Chapter 2: Implications

We prove a mathematical proposition by starting from some known (true) propositions and drawing a sequence of conclusions implied by these starting propositions, ending with the desired proposition as a conclusion.

Definition: Let P and Q be statements. "$P \Rightarrow Q$" ("P implies Q") is the statement "if P is true then Q is true".

Is this definition precise? What if P is a false statement? Would $P \Rightarrow Q$ be true or false? (Does it depend on Q?)

It will help to think about an example. Consider the statement:

For integers n, $n > 2 \Rightarrow n > 0$.

- This statement is a true proposition.
- In the book "$n > 2 \Rightarrow n > 0$" is called a universal implication, because it is true for

all integers n. We'll discuss universal statements later (Chapter 7).

Let's analyze this statement using a truth table:

For integers n, $\underbrace{n>2}_{P(n)} \Rightarrow \underbrace{n>0}_{Q(n)}$.

(a predicate) (another predicate)

also a predicate

	$P(n)$	$Q(n)$	$P(n) \Rightarrow Q(n)$
$n<0$	F	F	T
$n=0$	F	F	T
$n=1$	F	T	T
$n=2$	F	T	T
$n>2$	T	T	T

$P(n) \Rightarrow Q(n)$ is true for all integers n.

$\rightsquigarrow$ Precise definition of $P \Rightarrow Q$:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Some Strange Consequences:

- The statement

$$1 = 2 \implies \text{the Goldbach conjecture is true}$$

is true!

- The statement

$$\underset{\substack{\uparrow \\ \text{hypothesis}}}{1 = 2} \implies \underset{\substack{\uparrow \\ \text{conclusion}}}{\text{the Goldbach conjecture is false}}$$

is true!

We cannot say whether the conclusion is true or false, because the hypothesis is not true.

- The statement

$$1 \neq 2 \implies \pi > 3$$

is true.

Note that the notion of causation is not built into the definition of "$\Rightarrow$".

- The statement

$$1 \neq 2 \implies \pi < 3$$

is false.

Reading Implications:

We read the statement $P \Rightarrow Q$ as:

(i) If P then Q.

(ii) P implies Q.

(iii) Q if P.

(iv) P only if Q. $\leftsquigarrow$ $(P \Rightarrow Q) \Leftrightarrow \underbrace{[(\neg Q) \Rightarrow (\neg P)]}_{\text{contrapositive}}$

(v) Q whenever P.

(vi) P is sufficient for Q.

(vii) Q is necessary for P. $\leftsquigarrow$ cf. (iv)

Negation: "$P \not\Rightarrow Q$" means "$\neg(P \Rightarrow Q)$".

Converse: The converse of the statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

(These are not equivalent: For integers n, $n > 3 \Rightarrow n > 0$ is true, but $n > 0 \Rightarrow n > 3$ is false!)

Logical Equivalence:

"$P \Leftrightarrow Q$" means "$(P \Rightarrow Q) \underset{\text{"and"}}{\wedge} (Q \Rightarrow P)$"

- For integers a, b,

$$a < b \iff 2a < 2b.$$

Ways to Read "$P \Leftrightarrow Q$":

(i) P is equivalent to Q.

(ii) P is necessary and sufficient for Q.

(iii) $\underbrace{\text{P if and only if Q}}_{\text{Abbreviated "P iff Q"}}$.