Chapter 10: Counting

Notation: Given a natural number n, we will write $\mathbb{N}_n = \{1, \ldots, n\} = \{i \in \mathbb{Z} \mid 1 \le i \le n\}$.

Def^n: Given a set X, if there is a bijection $f: \mathbb{N}_n \to X$ then we say that X has cardinality n, written $|X| = n$.

We say that the empty set $\emptyset$ has cardinality 0, $|\emptyset| = 0$.

cardinality means number of elements (or "size" of the set)

If X is empty, or has cardinality $n \in \mathbb{N}$, then we say X is finite; otherwise, we say X is infinite.

Examples:

- $|\{1,2,3\}| = |\{2, \pi, 17\}| = |\{a,b,c\}| = 3$

  three distinct "objects"

- $\mathbb{N}$ is infinite.

- Let $A = \{p \in \mathbb{N} \mid p \text{ is prime and } p < 100\}$.
  Then $|A| = 25$.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Theorem 10.2.1 (The addition principle):

Let X, Y be disjoint finite sets. Then $X \cup Y$ is finite and

$$|X \cup Y| = |X| + |Y|.$$

Proof: Easy exercise. $\square$

Idea:

$$\{1, \dots, n\} \xleftrightarrow{\text{bijection}} X$$

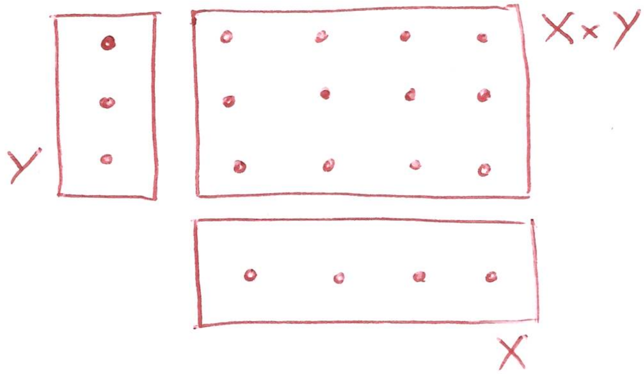
$$\{\overbrace{1, \dots, n}, \underbrace{n+1, \dots, n+m}\}$$

$$\{1, \dots, m\} \xleftrightarrow[\text{bijection}]{} Y$$

Theorem 10.2.3 (The multiplication principle):

Let X and Y be finite sets with $|X| = n$ and $|Y| = m$. Then $X \times Y$ is finite and $|X \times Y| = nm$.

Proof: Easy exercise. $\square$



Proposition A: For any set X, if there is an injection $f: X \to \mathbb{N}_n$, then X is finite and $|X| \le n$.

This proposition should be highly intuitive, but we include a proof here for completeness.

Proof: We use induction on n.

Base case $(n=1)$: If $f: X \to \{1\}$ is an injection, then X must have only one element. So the base case holds.

Inductive step: Let $k \in \mathbb{N}$ and suppose that Proposition A holds when $n=k$. Let $f: X \to \mathbb{N}_{k+1}$ be an injection.

for all sets X

Case 1: If $k+1 \notin f(X)$ then we can restrict the codomain of f to obtain an injection $X \to \mathbb{N}_k$. Hence, by the inductive hypothesis, X is finite and $|X| \le k \le k+1$.

Case 2: If $k+1 \in f(X)$ then there is a unique element $x_0 \in X$ s.t. $f(x_0) = k+1$. Let $\hat{X} = X - \{x_0\}$. Let $\hat{f}: \hat{X} \to \mathbb{N}_k$ be given by $\hat{f}(x) = f(x)$ $\forall x \in \hat{X}$. Then $\hat{f}$ is an injection. So, by the inductive hypothesis, $\hat{X}$ is finite and $|\hat{X}| \le k$. It follows that $X = \hat{X} \cup \{x_0\}$ is finite and $|X| \le k+1$, as required.

This proves the inductive step.

The result then follows for all $n \in \mathbb{N}$ by induction. $\square$

<u>Proposition B</u>: For any set X, if there is a surjection $f: \mathbb{N}_n \to X$, then X is finite and $|X| \le n$.

The proof is similar to the proof of Proposition A. Again this is included for completeness.

<u>Proof</u>: We use induction on n.

Base case $(n=1)$: If $f:\{1\} \to X$ is a surjection, then X must have only one element. So the base case holds.

Inductive step: Let $k \in \mathbb{N}$ and suppose that <u>Proposition B</u> <u>holds</u> when $n=k$. Let $f: \mathbb{N}_{k+1} \to X$ be a surjection.

for all <u>sets X</u>

Case 1: If $f|_{\mathbb{N}_k}$ is a surjection, then by the inductive hypothesis X is finite and $|X| \le k \le k+1$.

Case 2: If $f|_{\mathbb{N}_k}$ is not a surjection, let $x_0 = f(k+1)$ and let $\hat{f}: \mathbb{N}_k \to \hat{X} = X - \{x_0\}$ be given by $\hat{f}(x) = f(x)$ $\forall x \in \mathbb{N}_k$. Then $\hat{f}$ is a surjection. So, by the inductive hypothesis, $\hat{X}$ is finite and $|\hat{X}| \le k$. Hence $X = \hat{X} \cup \{x_0\}$ is finite and $|X| \le k+1$.

This proves the inductive step.

The result then follows for all $n \in \mathbb{N}$ by induction. $\square$

We stated Propositions A & B in this way to make them easy to prove by induction. What we really want to observe are the following consequences.

Corollary A: Let X, Y be sets. If Y is finite and there is an injection $f: X \to Y$, then X is finite and $|X| \le |Y|$.

Corollary B: Let X, Y be sets. If X is finite and there is a surjection $f: X \to Y$, then Y is finite and $|X| \ge |Y|$.

We also observe:

Proposition: Let X, Y be finite sets. If there is a bijection $f: X \to Y$, then $|X| = |Y|$.

The Inclusion-Exclusion Principle

An extremely useful combinatorial technique.

Proposition (The inclusion-exclusion principle for two sets): Let X, Y be finite sets. Then $X \cup Y$ is finite and

$$|X \cup Y| = |X| + |Y| - \underbrace{|X \cap Y|}_{\text{avoids double counting the overlap.}}$$

Proof: The sets X, Y and $X \cup Y$ can be written as disjoint unions as follows:

$$X = (X - Y) \cup (X \cap Y),$$

$$Y = (Y-X) \cup (X \cap Y)$$
$$X \cup Y = (X-Y) \cup (Y-X) \cup (X \cap Y).$$

The result then follows easily from the addition principle. $\square$

Example Application:

How many integers n, $1 \le n \le 500$, are divisible by 7 or 11?

Solution:

Let $A = \{n \in \mathbb{Z} \mid 1 \le n \le 500 \text{ and } n \text{ is divisible by } 7\}$,

and $B = \{n \in \mathbb{Z} \mid 1 \le n \le 500 \text{ and } n \text{ is divisible by } 11\}$.

Then $A \cup B = \{n \in \mathbb{Z} \mid 1 \le n \le 500 \text{ and } n \text{ is divisible by } 7 \text{ or } 11\}$.

We need to find $|A \cup B|$.

Now $A = \{n \in \mathbb{Z} \mid n = 7k \text{ for some } k \in \mathbb{Z} \text{ with } 1 \le k \le 71\}$

so $|A| = 71$.

$7 \times 71 = 497$

Also $B = \{n \in \mathbb{Z} \mid n = 11k \text{ for some } k \in \mathbb{Z} \text{ with } 1 \le k \le 45\}$

so $|B| = 45$.

$11 \times 45 = 495$

Note that $A \cap B = \{n \in \mathbb{Z} \mid 1 \le n \le 500 \text{ and } n \text{ is divisible by } 7 \text{ and } 11\}$,

so $A \cap B = \{n \in \mathbb{Z} \mid 1 \le n \le 500 \text{ and } n \text{ is divisible by } 77\}$

which equals $\{n \in \mathbb{Z} \mid n = 77k \text{ for some } k \in \mathbb{Z} \text{ with } 1 \le k \le 6\}$.

So $|A \cap B| = 6$.

$77 \times 6 = 462$

Hence $|A \cup B| = 71 + 45 - 6 = 110$.

Proposition (The inclusion-exclusion principle for three sets):

Let X, Y, Z be finite sets. Then $X \cup Y \cup Z$ is finite and

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| \underbrace{- |X \cap Y| - |X \cap Z| - |Y \cap Z|}_{\text{corrects for double counting}} \underbrace{+ |X \cap Y \cap Z|}.$$

corrects for the fact that $-|X \cap Y| - |X \cap Z| - |Y \cap Z|$ is an overcorrection in the case where $X \cap Y \cap Z \neq \emptyset$.

Proof: We compute, using the inclusion exclusion principle for two sets and the distributive law,

$$\begin{aligned}
|X \cup Y \cup Z| &= |(X \cup Y) \cup Z| \\
&= |X \cup Y| + |Z| - |(X \cup Y) \cap Z| \\
&= |X| + |Y| - |X \cap Y| + |Z| \\
&\quad - |(X \cap Z) \cup (Y \cap Z)| \\
&= |X| + |Y| + |Z| - |X \cap Y| \\
&\quad - \left(|X \cap Z| + |Y \cap Z| - |\underbrace{X \cap Z \cap Y \cap Z}_{X \cap Y \cap Z}|\right) \\
&= |X| + |Y| + |Z| - |X \cap Y| \\
&\quad - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|.
\end{aligned}$$

$\square$

Question: Why the $+|X \cap Y \cap Z|$?

Note that if $x \in X \cap Y \cap Z$ then

- $|X| + |Y| + |Z|$ counts x 3 times;
- $-|X \cap Y| - |Y \cap Z| - |Z \cap X|$ counts x -3 times;
- $+|X \cap Y \cap Z|$ counts x 1 time.

Example Application:

Suppose a class has 90 students, and three discussion sections (9am, 10am, 11am). Suppose 3 students go to the 9am section, 21 go to the 10am section, and 25 go to the 11am section. If 2 students go to both the 9am & 10am sections, 1 student goes to both the 9am & 11am sections, 7 students go to both the 10am & 11am sections, and 1 student goes to all three sections, how many students don't bother going to discussion?

Solution:

A: set of students attending 9am section

B: ——— " ——— 10am section

C: ——— " ——— 11am section.

$|A \cup B \cup C|$ = # of students attending discussion.

$|A \cup B \cup C| = 3 + 21 + 25 - 2 - 1 - 7 + 1 = 40.$

$\rightarrow$ So 50 students do not attend.

Disclaimer: These numbers are completely made up.

Inclusion-Exclusion Principle for Four Sets:
Let X, Y, Z, W be finite sets. Then $X \cup Y \cup Z \cup W$ is finite and

$$
\begin{aligned}
|X \cup Y \cup Z \cup W| = {} & |X| + |Y| + |Z| + |W| \\
& - |X \cap Y| - |X \cap Z| - |X \cap W| - |Y \cap Z| \\
& - |Y \cap W| - |Z \cap W| + |X \cap Y \cap Z| \\
& + |X \cap Y \cap W| + |X \cap Z \cap W| + |Y \cap Z \cap W| \\
& - |X \cap Y \cap Z \cap W| .
\end{aligned}
$$

The pattern continues for five, six, or $\underline{\underline{n}}$ sets.