Chapter 5: The Induction Principle

Definition: The natural numbers are the positive integers:

$$1, 2, 3, 4, 5, 6, 7, 8, \ldots .$$

Suppose we need to prove a list of propositions, one for every natural number:

- $P(1)$
- $P(2)$
- $P(3)$
- $P(4)$
- $P(5)$
- $P(6)$

$\vdots$

Principle of Mathematical Induction:

$P(n)$ is true for all natural numbers n if

(i) $P(1)$ is true, and

(ii) $P(k) \Rightarrow P(k+1)$ for all natural numbers k.

Domino effect: $\underset{\checkmark}{P(1)} \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \cdots$

Template for proof by induction:

Problem: Prove $P(n)$ for all natural numbers $n \geq n_0$. ← Usually n_0 will be 1.

Proof: We use induction on n.

[If it is not clear from the context, it is a good idea to clarify what the statement $P(n)$ is.]

omit if $n_0 = 1$

- Base case: [Verify $P(n)$ for $n = n_0$.]
- Inductive step: Let k be a natural number, $k \geq n_0$, and suppose $\underbrace{P(n) \text{ is true when } n = k}_{\text{i.e. } P(k) \text{ is true.}}$ ← "Inductive Hypothesis"

Then [Prove that $P(k+1)$ is true].

This proves the inductive step.

Hence, by induction, the statement is true for all natural numbers n. □

Examples:

Proposition: For natural numbers n, $n \leq 2^n$.

Proof: We use induction on n.

Base case ($n=1$): When $n=1$, $2^n = 2$. Note $1 \leq 2$. So the base case holds.

Inductive step: Let k be a natural number and suppose $k \le 2^k$. Then

$$k+1 \le 2^k + 1 \le 2^k + 2^k = 2^{k+1},$$

in particular, $k+1 \le 2^{k+1}$. This proves the inductive step.

Hence, by induction, we have proved that $n \le 2^n$ for all natural numbers n. $\square$

<u>Proposition:</u> For integers $n \ge 4$, $n^2 \le 2^n$.

<u>Remark:</u> Note that $n^2 \le 2^n$ is false when $n=3$.

<u>Proof:</u> We use induction on n.

Base case ($n=4$): When $n=4$ the desired statement becomes $16 \le 16$, which is true. So the base case holds.

Inductive step: Let k be a natural number, $k \ge 4$, and suppose $k^2 \le 2^k$. Then

$$(k+1)^2 = k^2 + 2k + 1 \le k^2 + 3k \le k^2 + k^2 \le 2^k + 2^k = 2^{k+1}.$$

This proves the inductive step.

Hence, by induction, $n^2 \le 2^n$ for all integers $n \ge 4$. $\square$

<u>Exercise:</u> Prove that for any integer $n \ge 4$,

$$n! > 2^n.$$

Proposition: For any natural number n, $6^{n+1}+7^{2n-1}$ is divisible by 43.

Proof: We use induction on n.

Base case ($n=1$): Note that when $n=1$, $6^{n+1}+7^{2n-1} = 6^2+7 = 43$. So the base case holds.

Inductive step: Let k be a natural number and suppose $6^{k+1}+7^{2k-1}$ is divisible by 43.

Then, when $n=k+1$,

$$\begin{aligned} 6^{n+1}+7^{2n-1} &= 6^{(k+1)+1}+7^{2(k+1)-1} = 6^{k+2}+7^{2k+1} \\ &= 6 \times 6^{k+1} + 49 \times 7^{2k-1} \\ &= 6\left(6^{k+1}+7^{2k-1}\right) + 43 \times 7^{2k-1}. \end{aligned}$$

By the inductive hypothesis $6(6^{k+1}+7^{2k-1})$ is divisible by 43, and clearly $43 \times 7^{2k-1}$ is divisible by 43. This proves the inductive step. Hence, by induction, the statement is true for all natural numbers n. $\square$

Question: Can you explain why the above argument works out?

Strong Induction:

The Principle of Mathematical Induction implies:

The Strong Induction Principle:

$P(n)$ is true for all natural numbers n if

(i) $P(1)$ is true, and

(ii) for all natural numbers k,

$$\begin{bmatrix} P(n) \text{ is true for all natural} \\ \text{numbers } n \leq k \end{bmatrix} \Rightarrow P(k+1) \text{ is true.}$$

Remark: A proof by strong induction can be turned into a proof by induction (prove the statement $Q(n)$ that "$P(m)$ is true for all natural numbers $m \leq n$" by ordinary induction) but we wouldn't usually do this.

A "Modified" Strong Induction Principle:

$P(n)$ is true for all natural numbers n if

(i) $P(1), P(2), \ldots, P(n_0)$ are true, and

(ii) for all natural numbers $k \geq n_0$

$$\begin{bmatrix} P(n) \text{ is true for all natural} \\ \text{numbers } n \leq k \end{bmatrix} \Rightarrow P(k+1) \text{ is true.}$$

Remark: We still just call this "strong induction". We are just starting the strong induction later, i.e. after n_0.

Examples:

Definition: Let $F_1 = 1$, $F_2 = 1$, and for each natural number $n \geq 3$ let

$$F_n = F_{n-1} + F_{n-2}.$$

The sequence of integers $\{F_n\}$ is called the Fibonacci sequence.

Proposition: For any natural number n, $F_n \leq 2^n$.

Proof: We use strong induction on n.

Base case ($n=1$ and $n=2$): When $n=1$, $F_n = 1 \leq 2 = 2^n$. When $n=2$, $F_n = 1 \leq 4 = 2^n$. So the base case holds.

Inductive step: Let k be a natural number, $k \geq 2$, and suppose $F_n \leq 2^n$ for all natural numbers $n \leq k$. Then

$$\begin{aligned} F_{k+1} = F_k + F_{k-1} &\leq 2^k + 2^{k-1} \\ &\leq 2^k + 2^k = 2^{k+1}. \end{aligned}$$

This proves the inductive step.

Hence, by strong induction, $F_n \le 2^n$ for all natural numbers n. $\square$

Proposition (Binet formula):
Let $\{F_n\}$ be the sequence of Fibonacci numbers. For natural numbers n,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Remark: The Fibonacci numbers satisfy what is called a second order linear homogeneous difference equation: For natural numbers n,

$$(*) \qquad \underbrace{F_{n+2}}_{\text{second order}} - F_{n+1} \underbrace{- F_n = 0}_{\text{homogeneous}}.$$

Just as the general solution of a second order homogeneous differential equation $ay'' + by' + cy = 0$ can be found by solving the characteristic equation $az^2+bz+c=0$ (for z), the general solution to $(*)$ can be found by solving the characteristic equation of $(*)$: $z^2 - z - 1 = 0$.
The solutions are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

The "general solution" to the difference equation "$y_{n+2} - y_{n+1} - y_n = 0$" is then $y_n = A\alpha^n + B\beta^n$ where A and B are any constants. The Fibonacci numbers correspond to the specific solution determined by the initial conditions $y_1 = 1$ and $y_2 = 1$ (these force us to have $A = \frac{1}{\sqrt{5}}$ and $B = \frac{1}{\sqrt{5}}$).

Proof of the Binet formula:

We use strong induction on n. We also note, for later use, that $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the two roots of the equation $z^2 - z - 1 = 0$.

Base case ($n=1$ and $n=2$): For $n=1$, note that

$$\frac{1}{\sqrt{5}}(\alpha^1 - \beta^1) = \frac{\alpha - \beta}{\sqrt{5}} = \frac{1+\sqrt{5}-(1-\sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1.$$

For $n=2$, note that

$$\frac{1}{\sqrt{5}}(\alpha^2 - \beta^2) = \frac{\alpha-\beta}{\sqrt{5}}(\alpha+\beta) = \alpha + \beta = 1.$$

So $F_1 = \frac{1}{\sqrt{5}}(\alpha^1 - \beta^1)$ and $F_2 = \frac{1}{\sqrt{5}}(\alpha^2 - \beta^2)$.

Inductive step: Let k be a natural number, $k \geq 2$, and suppose $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ for all natural numbers $n \leq k$. Then

$$F_{k+1} = F_k + F_{k-1} = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k) + \frac{1}{\sqrt{5}}(\alpha^{k-1} - \beta^{k-1})$$
$$= \frac{1}{\sqrt{5}}\left(\alpha^{k-1}(\alpha+1) - \beta^{k-1}(\beta+1)\right).$$

Since $\alpha^2-\alpha-1=0$ and $\beta^2-\beta-1=0$ we have $\alpha^2=\alpha+1$ and $\beta^2=\beta+1$, so

$$F_{k+1}=\frac{1}{\sqrt{5}}\left(\alpha^{k-1}\alpha^2-\beta^{k-1}\beta^2\right)=\frac{1}{\sqrt{5}}\left(\alpha^{k+1}-\beta^{k+1}\right)$$

as required. This proves the inductive step.

Hence, by strong induction, the Binet formula holds for all natural numbers n. $\square$

Let's consider a different kind of problem:

<u>Question</u>: Can every integer amount of postage (in cents) that is at least 12 cents be made from 4 and 5 cent stamps?

→ We'll prove that the answer is <u>yes</u>!

<u>Proposition</u>: Every integer amount of postage that is at least 12 cents can be made from 4 and 5 cent stamps.

<u>Proof</u>: We use strong induction on n, where n denotes the amount of postage.

Base case ($n=12, n=13, n=14, n=15$):
Note that $12=4+4+4$, $13=4+4+5$, $14=4+5+5$, $15=5+5+5$. So the base case holds.

Inductive step: Let k be a natural number,

$k \geq 15$, and suppose that any amount n of postage can be covered for $12 \leq n \leq k$. Then, since $k \geq 15$ implies $12 \leq k-3 \leq k$, $(k-3)$ cents can be covered by 4 and 5 cent stamps. To cover $(k+1)$ cents we simply need to add one more 4 cent stamp, so $(k+1)$ cents postage can also be covered.

This proves the inductive step.

Hence, by strong induction, any integer amount of postage that is at least 12 cents can be covered. $\square$

<u>An alternate proof:</u>

We use induction on n, where n is the amount of postage.

Base case $(n=12)$: Note $12 = 4+4+4$.

Inductive step: Let k be a natural number, $k \geq 12$, and suppose that k cents postage can be covered using only 4 and 5 cent stamps.

Case 1: If at least one 4 cent stamp is used to cover k cents, then one may simply replace one 4 cent stamp with a 5 cent stamp to cover $(k+1)$ cents.

Case 2: If no 4 cent stamps are used

(i.e. only 5 cent stamps are used) to cover k cents, then k is divisible by 5. Hence $k \geq 15$ (since $k \geq 12$) and it requires at least three 5 cent stamps to cover k cents postage. In this case one can cover $k+1$ cents postage by replacing three of the 5 cent stamps by four 4 cent stamps.

This proves the inductive step.

Hence, by induction, any integer amount of postage that is at least 12 cents can be covered. $\square$

Prime Factorizations

<u>Def^n</u>: A natural number n is said to be <u>prime</u> if $n > 1$ and the only positive (integer) divisors of n are 1 and n. If an natural number $n > 1$ is not prime then it is said to be <u>composite</u>.

<u>Primes</u>: 2, 3, 5, 7, 11, 13, 17, 19, 23, ...

<u>Note</u>: An integer $n > 1$ is composite if and only if it can be written as a product $n = ab$ of integers a and b such that $1 < a < n$ and $1 < b < n$.

Proposition 23.1.2: Every integer greater than 1 can be written as a product of prime numbers.

Proof: For each natural number $n > 1$ let $P(n)$ be the statement that n may be written as a product of primes. We use strong induction on n to prove $P(n)$ for all integers $n > 1$.

Base case ($n = 2$): The number 2 is prime, so is a product of one prime (itself).

Inductive step: Let k be a natural number, $k \geq 2$, and suppose $P(n)$ is true for all integers n with $2 \leq n \leq k$.

Case 1: If $k+1$ is prime, then $P(k+1)$ is true as $k+1$ is a product of one prime (itself).

Case 2: If $k+1$ is composite then $k+1 = ab$ for some integers a and b with $1 < a < k+1$ and $1 < b < k+1$. Since $1 < a \leq k$ and $1 < b \leq k$, by the inductive hypothesis, a and b may be written as products of primes. It follows that $k+1$ may be written as a product of primes.

This proves the inductive step.

Hence, by strong induction, $P(n)$ holds for all integers $n > 1$. $\square$

Some facts to be proved later:

<u>Theorem</u>: Suppose p is a prime number and a, b are positive integers. Then

$$p \mid ab \implies (p \mid a \text{ or } p \mid b).$$

<u>Remark</u>: The above property is sometimes used to <u>define</u> prime numbers. This is also how one defines "prime" in the more general setting of <u>abstract algebra</u>.

<u>Fundamental Theorem of Arithmetic</u>:

Every integer greater than 1 can be written <u>uniquely</u> as a product of prime numbers.