Chapter 12: Couting Functions and Subsets

Def^n: Given sets X and Y we denote the set of functions from X to Y by $\mathrm{Fun}(X,Y)$, or by Y^X.

Proposition 12.1.2: Suppose that X and Y are non-empty finite sets, then

$$|Y^X| = |Y|^{|X|}.$$

i.e. if $|X| = m$ and $|Y| = n$ then $|\mathrm{Fun}(X,Y)| = n^m$.

Proof (sketch):

For each element $x \in X$, there are $|Y|$ possible choices for the value of x when we choose/construct a function $X \to Y$. There are $|X|$ elements in X so in choosing a function we have

$$\underbrace{|Y| \times |Y| \times \cdots \times |Y|}_{|X| \text{ copies of } |Y|} = |Y|^{|X|}$$

choices. Thus $|X^Y| = |Y|^{|X|}$. $\square$

See the book for a formal proof by induction (based on the same idea).

Proposition 12.1.4: Suppose that X and Y are non-empty sets with $|X| = m$ and $|Y| = n$, then the number of injections $X \to Y$ is given by $\underbrace{n(n-1)\cdots(n-m+1)}$.

← one of the factors is zero if $m > n$ (pigeonhole principle).

Proof (sketch):

Let $X = \{x_1, \ldots, x_m\}$ be a set with m elements and let Y be a set with n elements ($m, n \in \mathbb{N}$).

When choosing a function $X \to Y$ we have

- n choices for where to send x_1, then
- $n-1$ choices for where to send x_2, since we have to avoid the value taken by x_1, then
- $n-2$ choices for where to send x_3, since we have to avoid the values taken by x_1 & x_2, then

⋮

- $n-(m-1) = n-m+1$ choices for where to send x_m, since we have to avoid the values taken by $x_1, x_2, \ldots, x_{m-1}$

It follows that there are

$$n(n-1)\cdot\ldots\cdot(n-m+1)$$

different injections $X \to Y$.

$\square$

Again see the book for a straightforward proof by induction based on this idea.

→ An important application of this is to the case where $m = n$.

Note: $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, ...

<u>Def^n</u>: Given a set X, a bijection $X \to X$ is called a <u>permutation</u> of X.

<u>Corollary 12.1.7</u>: Given a finite non-empty set X with $|X| = n$, the number of permutations of X is

$$\underbrace{n!}_{} \overset{\text{def}^n}{=} n(n-1)(n-2)\cdot \dots \cdot 1.$$

"<u>n factorial</u>"

Recall that for a set X,

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}.$$

"power set of X"

<u>Proposition 12.2.1</u>:
Let X be a set with cardinality $n \in \mathbb{N} \cup \{0\}$.
Then

$$|\mathcal{P}(X)| = 2^n.$$

For each element of X we have two choices when defining a subset of X: <u>in</u> the subset <u>or</u> <u>out</u> of the subset.

<u>Remark</u>: There is a natural bijection $\mathcal{P}(X) \to \{0,1\}^X$ which sends $A \subseteq X$ to the <u>characteristic function</u> χ_A.

Binomial Coefficients:

Def^n: Given a set X with $|X| = n \in \mathbb{N} \cup \{0\}$ we define (for any $r \in \mathbb{N} \cup \{0\}$)

$$P_r(X) := \{ A \subseteq X \mid |A| = r \}.$$

[Note $P(X) = P_0(X) \cup P_1(X) \cup \cdots \cup P_n(X)$.]

We then define the binomial coefficient

$$\binom{n}{r} := |P_r(X)|.$$

"n choose r"

the number of different subsets of $X = \{x_1, \ldots, x_n\}$ of cardinality r.

Remark: $\binom{n}{r}$ does not depend on the actual set X, only on $n = |X|$ (and r).

Example: Let $X = \{a, b, c, d\}$. Then

$$P_0(X) = \{\emptyset\};$$

$$P_1(X) = \{\{a\}, \{b\}, \{c\}, \{d\}\};$$

$$P_2(X) = \{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}\};$$

$$P_3(X) = \{\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\};$$

$$P_4(X) = \{\{a,b,c,d\}\} = \{X\}.$$

and $P_r(X) = \emptyset$ if $r > 4$

So $\binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, $\binom{4}{3} = 4$, $\binom{4}{4} = 1$, $\binom{4}{r} = 0$ if $r > 4$.

Proposition: For $n, r \in \mathbb{N} \cup \{0\}$,

(i) $\binom{n}{r} = 0$ if $r > n$,

(ii) $\underbrace{\binom{n}{0} = 1}$, $\binom{n}{1} = n$, $\binom{n}{n} = 1$,

$P_0(X) = \{\emptyset\}$ so $|P_0(X)| = 1$.

(iii) $\underbrace{\binom{n}{r} = \binom{n}{n-r}}$ for $0 \leq r \leq n$.

choosing which elements to include is the same as choosing which to leave out

Proof:

$$\begin{aligned} P_r(X) &\to P_{n-r}(X) \\ A &\mapsto A^c \end{aligned}$$

is a bijection. $\square$

Remark: Note that $P(X)$ may be written as the disjoint union $P_0(X) \cup P_1(X) \cup \cdots \cup P_n(X)$ where $n = |X|$. Hence:

$\longrightarrow$ **Proposition 12.2.7:** $\displaystyle\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Theorem 12.2.10: For $n, r \in \mathbb{N} \cup \{0\}$ with $0 \leq r \leq n$ we have

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)\cdot\ldots\cdot(n-r+1)}{r!}.$$

Proof:

Let X be a set with n elements. For each subset A of X with $|A| = r$ there is a bijection of A with $\mathbb{N}_r$ (in fact there are $r!$ of them, as the bijection can be composed with any permutation of A to give another bijection and all bijections are obtained this way), hence there is an injection $\mathbb{N}_r \to X$ whose image is A, in fact there are precisely $r!$ injections $\mathbb{N}_r \to X$ whose images are A.

Since

1. each injection $\mathbb{N}_r \to X$ defines some subset $A \subseteq X$ with $|A| = r$ (the image of the injection),

   and

2. for each subset $A \subseteq X$ there are $r!$ injections $\mathbb{N}_r \to X$ with image A

we have:

$$\underbrace{\binom{n}{r}}_{\substack{\#\text{ of subsets } A \subseteq X \\ \text{with } |A| = r}} = \frac{\#\text{ of injections } \mathbb{N}_r \to X}{r!} = \frac{n(n-1)\cdot\ldots\cdot(n-r+1)}{r!}.$$

$\square$

Proposition 12.2.8: For $n, r \in \mathbb{N}$ with $r \leq n$ we have

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Remark: This is the rule we use to generate "Pascal's triangle":

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}$$

Proof: (See the book for a different proof which is equally straightforward.)

Given $n, r \in \mathbb{N}$ with $r \leq n$

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-1-r)!}$$

$$= (n-1)! \left[\frac{1}{(r-1)!\,(n-r)!} + \frac{1}{r!\,(n-1-r)!} \right]$$

$$= (n-1)! \left[\frac{r + (n-r)}{r!\,(n-r)!} \right]$$

put the fractions over a common denominator

$$= (n-1)! \cdot \frac{n}{r!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!} = \binom{n}{r}. \qquad \square$$

Theorem 12.3.1 (The Binomial Theorem):

For $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

(n = 0 is also fine here)

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

since $\binom{n}{n-i} = \binom{n}{i}$

Proof: Let $a, b \in \mathbb{R}$. We use induction on n.

Base case ($n=1$):

$$(a+b)^1 = a+b \quad \text{and}$$

$$\sum_{i=0}^{1} \binom{1}{i} a^{1-i} b^i = \binom{1}{0} a + \binom{1}{1} b = a+b$$

so the base case holds.

Inductive step: Let $k \in \mathbb{N}$ and suppose that $(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$. Then

$$(a+b)^{k+1} = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i (a+b)$$

$$= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1}$$

writing j for $i+1$

$$= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{j=1}^{k+1} \binom{k}{j-1} a^{k+1-j} b^j$$

writing i for j

$$= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=1}^{k+1} \binom{k}{i-1} a^{k+1-i} b^i$$

$$= \binom{k}{0} a^{k+1} b^0 + \underbrace{\sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^i}_{\text{use Prop. 12.2.8}} + \binom{k}{k} a^0 b^{k+1}$$

$$= a^{k+1} + \sum_{i=1}^{k} \binom{k+1}{i} a^{k+1-i} b^i + b^{k+1}$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i ,$$

as required. This proves the inductive step.

Hence, by induction, the binomial formula holds for all $n \in \mathbb{N}$, and since $a, b \in \mathbb{R}$ were arbitrary, it holds for all $a, b \in \mathbb{R}$. $\square$

Examples:

$(a+b)^1 = a + b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$\ldots\ldots$

The coefficient of a^6b^4 in $(a+b)^{10}$ is given by $\binom{10}{6} = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} = \frac{10 \times 3 \times 7}{1} = 210$.

(cancelling 6! top & bottom)