Chapter 4: Proof by Contradiction

In a proof by contradiction we prove that $(\neg P) \Rightarrow R$ where R is false, so P must be true.

A template for proof by contradiction:

Problem: Prove the statement P.

Proof: Suppose, with a view to obtaining a contradiction, that P is false.

Always indicate in your proof when you are using proof by contradiction.

Then ... [Present here some argument leading to a contradiction].

e.g., "$1=0$" or "Q is true and Q is false"

Hence our assumption that P is false cannot be true.

We therefore conclude that P must be true, as required. $\square$

When is this useful?

- Surprisingly often!
- Particularly when proving a "nonexistence statement".

But it is much better to give a direct proof when possible.

Examples:

Proposition: There do not exist integers m and n such that $14m + 20n = 101$.

Proof: Suppose, with a view to obtaining a contradiction, that there are integers n, m with $14m + 20n = 101$. Then, since $14m + 20n = 2(7m + 10n)$, 101 is even. But this contradicts the fact that 101 is even. Hence our assumption on the existence of such n, m cannot be true.
We conclude that there do not exist integers n, m such that $14m + 20n = 101$. $\square$

Proposition: For any real numbers a, b,

$$2 + a^2 + b^2 \neq 1 + 2ab.$$

Proof (by contradiction):
Suppose that there exist real numbers a, b, such that $2 + a^2 + b^2 = 1 + 2ab$.

Then

$$0 = (2 + a^2 + b^2) - (1 + 2ab) = 1 + a^2 + b^2 - 2ab = 1 + (a-b)^2.$$

But $1 + (a-b)^2 \geq 1$, so $1 + (a-b)^2 \neq 0$ and we have obtained a contradiction.
Hence our assumption on the existence of such a and b cannot be true.
We conclude that for any real numbers a, b,
$2 + a^2 + b^2 \neq 1 + 2ab$. $\square$

Exercise: Let $f(x) = \frac{2x+3}{x+2}$. Prove that for any real number x, $f(x) \neq 2$.

Proof by Contrapositive

Homework: The statement "$P \Rightarrow Q$" is logically equivalent to the statement $\underbrace{\text{"}(\neg Q) \Rightarrow (\neg P)\text{"}}$.

contrapositive of "$P \Rightarrow Q$"

A template for proof by contrapositive:

Problem: Prove $P \Rightarrow Q$.

Proof: We will prove here the contrapositive of the desired statement: [State the contrapositive here.]

Suppose $\underbrace{Q \text{ is false}}$.

i.e. $\neg Q$ is true.

Then ... [Prove that P is false]. $\square$

Remark: Any proof by contrapositive can also be done as a proof by contradiction, but it is better style to use contraposition when this is possible.

Examples:

Recall that last time (notes on Chapter 3) we proved that the square of an even integer is even.

Proposition: For integers a,

$$a^2 \text{ is odd} \implies a \text{ is odd.}$$

Proof: Let a be an integer.

We have to prove an implication between two predicates so we fix the free variable "a" first.

We will prove here the contrapositive of the desired statement:

$$a \text{ is even} \implies a^2 \text{ is even.}$$

We proved this earlier (see page 4 of the notes on Chapter 3).

Since a was arbitrary, we conclude that the result holds for all integers a. $\square$

Proposition: For integers a, b,

$$ab = 0 \implies (a = 0 \text{ or } b = 0).$$

Proof: Let a and b be integers. Consider the contrapositive of the desired statement:

$$(a \neq 0 \text{ and } b \neq 0) \implies ab \neq 0.$$

This is clearly true, so the desired result holds. $\square$

Proposition: For integers a, b,

$$a+b \geq 11 \implies (a \geq 6 \text{ or } b \geq 6).$$

Proof: Let a and b be integers. We will prove the contrapositive of the desired statement:

$$(a < 6 \text{ and } b < 6) \implies a+b < 11.$$

To see this, observe that

$$\begin{aligned}(a<6 \text{ and } b<6) &\implies (a \leq 5 \text{ and } b \leq 5)\\ &\implies a+b \leq 10\\ &\implies a+b < 11.\end{aligned}$$

Since a and b were arbitrary we conclude that the result holds for all integers a, b. $\square$

Exercise: Give a proof of the above by contradiction. Note that for statements P, Q,

$$\neg(P \Rightarrow Q) \iff P \wedge (\neg Q).$$

So the proof would start out as follows.

Proof: Let a and b be integers. Suppose, with a view to obtaining a contradiction, that $a+b \geq 11$ and $(a<6 \text{ and } b<6)$.

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