Chapter B: Number Systems
($\mathbb{Q}$ & $\mathbb{R}$)

Rational Numbers

The rational numbers are defined in a straightforward way from the integers:

$$\mathbb{Q} = \{ \tfrac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N} \}.$$

- It is important to note that, e.g., $\frac{3}{2} = \frac{6}{4} = \frac{15}{10}$. Generally, for $a, c \in \mathbb{Z}$, $b, d \in \mathbb{N}$

$$\frac{a}{b} = \frac{c}{d} \iff \underline{ad = bc}.$$

Really this defines what it means for $\frac{a}{b}$ to be equal to $\frac{c}{d}$. We'll come back to this when we talk about equivalence relations.

- Any rational number q can be expressed in lowest terms, that is, written as $q = \frac{a}{b}$ where $a \in \mathbb{Z}, b \in \mathbb{N}$ and $\underline{\gcd(a,b) = 1}$.

(we say a and b are coprime)

Proof: Let $q \in \mathbb{Q}$. Then $q = \frac{c}{d}$ for some $c \in \mathbb{Z}$ and $d \in \mathbb{N}$. Let $a = \frac{c}{\gcd(c,d)}$ and $b = \frac{d}{\gcd(c,d)}$.

Then $\frac{a}{b} = \frac{c}{d} = q$ and (Exercise 11.4)

$\gcd(a,b) = 1$. $\square$

- Starting with only the integers one should carefully define the addition of rational numbers (by first putting the numbers over a common denominator) and the multiplication of rational numbers, and check that these definitions do not depend on whether we write, e.g., $q = \frac{3}{2}$ or $q = \frac{15}{10}$. But this is, of course, elementary.

Decimal Expansions of Real Numbers

- One way to define the real numbers (though not the most elegant) is in terms of infinite decimal expansions, so, e.g.,

$$\sqrt{2} = 1.4142135623730950488\ldots$$

the decimal expansion continues forever, and never starts to repeat itself.

- Numbers with recurring decimal expansions are rational.

$$0.\dot{3} = 0.33333\ldots = \frac{1}{3}$$

$$8.\overline{657} = 8.657657\ldots = \frac{321}{37}$$

Idea: $8.\overline{657} = 8 + 0.\overline{657}$

Now $1000 \times 0.\overline{657} = 657.\overline{657}$

so $1000 \times 0.\overline{657} - 0.\overline{657} = 999 \times 0.\overline{657}$
$= 657.$

Hence $0.\overline{657} = \dfrac{657}{999}$, so $8.\overline{657} = 8 + \dfrac{657}{999}$

and one can check that $8 + \dfrac{657}{999} = \dfrac{321}{37}$ in lowest terms.

- Subtlety: Decimal expansions are not unique.

  (for rational numbers with a terminating decimal expansion)

↳ $0.\dot{9} = 0.999\ldots = 1$

Proof: Let $x = 0.\dot{9}$. Then $10x = 9.\dot{9}$,
so $10x - x = 9.\dot{9} - 0.\dot{9} = 9$, i.e. $9x = 9$.
Hence $x = 1$. $\square$

↳ Similarly $7.292 = 7.291\dot{9}$
$= 7.2919999\ldots.$