are given by a formula!

  Here $f(1) = 0$, $f(2) = 0$, $f(3) = 1$, $f(4) = 2$, $f(5) = 2$, $f(6) = 3$, ...

<u>Remark</u>: Sometimes (especially in calculus courses) a function is given by a formula (like $f(x) = \frac{x^2}{x-1}$) and the domain and codomain are not specified. In this case, the convention is to take the domain to be the subset of $\mathbb{R}$ where the formula makes sense, and the codomain to be $\mathbb{R}$. E.g., for $f(x) = \frac{x^2}{x-1}$ we take the domain to be $\mathbb{R} - \{1\}$ and the codomain to be $\mathbb{R}$.

<u>Def^n</u>: Let f, g be two functions. We say that f and g are <u>equal</u> ($f = g$) if they have the same domain X & codomain Y, and $\forall x \in X$, $f(x) = g(x)$.

Def^n: Let $f: X \to Y$ be a function and $A \subseteq X$. Then we define a new function $g: A \to Y$ by $g(a) = f(a)$ $\forall a \in A$. We call g the restriction of f to A, denoted $g = f|_A$.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$, and $h: \mathbb{R} \to \mathbb{R}$ be given by $h(x) = |x|$. Then

$$f|_{\mathbb{R}^{\geq 0}} = h|_{\mathbb{R}^{\geq 0}}.$$

Here $\mathbb{R}^{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$.

Def^n: Let $f: X \to Y$ be a function. The image of f, denoted $f(X)$, is defined by

$$f(X) = \{f(x) \mid x \in X\}.$$

Note: $f(X) \subseteq Y$

$\to$ When $f(X) = Y$ we say f is surjective or onto.

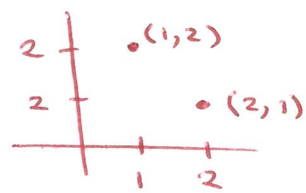
Example: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin(x)$. Then $f(\mathbb{R}) = \underbrace{[-1, 1]}_{\{x \in \mathbb{R} \mid -1 \leq x \leq 1\}}$.

The Cartesian Product of Two Sets

Def^n: Given sets X, Y, the product of X and Y, denoted $X \times Y$, is the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. That is,

$$X \times Y = \{ (x,y) \mid x \in X, y \in Y \}.$$

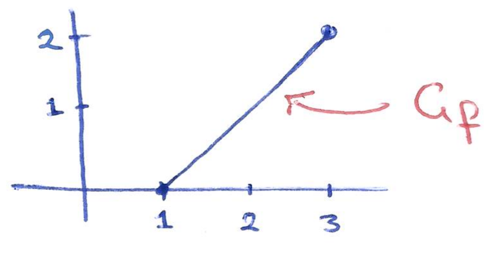
Example: $\mathbb{R} \times \mathbb{R}$ is denoted $\mathbb{R}^2$. Since we take the set of all ordered pairs of real numbers $(1,2) \neq (2,1)$.



Def^n: Let $f: X \to Y$ be a function. The graph of f, denoted G_f, is the set

$$G_f = \{ (x, \underbrace{f(x)}_{\in Y}) \mid x \in X \} \subseteq X \times Y.$$

Example: Let $f: [1,3] \to \mathbb{R}$ be given by $f(x) = x - 1$.



Composition of Functions

Def^n: Given two functions $f: X \to Y$ and $g: Y \to Z$, the composite of f and g, denoted by $g \circ f : X \to Z$, is defined by

$$(g \circ f)(x) = g(f(x)) \qquad \forall x \in X.$$

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^4$ and $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x+1$. Then, $\forall x \in \mathbb{R}$,

order matters!

$$f \circ g\,(x) = (x+1)^4,$$
$$g \circ f\,(x) = x^4 + 1,$$
$$f \circ f\,(x) = x^{16}, \quad \text{and}$$
$$g \circ g\,(x) = x + 2.$$

Proposition: Let $f: X \to Y$, $g: Y \to Z$, and $h: Z \to W$ be functions. Then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof: Both functions are maps $X \to W$, and both map $x \in X$ to $h(g(f(x)))$. $\square$