Chapter 14: Counting Infinite Sets

Def^n: Two sets X, Y are equipotent (or equinumerous or have the same cardinality) if there is a bijection $X \to Y$.

Example: $\{a, b, c\}$ and $\{1, 2, 3\}$ are equipotent.

Remark:

1. Two finite sets are equipotent if and only if they have the same number of elements.
2. Let Y be a finite set, and X a proper subset of Y. Then $|X| < |Y|$ and the two sets are not equipotent.

Examples:

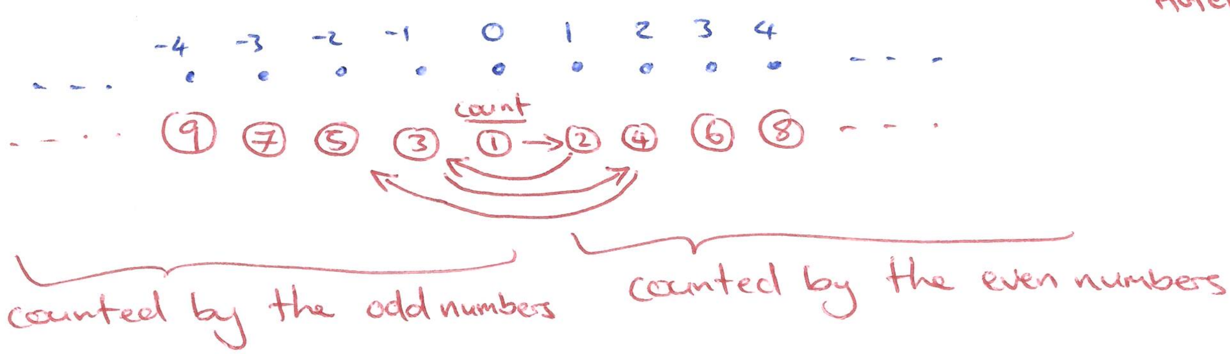
- $\mathbb{N}$ and $\mathbb{Z}^{\geq 0} = \mathbb{N} \cup \{0\}$ are equipotent; the map $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ given by $f(n) = n-1$ is a bijection.
- $(0, 1)$ and $(0, 2)$ are equipotent; the map $x \mapsto 2x$ is a bijection from $(0,1)$ to $(0,2)$.
- $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$ are equipotent; the map $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a bijection.

Defⁿ: A set X is said to be <u>denumerable</u> (or <u>enumerable</u> or <u>countably infinite</u>) if it is equipotent with $\mathbb{N}$ (can be put in bijection with $\mathbb{N}$).

Defⁿ: We say that a set X is <u>countable</u> if X is finite or countably infinite. Otherwise, we say X is <u>uncountable</u>.

<u>Examples:</u>

- $\{a, b, c, d\}$ is finite and therefore countable.
- $\mathbb{N}$ is denumerable (by defⁿ) so is countable.
- $\mathbb{Z}^{\geq 0} = \mathbb{N} \cup \{0\}$ is denumerable.
- $\mathbb{Z}$ is denumerable. ← cf. "Hibert's Hotel"



Remark: A set X is denumerable if and only if we can list the elements of X in an infinite sequence such that each element appears exactly once.

i.e. "enumerate the elements".

Theorem 14.1.4: A set X is infinite if and only if it is equipotent to a proper subset of itself.

Proof (sketch): Let X be a set.

Forward implication: If X is infinite, then we can argue by induction that X has a denumerable subset $\{x_1, x_2, x_3, \ldots\} = \{x_n \mid n \in \mathbb{N}\}$ (with $x_i \neq x_j$ when $i \neq j$). So we may define a bijection $f: X \to X - \{x_1\}$ by

This is all that needs to be done to make the argument formal.

$$f(x) = \begin{cases} x_{n+1} & \text{if } x = x_n \text{ (for some } n \in \mathbb{N}) \\ x & \text{if } x \notin \{x_1, x_2, x_3, \ldots\}. \end{cases}$$

Reverse implication (by contraposition): If X is finite, then any proper subset of X has cardinality less than $|X|$ so cannot be equipotent with X.

$\square$

Exercise: Find a bijection from $[0,\infty) = \{x \in \mathbb{R} \mid x \geq 0\}$ to $(0,\infty) = \{x \in \mathbb{R} \mid x > 0\}$.

Solution: Let $f: [0,\infty) \to (0,\infty)$ be given by

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is an integer} \\ x & \text{if } x \text{ is not an integer.} \end{cases}$$

Denumerable Sets:

Proposition 14.2.1: Given a denumerable set X, a set Y is denumerable if and only if it is equipotent to X.

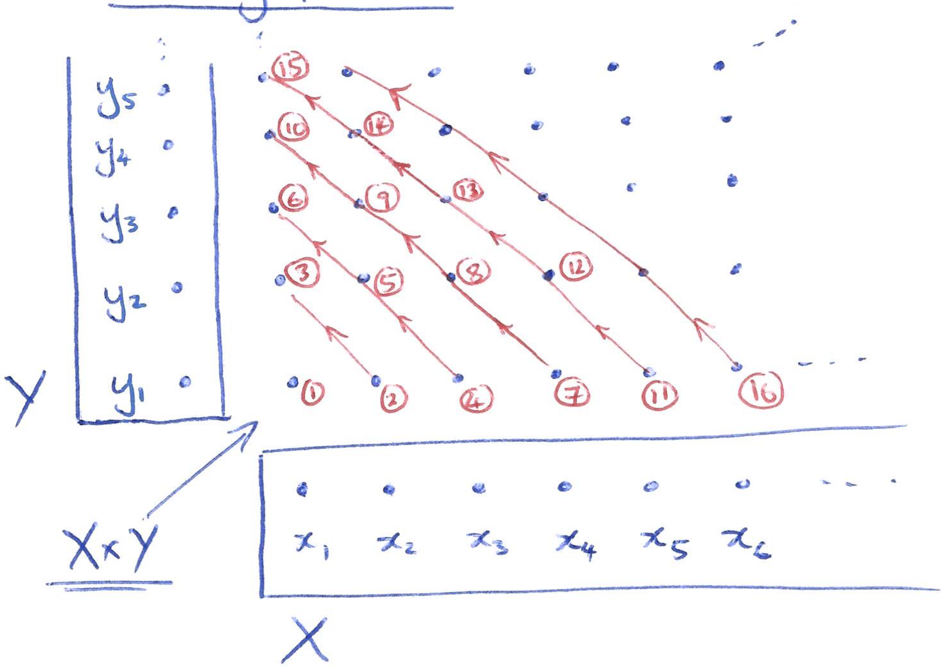
Proof: Easy exercise. ("Equipotency is transitive".) $\square$

Proposition: If X and Y are denumerable sets, then so is $X \cup Y$.

use the odd natural numbers to count/enumerate X and the even natural numbers for Y.

Proposition 14.2.3: If X and Y are denumerable, then $X \times Y$ is denumerable.

"Proof by picture:"



$\square$

Corollary: If A is a denumerable set, then so is $A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ copies}}$ for any $n \in \mathbb{N}$.

Proposition:

(i) An infinite subset of a denumerable set is denumerable.

(ii) A subset of a denumerable set is countable.

Proof: Part (i) implies part (ii), so we prove part (i).

$$\begin{array}{ccc} X & \xrightarrow[\text{bijection}]{f} & \mathbb{N} \\ \cup\shortmid & & \cup\shortmid \\ B & \longrightarrow & A \\ & & \| \\ & & f(B) \end{array}$$

It suffices to consider infinite subsets of $\mathbb{N}$, since any denumerable set can be put in bijection with $\mathbb{N}$. Let A be an infinite subset of $\mathbb{N}$.

Define a map $g : \mathbb{N} \to A$ inductively as follows.

"Base case" $\longrightarrow$ Let $g(1)$ be the least element in A.

"Inductive step" $\longrightarrow$ Let $k \in \mathbb{N}$ and suppose $g(1), \ldots, g(k)$ have been defined so that $g(1) < g(2) < \cdots < g(k)$. Then let $g(k+1)$ be the least element in $A - \{g(1), \ldots, g(k)\}$.

This defines the map $g : \mathbb{N} \to A$. By def^n,

$$g(1) < g(2) < g(3) < \cdots$$

so g is injective. To see that g is surjective note that if g were not surjective there would (by def^n) exist $a \in A$ s.t. $a \notin g(\mathbb{N})$, but then by the def^n of g we must have $g(n) \leq a \;\forall n \in \mathbb{N}$. But this is impossible since then $g(\mathbb{N})$ would be finite.

$\uparrow$ not possible since g is injective

Hence $g : \mathbb{N} \to A$ is a bijection. $\square$

Theorem 14.2.6: The set of rational numbers is countably infinite.

Proof: We apply the previous proposition.
Define a map $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $f(q) = (a, b)$ where q is expressed in lowest terms by $\frac{a}{b}$. ← unique

This map is clearly injective, so $\mathbb{Q}$ is equipotent with $f(\mathbb{Q})$. By the previous proposition, since $\mathbb{Z} \times \mathbb{N}$ is denumerable, $f(\mathbb{Q})$ is denumerable.
Hence $\mathbb{Q}$ is denumerable. $\square$

Cantor's Diagonal Slash

Theorem 14.3.1 (Cantor 1874):

The set of real numbers is uncountable.

Proof (sketch):

Suppose, with a view to obtaining a contradiction that the set of real numbers can be enumerated, that is, the real numbers can be listed

decimal expansion

$$
\begin{aligned}
x_1 &= a_{10} . a_{11} a_{12} a_{13} a_{14} \cdots a_{1n} \cdots , \\
x_2 &= a_{20} . a_{21} a_{22} a_{23} a_{24} \cdots a_{2n} \cdots , \\
x_3 &= a_{30} . a_{31} a_{32} a_{33} a_{34} \cdots a_{3n} \cdots , \\
&\vdots \\
x_m &= a_{m0} . a_{m1} a_{m2} a_{m3} a_{m4} \cdots a_{mn} \cdots , \\
&\vdots
\end{aligned}
$$

where $a_{m0} \in \mathbb{Z}$ and $a_{mn} \in \{0, 1, 2, \dots, 9\}$ for all $m, n \in \mathbb{N}$ (and the decimal expansion of each x_m, $m \in \mathbb{N}$, is chosen not to end in repeating 9's).

Take $b = 0.b_1 b_2 b_3 \dots b_n \dots$ where

to make the decimal expansions unique.

$$b_n = \begin{cases} 1 & \text{if } a_{nn} = 0, \\ 0 & \text{if } a_{nn} \neq 0. \end{cases}$$

Then the decimal expansion of b (does not end in repeating 9's and) differs from the decimal expansion of x_m at the m^{th} decimal place for every $m \in \mathbb{N}$. So $b \in \{x_1, x_2, x_3, \dots\} = \{x_m \mid m \in \mathbb{N}\}$, a contradiction. We conclude that the real numbers must be uncountable.

$\square$

We conclude with the following highly non-trivial fact:

Theorem (Schröder - Bernstein Theorem):
Let X, Y be two sets. Assume there are two injections $f : X \to Y$ and $g : Y \to X$. Then there is a bijection $h : X \to Y$ and thus X and Y have the same cardinality.