Chapter 6: Set Theory

Def^n: A set is a well-defined collection of objects. By well-defined we mean that it is clearly specified which "objects" are part of the set, and which are not.

Examples:

- The collection of all even integers is a set.
- The collection of all tall trees is not a set. (Which trees are "tall"?)

Some Important Sets:

- $\mathbb{N}$ denotes the set of all natural numbers (counting numbers).
- $\mathbb{Z}$ denotes the integers. (Zählen)
- $\mathbb{Q}$ denotes the rational numbers.
- $\mathbb{R}$ denotes the real numbers.
- $\mathbb{C}$ denotes the complex numbers
- ϕ denotes the empty set, that is, the unique set without any objects.

The empty set is also denoted $\{\ \}$ or $\emptyset$ (a zero with a line through it).

The symbol $\emptyset$ which I use for the empty set is not a "phi", but a circle with a line through it.

Def^n: The objects in a set are called the elements (or members or points) of the set.

Notation: Let S be a set.

"$x \in S$" means "x is an element of S".
("x is in S")

If y is not an element of S, we write "$y \notin S$". ("y is not in S")

Examples: $5 \in \mathbb{N}$, $\frac{1}{2} \in \mathbb{Q}$, $\sqrt{2} \notin \mathbb{Q}$.

How do we describe/specify a set?

Ways to define a set:

(1) Use words:

- The set of all integers.
- The set of odd integers betwen -4 and 10.

(2) List the elements of the set (between curly braces):

- $\{3, 5, 7, 9\}$

- $\{1, 4, \pi, \sqrt{2}\}$ (order is unimportant)

- $\{1, 2, 3, 4, 5, \ldots\}$ "and so on"

(3) Set Builder Notation (conditional):

- $\{x \in \mathbb{R} \mid x > 2\}$ — $\mid$ "such that"

  "The set of $x \in \mathbb{R}$ such that $x>2$."
  In other words: "The set of real numbers x such that x is greater than 2."

Remark: Sometimes people write : rather than | to mean "such that".

- $\{x \in \mathbb{Z} \mid x \text{ is odd and } -4 < x < 10\}$

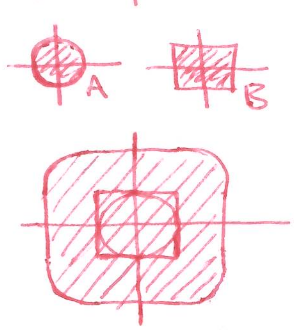
- $\{x \in \mathbb{R} \mid x^2 - x - 2 = 0\}$.

(4) Set Builder Notation (constructive):

- $\{n^2 \mid n \in \mathbb{Z}\} = \{0, 1, 4, 9, \ldots\}$

- $\{a+b \mid a \in A, b \in B\}$, where A and B are some specified sets of complex numbers.

Example:



- $\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\} = \mathbb{Q}$

Def^n (Set Equality): Let A and B be two sets. We say A and B are equal ($A = B$) if A and B have precisely the same elements.

Examples:

- $\{x \in \mathbb{R} \mid x^2 - x - 2 = 0\} = \{-1, 2\}$

- $\{x \in \mathbb{R} \mid \underline{x^2 + x + 2 = 0}\} = \emptyset$.

  no real solutions

Let A and B be two sets.

Def^n: We say that A is a subset of B ($A \subseteq B$) if every element of A is also an element of B.

Def^n: We say that A is a proper subset of B ($A \subset B$) if $A \subseteq B$ and $A \neq B$. Often the notation $A \subsetneq B$ is preferred for clarity.

Remarks:

- The empty set is a subset of any set A ($\emptyset \subseteq A$). In particular, $\emptyset \subseteq \emptyset$. The empty set is a proper subset of any nonempty set.

- Let A be a set. $A \subseteq \emptyset$ if and only if $A = \emptyset$.

- $x \in A \iff \{x\} \subseteq A$.

- If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proving that A is a subset of B (template):

Proof: Let $x \in A$. Then [Prove that $x \in B$].

Since $x \in A$ was arbitrary, we conclude that $A \subseteq B$. $\square$

Proving that A = B:

Usually one breaks this up into two steps:

- Prove that $A \subseteq B$.
- Prove that $B \subseteq A$.

Operations on Sets

Def^n: Let A and B be two sets.

- The union of A and B, denoted $A \cup B$, is defined by

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

- The intersection of A and B, denoted $A \cap B$, is defined by

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

- We say that A and B are disjoint if $A \cap B = \emptyset$. (That is, if A and B have no elements in common.)

Examples:

- Let $A = \{1,2\}$, $B = \{2,3\}$.
  Then $A \cup B = \{1,2,3\}$ and
  $A \cap B = \{2\}$.

- Let $A = \{1,3\}$, $B = \{2,4\}$.
  Then A and B are disjoint ($A \cap B = \emptyset$).

- Let $A = \{x \in \mathbb{R} \mid x < 10\}$,
  $B = \{x \in \mathbb{R} \mid x > 2\}$.
  Then $A \cap B = \{x \in \mathbb{R} \mid 2 < x < 10\}$ and
  $A \cup B = \mathbb{R}$.

Remark: For any two sets A, B, it is always true that

$$A \cap B \subseteq A, \qquad A \cap B \subseteq B,$$

$$A \subseteq A \cup B, \qquad B \subseteq A \cup B,$$

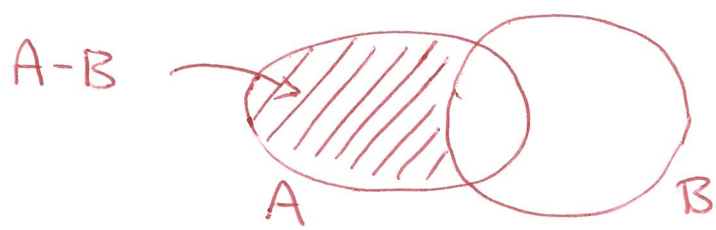
$$A \cap A = A = A \cup A, \qquad A \cap \emptyset = \emptyset,$$

$$A \cup \emptyset = A$$

Set Difference: The difference of A and B, denoted $A-B$ (or $A \setminus B$), is defined by

$$A-B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Remark: In the definition we do not require that B be a subset of A:



Remark: For any set A, it is always true that

$$A-A=\emptyset, \qquad A-\emptyset = A.$$

Examples:

- Let $A=\{1,2,3,4\}$, $B=\{2,3,4,5\}$.
  Then $A-B=\{1\}$ and $B-A=\{5\}$.

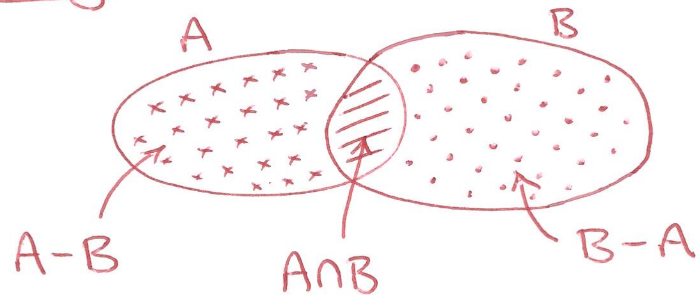
- Let $A=\{x\in\mathbb{R} \mid x<10\}$, $B=\{x\in\mathbb{R} \mid x>2\}$.
  Then $A-B=\{x\in\mathbb{R} \mid x\le 2\}$ and
  $B-A=\{x\in\mathbb{R} \mid x\ge 10\}$.

Proposition 6.2.4: Given two sets A, B,

(i) the three sets $A \cap B$, $A - B$, $B - A$ are pairwise disjoint;

(ii) $A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$.

Venn Diagram:



Proof: (See the textbook for a proof using truth tables.) Let A, B be two sets.

For (i) we need to prove

① $(A \cap B) \cap (A - B) = \emptyset$,

② $(A \cap B) \cap (B - A) = \emptyset$, and

③ $(B - A) \cap (A - B) = \emptyset$.

We prove ① by contradiction. Suppose $x \in (A \cap B) \cap (A - B)$. Since $x \in A \cap B$, $x \in B$. Since $x \in A - B$, $x \notin B$. Clearly we have a contradiction, so no such x exists. We conclude that $(A \cap B) \cap (A - B) = \emptyset$.

The proof of ② is similar, as is the proof of ③.

For (ii) we note that $(A \cap B) \cup (A - B) \cup (B - A)$ is clearly contained in $A \cup B$, so it suffices to prove that $A \cup B \subseteq (A \cap B) \cup (A - B) \cup (B - A)$.

Suppose $x \in A \cup B$. (We need to show that this implies $x \in (A \cap B) \cup (A - B) \cup (B - A)$.)

If $x \in A \cap B$ then $x \in (A \cap B) \cup (A-B) \cup (B-A)$.

Suppose $x \notin A \cap B$. Then x is in A or B, but not both. So, if x is in A, then $x \in A-B$, and if x is in B, then $x \in B-A$. We conclude that in any case $x \in (A \cap B) \cup (A-B) \cup (B-A)$, as required.

$\square$

Set Complement:

Often when we are doing mathematics, the sets we are dealing with are all subsets of some fixed "universal set" (e.g., the set of real numbers).

Def^n: When there is a fixed "universal set" $\mathcal{U}$ (which is understood from the context) we define the complement of a subset A of $\mathcal{U}$ in $\mathcal{U}$ to be

$$A^c = \mathcal{U} - A = \{x \in \mathcal{U} \mid x \notin A\}.$$

Examples:

- $A = \{x \in \mathbb{R} \mid x > 2\}$, $\mathcal{U} = \mathbb{R}$,
  $A^c = \{x \in \mathbb{R} \mid x \leq 2\}$.
- Let A be the set of even integers, and $\mathcal{U} = \mathbb{Z}$. Then $A^c = \mathbb{Z} - A$ is the set of all odd integers.

$\underline{\text{Def}^n}$ (Power Set): Let X be a set. The power set of X, denoted $\mathcal{P}(X)$, is the set of all subsets of X.

Examples:

- Let $X = \{1,2,3\}$. Then

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

- Let $A = \{1\}$. Then

$$\mathcal{P}(A) = \{\emptyset, \{1\}\}.$$

- $\mathcal{P}(\emptyset) = \{\emptyset\} \neq \emptyset.$

Remark: If X has n elements (n a natural number) then $\mathcal{P}(X)$ has 2^n elements. This is where the name "power set" comes from. Another notation for the power set is $\mathcal{P}(X) = 2^X$. One can think of the set $\mathcal{P}(X) = 2^X$ as the set of functions mapping from X to the set $\{0,1\}$ (which has 2 elements)...

Theorem 6.3.4: Let A, B, C be sets. Then

(i) $A \cup (B \cup C) = (A \cup B) \cup C,$
$A \cap (B \cap C) = (A \cap B) \cap C.$ (Associativity)

(ii) $A \cup B = B \cup A, \quad A \cap B = B \cap A.$ (Commutativity)

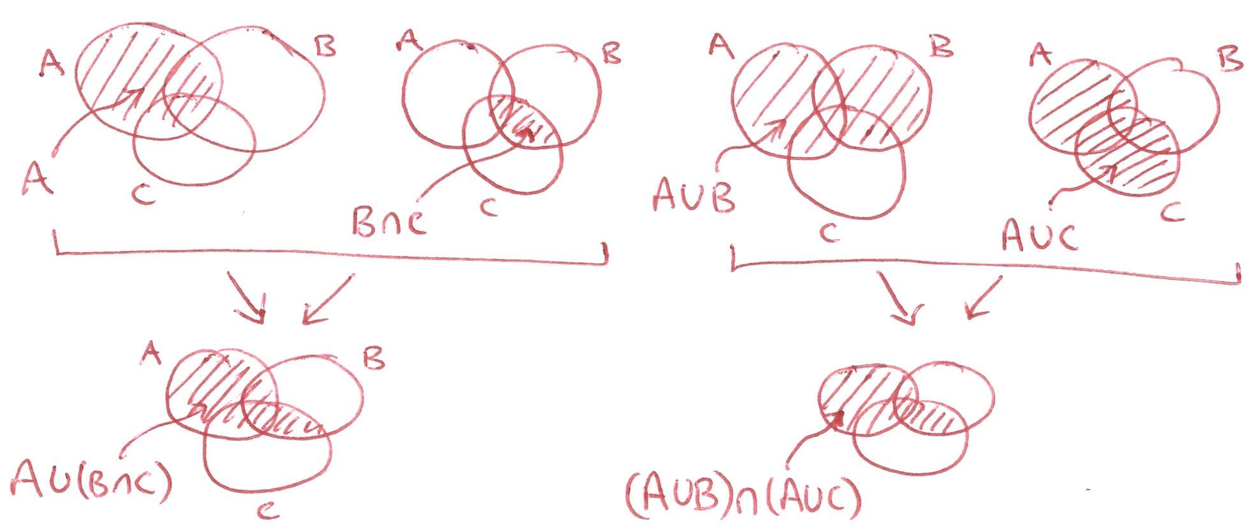
(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$ (Distributivity)

(iv) (Assuming a universal set $\mathcal{U}$ is fixed.) (same for (v))
$(A \cup B)^c = A^c \cap B^c,$
$(A \cap B)^c = A^c \cup B^c.$ (De Morgan's laws)

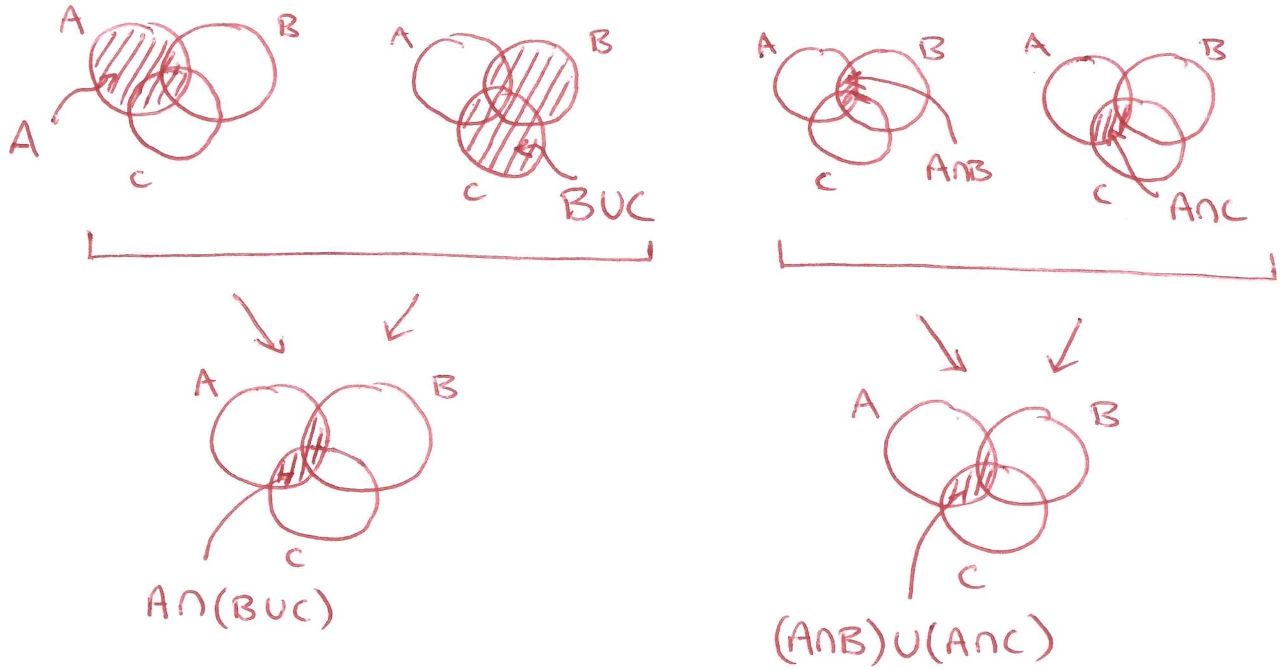
(v) $A \cup A^c = \mathcal{U}, \quad A \cap A^c = \phi.$ (Complementation)

(vi) $(A^c)^c = A.$ (Double Complementation)

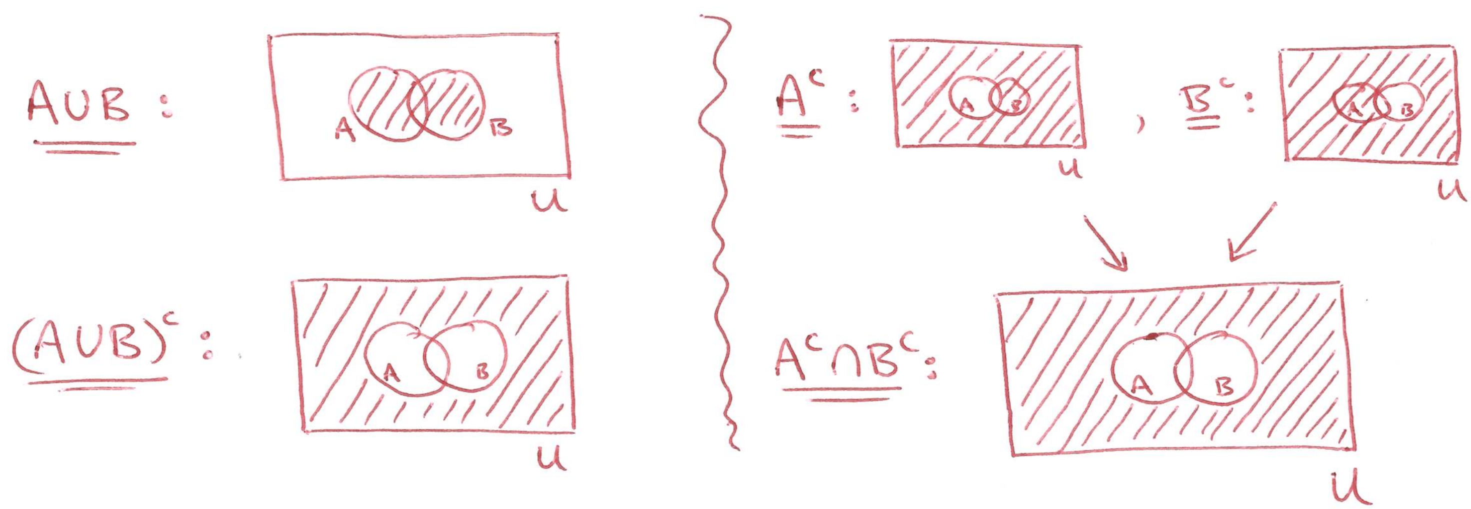
Venn Diagrams for (iii): $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



Venn diagrams for (iii): $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



Venn diagrams for (iv): $(A \cup B)^c = A^c \cap B^c$



The textbook has a written proof of (iii): $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Let's do a proof of (iv): $(A \cap B)^c = A^c \cup B^c$.

Proof of $(A \cap B)^c = A^c \cup B^c$:

- Proof that $(A \cap B)^c \subseteq A^c \cup B^c$:

  Let $x \in (A \cap B)^c = U - (A \cap B)$. Then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$. If $x \notin A$, then $x \in A^c$ (note $x \in U$), and if $x \notin B$, then $x \in B^c$.
  Hence, in any case, $x \in A^c \cup B^c$.
  We conclude that $(A \cap B)^c \subseteq A^c \cup B^c$.

- Proof that $A^c \cup B^c \subseteq (A \cap B)^c$:

  Let $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$.
  If $x \in A^c$ then $x \notin A$ so $x \notin A \cap B$ and therefore $x \in (A \cap B)^c$ (again note $x \in U$). Similarly, if $x \in B^c$ then $x \notin B$ so $x \notin A \cap B$ and therefore $x \in (A \cap B)^c$. Hence, in any case, $x \in (A \cap B)^c$.
  We conclude that $A^c \cup B^c \subseteq (A \cap B)^c$.

$\square$