Chapter 7: Quantifiers

Let $P(a)$ be a predieate.

Common questions:

- When is $P(a)$ tue? (What is the set of elements a such that $P(a)$ is true?)
- Can $P(a)$ ever be false? (Or is it always true?)
- Can $P(a)$ ever be true? (Or is it always false?)

→ We use quantifiers when answering these questions about $P(a)$.

Universal Statements: Let A be a set.

Defⁿ: The notation "$\forall a \in A, P(a)$" means

(i) For each element a in the set A, the statement $P(a)$ is true;

or, equivalently,

(ii) $P(a)$ is true for all $a \in A$;

or, equivalently,

(iii) $\{a \in A \mid P(a)\} = A$;

or, equivalently,

(iv) $a \in A \Rightarrow P(a)$.

Examples:

- $\forall a \in \mathbb{R} - \{0\}, \quad a^2 > 0.$
- $\forall a \in \mathbb{N}, \quad 2a+1 \geq 3.$
- ("Universal Implication") $\forall n \in \mathbb{N}, \quad n > 2 \Rightarrow n > 0.$

Existence statements:

Def^n: The notation "$\exists a \in A, P(a)$" means

(i) For some (ie. at least one) element a in the set A, $P(a)$ is true;

or, equivalently,

(ii) There exists $a \in A$ such that $P(a)$ is true;

or, equivalently,

(iii) $\{a \in A \mid P(a)\} \neq \emptyset$;

or, equivalently,

(iv) $x \in A \not\Rightarrow \neg P(a)$.

Examples:

- $\exists a \in \mathbb{R}, \quad a^2 = 2.$ ⇜ $(a = \pm\sqrt{2})$
- $\exists b \in \mathbb{Z}, \left(\forall a \in \mathbb{Z}, a+b=a\right).$ ⇜ $(\underbrace{b=0}_{\in})$

In brief:

- $\forall$ reads as "for all".
- $\exists$ reads as "there exists".
- $\nexists$ reads as "there does not exist."

↳ Example: $\nexists\, a \in \mathbb{Q},\ a^2 = 2$.

Remark: You should use these symbols often. There are 3 ways to use a predicate $P(a)$ in a sentence:

1. Quantify that $P(a)$ holds for all $a \in A$.
2. Quantify that $P(a)$ holds for some $a \in A$.
3. Fix the free variable "a" so that $P(a)$ becomes a proposition.

↳ Examples:

- Let $a = 6$. $\underbrace{\text{The number } a \text{ is divisible by } 3}_{P(a)}$.
- Let a be an integer. $\underbrace{[\ \cdots\ ]}_{P(a)}$ ← some statement about integers.

How do we negate statements involving quantifiers?

→ Recall that to disprove a "for all" claim, one just needs to come up with a counterexample. To disprove a "there exists" claim one has to prove that in every case the desired property fails to hold.

(To disprove a statement is to prove its negation.)

Negation of Statements Involving Quantifiers

Proposition	Negation
$\forall a \in A,\ P(a)$	$\exists a \in A,\ \neg P(a)$
$\exists a \in A,\ P(a)$	$\forall a \in A,\ \neg P(a)$
$\forall a \in A, \exists b \in B,\ P(a,b)$	$\exists a \in A, \forall b \in B,\ \neg P(a,b)$
$\exists a \in A, \forall b \in B,\ P(a,b)$	$\forall a \in A, \exists b \in B,\ \neg P(a,b)$
$\forall a \in A, \forall b \in B,\ P(a,b)$	$\exists a \in A, \exists b \in B,\ \neg P(a,b)$
$\exists a \in A, \exists b \in B,\ P(a,b)$	$\forall a \in A, \forall b \in B,\ \neg P(a,b)$

Note: Order matters! (Dependence)

$$\neg(\forall a \in A, \exists b \in B, P(a,b)) \iff \exists a \in A, \neg(\exists b \in B, P(a,b))$$
$$\iff \exists a \in A, \forall b \in B, \neg P(a,b)$$

Examples

	Proposition	Negation	
False. →	$\forall x\in\mathbb{R},\ x>0$	$\exists x\in\mathbb{R},\ x\leq 0$	← True.
True. →	$\exists x\in\mathbb{R},\ x>0$	$\forall x\in\mathbb{R},\ x\leq 0$	← False.
False. →	$\forall x\in\mathbb{R},\exists y\in\mathbb{R},\ xy>0$	$\exists x\in\mathbb{R},\forall y\in\mathbb{R},\ xy\leq 0$	← True. $(x=0)$
False. →	$\exists x\in\mathbb{R},\forall y\in\mathbb{R},\ xy>0$	$\forall x\in\mathbb{R},\exists y\in\mathbb{R},\ xy\leq 0$	← True. (given $x\in\mathbb{R}$ you can take, e.g., $y=-x$)
False. →	$\underbrace{\forall x,y\in\mathbb{R}}_{\forall x\in\mathbb{R},\forall y\in\mathbb{R}},\ xy>0$	$\underbrace{\exists x,y\in\mathbb{R}}_{\exists x\in\mathbb{R},\exists y\in\mathbb{R}},\ xy\leq 0$	← True.
True. →	$\exists x,y\in\mathbb{R},\ xy>0$	$\forall x,y\in\mathbb{R},\ xy\leq 0$	← False.

Proving / Disproving Statements Involving Quantifiers:

- To prove $\forall a\in A, P(a)$ we fix an arbitrary element $a\in A$ ("Let $a\in A$.") and prove that $P(a)$ is true.
- To prove $\exists a\in A, P(a)$ we give an example of an element $a\in A$ such that $P(a)$ is true.

- To disprove $\forall a \in A, P(a)$ we need to prove its negation $\exists a \in A, \neg P(a)$. That is, we need to give a counterexample.

- To disprove $\exists a \in A, P(a)$ we need to prove its negation $\forall a \in A, \neg P(a)$.

Examples:

- Prove or disprove: $\forall m \in \mathbb{N}\ \exists n \in \mathbb{N}, m < n$.

The statement is true. Proof: Given $m \in \mathbb{N}$, if we let $n = m+1$, then $n \in \mathbb{N}$ and $m < n$. $\square$

- Prove or disprove: $\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}, m < n$.

The statement is false. Proof: We prove that $\underbrace{\exists n \in \mathbb{N}\ \forall m \in \mathbb{N}, m \geq n}$. Take $n = 1$. Then $\forall m \in \mathbb{N}$ $m \geq n$. $\square$

the negation

- Prove or disprove: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy = 0$.

The statement is true. Proof: Take $x = 0 \in \mathbb{R}$. Then $\forall y \in \mathbb{R}, xy = 0$. $\square$

- Prove or disprove: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy = 1$.

The statement is false. Proof: We prove that $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy \neq 1$. Given any $x \in \mathbb{R}$, take $y = 0$, then $xy = 0 \neq 1$. $\square$

the negation