

A SMOOTH COUNTEREXAMPLE TO THE TRAUTMAN CONJECTURE

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ABSTRACT. The Trautman conjecture asserted that a smooth three-dimensional CR manifold admitting a nowhere-zero closed section of its canonical bundle must be locally embeddable in $\mathbb{C}^2$. We modify a standard construction of nonembeddable smooth strongly pseudoconvex CR 3-manifolds so that the condition of having a nowhere-zero closed section of the canonical bundle is preserved, thus providing a strongly pseudoconvex counterexample to the Trautman conjecture.

1. INTRODUCTION AND MAIN RESULTS

Let M be a smooth 3-dimensional CR manifold with CR line bundle $T^{0,1} \subset \mathbb{C}TM$. Its *canonical bundle* is the complex line bundle

$$K = \{ \Omega \in \mathbb{C}\Lambda^2 T^*M : \iota_L \Omega = 0 \text{ for every } L \in T^{0,1} \}.$$

This paper constructs a counterexample to the following well-known conjecture:

The Trautman Conjecture ([34]). *If a smooth CR 3-manifold admits a nowhere-zero closed section of the canonical bundle, then it is locally CR embeddable in $\mathbb{C}^2$.*

1.1. Origins of the Conjecture. The Trautman conjecture arose naturally out of the work of Robinson, Trautman and many others on exact solutions of the Maxwell and Einstein equations via dimensional reduction, going back to the late 1950s and early 1960s [25, 26, 27, 28]. The link to Cauchy–Riemann (CR) geometry was provided by the key observation that a 4-dimensional Lorentzian manifold admitting a shear-free congruence of null geodesics induces a CR structure on the local 3-dimensional leaf space of this congruence [24, 29, 35]. This CR structure is strongly pseudoconvex when the congruence is *twisting* (which is, locally, a generic condition). Under suitable hypotheses, the Maxwell and Einstein equations can then be reinterpreted in terms of the Cauchy–Riemann operator on the underlying CR 3-manifold [34, 11, 19, 20, 31]. When this CR 3-manifold is taken to be embedded in $\mathbb{C}^2$, the prevalence of CR functions (and closed sections of the canonical bundle) coming from ambient holomorphic functions allows for the construction of many explicit solutions. This framework is natural in the sense that every algebraically special vacuum Einstein spacetime admits such a geodesic

congruence (the Goldberg-Sachs theorem [9, 10]), as does every spacetime admitting a null electromagnetic field (the Robinson theorem [26, 32]). Celebrated examples of metrics which arise from shear-free null geodesic congruences in this way include the Kerr and Kerr–Newman black holes.

The Trautman conjecture concerns a specific construction of 4-dimensional Lorentzian manifolds admitting null Maxwell fields [29, 34, 33], which starts from a CR 3-manifold together with a nowhere-zero closed section of its canonical bundle. Trautman conjectured that all such Lorentzian 4-manifolds locally arise from the data of an embedded hypersurface in $\mathbb{C}^2$ (and a closed section of the canonical bundle obtained via the embedding). From this point of view, the counterexample construction below therefore shows that the scope for using 3-dimensional CR manifolds to construct 4-dimensional Lorentzian manifolds admitting null electromagnetic fields is larger than was expected.

When formulating his conjecture, Trautman was influenced by the then-current state of knowledge concerning embeddability of CR structures [34]. In particular, he noted that Rosay had constructed nonembeddable CR 3-manifolds for which all local CR functions depend on a single CR function [30] and that, by the results of Tafel and Jacobowitz, these examples cannot carry a nowhere-zero closed section of the canonical bundle [32, 14, 34]; he credited C. Denson Hill with pointing out the relevance of Rosay's examples and Jacobowitz's results. The known 3-dimensional examples therefore did not contradict the proposed implication. On the other hand, Jacobowitz had constructed nonembeddable CR manifolds of real dimension seven whose canonical bundles do admit nowhere-zero closed sections [14]. Trautman nevertheless conjectured that the existence of a nowhere-zero closed section of the canonical bundle would imply local embeddability in the exceptional 3-dimensional case.

1.2. Main Results. We now state our main result. Let (z, t) be the coordinates on $\mathbb{C} \times \mathbb{R}$ and define

$$L_0 = \partial_{\bar{z}} - \frac{i}{2}z\partial_t, \quad w = t + \frac{i}{2}|z|^2.$$

Theorem 1.1. *For every neighborhood U of the origin in $\mathbb{C} \times \mathbb{R}$ and every $\varepsilon > 0$, there is a nonnegative function $\phi \in C^\infty(\mathbb{C} \times \mathbb{R})$ supported in U , with $\|\phi\|_{C^2} < \varepsilon$, such that the complex line field*

$$(1) \quad T_\phi^{0,1} = \text{span}_{\mathbb{C}}\{L_\phi\}, \quad L_\phi = L_0 + \phi_t\partial_z - \phi_z\partial_t,$$

has the following properties:

- (i) $T_\phi^{0,1}$ is a smooth strongly pseudoconvex CR structure;

(ii) *its canonical bundle has the nowhere-zero closed section*

$$\Omega_\phi = \iota_{L_\phi}(dz \wedge d\bar{z} \wedge dt) = -dz \wedge dw - d(\phi d\bar{z});$$

(iii) *every C^1 CR function h defined near the origin satisfies $dh(0) = 0$.*

In particular, $T_\phi^{0,1}$ is not locally CR embeddable at the origin.

Remark 1.2. In [14] (cf. [15]) Jacobowitz has shown that a strongly pseudoconvex CR 3-manifold which admits a nowhere-zero closed section of the canonical bundle is locally embeddable in a neighborhood of a given point if and only if it admits a single strongly independent CR function in a neighborhood of that point. Note that while Theorem 1.1(iii) directly implies nonembeddability at the origin, it also implies the absence of a strongly independent CR function near 0.

In the family of examples that we construct, ϕ vanishes to infinite order at the origin. Hence $T_\phi^{0,1}$ agrees with the Heisenberg CR structure outside U and has the same infinite jet as the Heisenberg structure at the origin. Our construction is very similar to those of Nirenberg [22, 23], Jacobowitz–Trèves [16] and Hill–Nacinovich [12] but differs in that the CR structure (1) depends on the derivatives of a single function ϕ (rather than simply on ϕ itself). The Hamiltonian-like structure of the perturbation term $\phi_t \partial_z - \phi_z \partial_t$ was chosen to ensure that the resulting CR structure admits a nowhere-zero closed section of the canonical bundle. The proof of nonembeddability is similar to that in [16, 22, 23], except that we are forced to introduce weight functions z and w and integrate by parts in order to remove the derivatives from the smooth bump function ϕ . Note that smoothness is essential here; it is well-known that a real analytic strongly pseudoconvex CR structure is always locally embeddable.

Since the existence of a nonvanishing closed section of the canonical bundle is locally equivalent to the existence of a pseudo-Einstein contact form, Theorem 1.1 has the following immediate corollary:

Corollary 1.3. *There is a smooth strongly pseudoconvex CR 3-manifold which admits a pseudo-Einstein contact form but is not locally embeddable in $\mathbb{C}^2$.*

Remark 1.4. In dimension three the standard definition of a pseudo-Einstein contact form comes from [4, 13]. This notion was first introduced in higher dimensions by Lee [18]. The existence of a pseudo-Einstein contact form is locally equivalent to the existence of a nowhere-zero closed section of the canonical bundle in all dimensions.

For embedded CR manifolds, the pseudo-Einstein condition is also related to Fefferman’s complex Monge–Ampère equation, whose exact solution on a bounded strongly

pseudoconvex domain gives the Cheng–Yau complete Kähler–Einstein metric [8, 5, 7, 13]. The pseudo-Einstein condition also simplifies the formal construction of a Kähler–Einstein structure with prescribed CR infinity in [2]. By results of [2, 6, 17], however, a smooth local Kähler–Einstein extension on the pseudoconvex side would yield a local two-sided extension and hence local CR embeddability. Theorem 1.1 therefore shows that the pseudo-Einstein condition alone does not guarantee such an extension.

A complex surface is called *unimodular* if it is endowed with a holomorphic volume form Ω . A holomorphic volume form (such as $dz \wedge dw$ on $\mathbb{C}^2$) pulls back to a nowhere-zero closed section of the canonical bundle on any embedded strongly pseudoconvex CR hypersurface. Consequently, Bryant [3] terms a strongly pseudoconvex CR 3-manifold endowed with a nowhere-zero closed section of the canonical bundle a *unimodular* CR structure. In [3] Bryant considered the natural embedding problem for unimodular CR structures and showed that if such a structure is real analytic then it is always locally realizable as a hypersurface in a unimodular complex surface. Theorem 1.1 implies that this result cannot be extended to smooth unimodular CR structures:

Corollary 1.5. *There is a smooth unimodular CR structure that is not locally realizable as a hypersurface in a unimodular complex surface.*

Remark 1.6. In [3] Bryant gives a geometric flow equation on a unimodular CR 3-manifold M whose solvability for $t \in (-\epsilon, \epsilon)$ implies that $M \times (-\epsilon, \epsilon)$ admits a unimodular complex surface structure inducing the given unimodular CR structure on $M \times \{0\}$. While for the counterexamples constructed in this paper the Bryant flow cannot have a smooth two-sided solution, it remains possible that a one-sided solution always exists realizing the unimodular CR structure as the pseudoconcave boundary of a unimodular complex surface (i.e. locally, there may always be a solution for $t < 0$ small).

In a more positive direction, the counterexample constructed shows that the class of strongly pseudoconvex CR 3-manifolds possessing a nowhere-zero closed section of the canonical bundle required for the various 4-dimensional spacetime constructions given in, e.g., [1, 11, 34, 35] is considerably larger than was previously expected.

1.3. Globalization. We conclude the introduction by noting that the examples constructed in Theorem 1.1 are standard at infinity and therefore extend, via the Cayley compactification, to S^3 as small nonembeddable perturbations of the standard spherical CR structure. Clearly such CR structures admit a nowhere-zero closed section of the canonical bundle on $S^3 \setminus \{\infty\}$. A slight modification of this compactification gives the following global result.

Theorem 1.7. *Given a point $p \in S^3$, there exists a smooth perturbation of the standard spherical CR structure on S^3 that admits a global nowhere-zero closed section of the canonical bundle but is not embeddable in any neighborhood of p . The perturbation can be taken to be arbitrarily small in the C^1 norm.*

In particular, it follows that there is a pseudo-Einstein CR structure near the standard structure on S^3 that is not locally embeddable. Moreover, from the proof in Section 6 below and the characterization of pseudo-Einstein contact forms in terms of volume normalization [4, 13, 18] it follows that the pseudo-Einstein contact form can be taken to be a small perturbation of the standard contact form.

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2. HAMILTONIAN PERTURBATIONS OF THE HEISENBERG CR STRUCTURE

In the presence of a fixed volume form, the Trautman conjecture can be rephrased in terms of a vector field dual to the closed section of the canonical bundle [34]. To see this we write $z = x + iy$ on $\mathbb{C} \times \mathbb{R}$ and fix the background volume form $dV = dx \wedge dy \wedge dt$. For convenience, we will make use of both dV and the complex volume form

$$\nu = dz \wedge d\bar{z} \wedge dt = -2i dV.$$

Since ν is a constant multiple of dV it defines the same notion of divergence as dV (namely the standard divergence operator on $\mathbb{R}^3$). Hence for any real or complex vector field X

$$\mathcal{L}_X \nu = (\operatorname{div} X) \nu.$$

By Cartan's formula, it follows that $\operatorname{div} X = 0$ if and only if $\iota_X \nu$ is closed. Thus, when X is a $(0,1)$ -vector field, $\operatorname{div} X = 0$ if and only if $\Omega = \iota_X \nu$ is a closed section of the canonical bundle.

For a smooth real function ϕ , set

$$(2) \quad X_\phi = \phi_t \partial_z - \phi_z \partial_t, \quad L_\phi = L_0 + X_\phi.$$

Proposition 2.1. *The field L_ϕ is divergence-free relative to ν . If ϕ is supported in a fixed precompact subset of $\mathbb{C} \times \mathbb{R}$ and $\|\phi\|_{C^2}$ is sufficiently small, then $T_\phi^{0,1}$ is a strongly pseudoconvex CR structure on $\mathbb{C} \times \mathbb{R}$ and*

$$\Omega_\phi := \iota_{L_\phi} \nu = -dz \wedge dw - d(\phi d\bar{z})$$

is then a closed section of the canonical bundle of $T_\phi^{0,1} = \text{span}_{\mathbb{C}}\{L_\phi\}$. Moreover, the form Ω_ϕ is nowhere zero.

Proof. Direct computation gives

$$\text{div } X_\phi = \partial_z \phi_t - \partial_t \phi_z = 0,$$

and that L_0 is also divergence-free. It follows that Ω_ϕ is closed.

When $\phi = 0$ one has

$$[L_0, \overline{L_0}] = i\partial_t.$$

Linear independence of L_0 and $\overline{L_0}$ and nondegeneracy of the Levi form are open conditions in the C^1 topology on the defining field, hence in the C^2 topology on ϕ . It follows that $T_\phi^{0,1}$ is a strongly pseudoconvex CR structure when $\|\phi\|_{C^2}$ is small.

Contracting with ν we have

$$\begin{aligned} \iota_{L_0} \nu &= -dz \wedge dt - \frac{i}{2} z dz \wedge d\bar{z} = -dz \wedge dw, \\ \iota_{X_\phi} \nu &= \phi_t d\bar{z} \wedge dt - \phi_z dz \wedge d\bar{z} = -d(\phi d\bar{z}), \end{aligned}$$

and hence $\Omega_\phi = \iota_{L_\phi} \nu = -dz \wedge dw - d(\phi d\bar{z})$. Clearly, $\iota_{L_\phi} \Omega_\phi = 0$ so Ω_ϕ is a section of the canonical bundle. The coefficient of $dz \wedge dt$ in Ω_ϕ is -1 , so Ω_ϕ never vanishes. $\square$

3. BUMP FUNCTIONS ON SHRINKING SOLID TORI

Here we construct the desired scalar perturbation function ϕ . For $j \geq 0$ let

$$a_j = 2^{-j}, \quad A_j = \left\{ w : |w - ia_j| < ca_j \right\}, \quad H_j = \left\{ w : |w - ia_j| < \frac{1}{2}ca_j \right\},$$

where $0 < c < \frac{1}{3}$. Note that every disk A_j lies in the upper half-plane and the disks accumulate only at the origin.

Consider the map $w : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}$ given by $w = t + \frac{i}{2}|z|^2$, as above. This map has circular fibers over points in the upper half-plane that shrink to points as one approaches the real axis. Abusing notation by writing w also for a point in the image plane, the fiber over a point $w = u + iv$ with $v > 0$ is then

$$(3) \quad \Gamma_w = \{t = u, |z|^2 = 2v\} \simeq S^1.$$

Thus

$$D_j = w^{-1}(A_j)$$

is the interior of a solid torus. The sets D_j are pairwise disjoint and shrink to the origin.

Choose a nonnegative, nonzero function $\rho \in C_c^\infty(\mathbb{C})$ with

$$\text{supp } \rho \subset \{\zeta : |\zeta| < 1/2\}.$$

Define the explicit bump functions

$$(4) \quad \phi_j(z, t) = e^{-1/a_j^2} \rho\left(\frac{w(z, t) - ia_j}{ca_j}\right), \quad \phi = \delta \sum_{j \geq j_0} \phi_j,$$

where j_0 will be taken large and $\delta > 0$ small. Then

$$\text{supp } \phi_j \subset w^{-1}(H_j) \Subset D_j.$$

Lemma 3.1. *The function ϕ in (4) is smooth and vanishes to infinite order at the origin. Given a prescribed neighborhood U of the origin and $\varepsilon > 0$, the parameters j_0 and δ can be chosen so that $\text{supp } \phi \subset U$ and $\|\phi\|_{C^2} < \varepsilon$.*

Proof. On D_j one has $|t| = O(a_j)$ and $|z| = O(a_j^{1/2})$. For every multi-index α , differentiation of the rescaled bump gives the bound

$$(5) \quad \|\partial^\alpha \phi_j\|_{L^\infty} \leq C_\alpha e^{-1/a_j^2} a_j^{-|\alpha|}.$$

The exponential factor dominates every inverse power of a_j . Since the distance of D_j from the origin is comparable to $a_j^{1/2}$, every derivative of ϕ_j tends to zero faster than every power of that distance. The supports are disjoint, so the sum is locally finite away from the origin and extends across the origin with zero infinite jet. Taking j_0 large puts all supports in U , and the final factor δ gives the asserted C^2 smallness. $\square$

4. WEIGHTED MOMENT IDENTITIES

Here we apply ideas from [21, 23, 16, 12] to establish a key family of weighted moment identities that will be used in the next section to prove that L_ϕ is not locally embeddable.

Given a point $w = u + iv$ in the upper half-plane and $\chi(z, \bar{z}, t)$ a C^1 function on $\mathbb{C} \times \mathbb{R}$, we define the fiber period to be

$$(6) \quad I_\chi(w) = \oint_{\Gamma_w} \chi dz,$$

where Γ_w is as in (3). Setting $r = (2v)^{1/2}$ we therefore have

$$(7) \quad I_\chi(w) = \int_0^{2\pi} i r e^{i\vartheta} \chi(r e^{i\vartheta}, r e^{-i\vartheta}, u) d\vartheta.$$

Lemma 4.1. *Let G be a connected open subset of the upper half-plane that contains a one-sided neighborhood $\{u + iv : u \in J, v \in (0, \eta)\}$ of an open interval J in the real line for some $\eta > 0$. Suppose that χ is C^1 in a neighborhood of $\overline{w^{-1}(G)}$ and satisfies*

$$(8) \quad L_0 \chi = 0 \quad \text{on } w^{-1}(G).$$

Then

$$(9) \quad I_\chi(w) = 0 \quad \text{for all } w \in G.$$

If, in addition, A is an open disk compactly contained in the upper half-plane, $\partial A \subset G$, and the same χ is C^1 in a neighborhood of $\overline{w^{-1}(A)}$, then

$$(10) \quad \int_{w^{-1}(A)} L_0 \chi \, dV = 0.$$

Proof. Differentiating (7) with respect to u we have

$$(11) \quad \partial_u I_\chi = \int_0^{2\pi} i z \chi_t \, d\vartheta.$$

Similarly, since $r = (2v)^{1/2}$ implies $\partial_v = \frac{1}{r} \partial_r$, we have

$$(12) \quad \partial_v I_\chi = \frac{i}{r} \int_0^{2\pi} \partial_r(z\chi) \, d\vartheta = \frac{i}{r^2} \int_0^{2\pi} (z\chi + z^2 \chi_z + r^2 \chi_{\bar{z}}) \, d\vartheta,$$

where $z = re^{i\vartheta}$. Now, integrating an angular derivative from 0 to 2π gives zero, and hence

$$(13) \quad 0 = \int_0^{2\pi} \partial_\vartheta(z\chi) \, d\vartheta = i \int_0^{2\pi} (z\chi + z^2 \chi_z - r^2 \chi_{\bar{z}}) \, d\vartheta.$$

Combining (12) and (13) gives

$$(14) \quad \partial_v I_\chi = i \int_0^{2\pi} 2\chi_{\bar{z}} \, d\vartheta.$$

Combining (11) and (14) we therefore obtain

$$(15) \quad 2\partial_{\bar{w}} I_\chi = \int_0^{2\pi} (iz\chi_t - 2\chi_{\bar{z}}) \, d\vartheta = -2 \int_0^{2\pi} L_0 \chi \, d\vartheta = 0.$$

Thus I_χ is holomorphic on G . When a fiber collapses to the real axis,

$$(16) \quad |I_\chi(u + iv)| \leq 2\pi(2v)^{1/2} \sup_{\Gamma_{u+iv}} |\chi| \longrightarrow 0.$$

In particular, I_χ extends continuously to zero on the real open interval J . Therefore, I_χ extends holomorphically across J by Schwarz reflection. The identity theorem then implies that I_χ is zero near J , and the connectedness of G gives (9).

To prove (10), we first note that by integrating fiber-wise and using (9) we have, up to a sign depending on the choices of orientations,

$$(17) \quad \int_{\partial w^{-1}(A)} \chi dz \wedge dw = \pm \int_{\partial A} \left(\oint_{\Gamma_w} \chi dz \right) dw = \pm \int_{\partial A} I_\chi(w) dw = 0.$$

Then, since

$$(18) \quad d(\chi dz \wedge dw) = -L_0 \chi dz \wedge d\bar{z} \wedge dt = -L_0 \chi \nu,$$

it follows by Stokes' theorem that

$$(19) \quad 0 = \int_{\partial w^{-1}(A)} \chi dz \wedge dw = \int_{w^{-1}(A)} d(\chi dz \wedge dw) = - \int_{w^{-1}(A)} L_0 \chi \nu.$$

Since $\nu = -2i dV$, this proves (10). $\square$

An important consequence of (10) is the following corollary:

Corollary 4.2. *Let ϕ be given by (4), and let h be a C^1 function defined in a neighborhood V of the origin in $\mathbb{C} \times \mathbb{R}$ satisfying $L_\phi h = 0$. Then, for every sufficiently large j ,*

$$(20) \quad \int_{D_j} z L_0 h dV = 0, \quad \int_{D_j} w L_0 h dV = 0.$$

Proof. Let U be an open disk centered at the origin in the complex plane small enough that $w^{-1}(\bar{U})$ is contained in V . Let U^+ be the intersection of U with the upper half-plane, and let G be U^+ minus the closure of $\bigcup_j H_j$. Note that for all j sufficiently large, $\bar{A}_j$ is contained in U^+ . Since the closed disks $\bar{H}_j$ are pairwise separated and accumulate only at the boundary point 0, their complement G is path connected and contains a one-sided neighborhood of an open interval in the real axis (in the sense of Theorem 4.1). By (4), $\phi = 0$ on $w^{-1}(G)$, and hence $L_0 h = 0$ there. The two standard CR functions z and $w = t + \frac{i}{2}|z|^2$ on the Heisenberg group satisfy

$$(21) \quad L_0 z = L_0 w = 0.$$

It follows that $L_0(zh) = L_0(wh) = 0$ on $w^{-1}(G)$, and that $L_0(zh) = zL_0 h$ and $L_0(wh) = wL_0 h$ on $w^{-1}(U^+)$. When j is sufficiently large, $\partial A_j \subset G$. Applying Theorem 4.1 with $\chi = zh$ and with $\chi = wh$ therefore gives (20). $\square$

5. WEIGHTED MOMENTS AND NONEMBEDDABILITY

We now use the two weighted identities to show that the differential of every local CR function must vanish at the origin.

Proposition 5.1. *Let ϕ be the function in (4). If h is a C^1 function satisfying $L_\phi h = 0$ near the origin, then*

$$(22) \quad dh(0) = 0.$$

Proof. For every sufficiently large j , the solid torus D_j lies in the domain of h . Because the supports are disjoint, only $\delta\phi_j$ is nonvanishing on D_j , and the equation $L_\phi h = 0$ can be written as

$$(23) \quad L_0 h = -\delta X_{\phi_j} h, \quad \text{where } X_{\phi_j} = (\phi_j)_t \partial_z - (\phi_j)_z \partial_t.$$

The vector field X_{ϕ_j} is divergence-free and compactly supported in D_j .

From the first identity in (20) we have

$$(24) \quad 0 = \int_{D_j} z L_0 h \, dV = -\delta \int_{D_j} z X_{\phi_j} h \, dV.$$

Integrating by parts (using that $\operatorname{div} X_{\phi_j} = 0$), using that $X_{\phi_j} z = (\phi_j)_t$ and then integrating by parts again, we obtain

$$(25) \quad 0 = \int_{D_j} z X_{\phi_j} h \, dV = - \int_{D_j} h X_{\phi_j} z \, dV = - \int_{D_j} h (\phi_j)_t \, dV = \int_{D_j} \phi_j h_t \, dV.$$

Define the positive mass

$$(26) \quad M_j := \int_{D_j} \phi_j \, dV > 0.$$

The probability measures $M_j^{-1} \phi_j dV$ have supports shrinking to the origin. Continuity of h_t therefore gives

$$(27) \quad h_t(0) = \lim_{j \rightarrow \infty} \frac{1}{M_j} \int_{D_j} \phi_j h_t \, dV = 0.$$

We now carry out a similar calculation, starting with the second identity in (20). Note first that

$$(28) \quad X_{\phi_j} w = \frac{i}{2} \bar{z} (\phi_j)_t - (\phi_j)_z.$$

Imitating the calculation in the previous paragraph therefore gives

$$(29) \quad \begin{aligned} 0 &= \int_{D_j} h X_{\phi_j} w \, dV \\ &= \int_{D_j} h \left(\frac{i}{2} \bar{z} (\phi_j)_t - (\phi_j)_z \right) dV \\ &= \int_{D_j} \phi_j \left(h_z - \frac{i}{2} \bar{z} h_t \right) dV. \end{aligned}$$

After division by M_j , the first term tends to $h_z(0)$. The second tends to zero because h_t is bounded and $\sup_{D_j} |z| \rightarrow 0$. Hence

$$(30) \quad h_z(0) = 0.$$

Finally, since $\phi_t(0) = \phi_z(0) = 0$ and $z(0) = 0$, we have $L_\phi|_0 = L_0|_0 = \partial_{\bar{z}}$. Thus $L_\phi h|_0 = 0$ gives $h_{\bar{z}}(0) = 0$. Together with (27) and (30), this proves (22). $\square$

Proof of Theorem 1.1. By Theorem 2.1 and Theorem 3.1 we may choose j_0 large enough such that the support of the smooth function ϕ given in (4) is contained in the prescribed neighborhood U , and then choose δ small enough to guarantee the required C^2 bound and strong pseudoconvexity. Theorem 2.1 then gives the required nowhere-zero closed canonical section Ω_ϕ , and Theorem 5.1 shows that every C^1 CR function has zero differential at the origin. It follows immediately that $T_\phi^{0,1}$ is not locally CR embeddable at the origin since any local CR embedding into $\mathbb{C}^2$ would have to have zero differential at the origin (its two components being C^1 CR functions) and would therefore fail to be an embedding. This proves Theorem 1.1. $\square$

6. GLOBALIZING TO S^3

Proof of Theorem 1.7. Regard S^3 as the unit sphere in $\mathbb{C}^2$, with complex coordinates (z_1, z_2) , and let $\iota: S^3 \rightarrow \mathbb{C}^2$ denote the inclusion. The complex 2-form $\Omega_S = \iota^*(dz_1 \wedge dz_2)$ is a global nowhere-zero closed section of the canonical bundle for the standard spherical CR structure on S^3 . Choose antipodal points $q, p \in S^3$ and a Cayley transform

$$\Psi: \mathbb{C} \times \mathbb{R} \longrightarrow S^3 \setminus \{q\}$$

that carries the origin to p and identifies the standard spherical CR structure with the Heisenberg CR structure. Recall that

$$\Omega_0 := \iota_{L_0} \nu = -dz \wedge dw.$$

Since $\Psi^* \Omega_S$ and Ω_0 are nowhere-zero sections of the same complex line bundle, there is a smooth nowhere-zero complex-valued function g on $\mathbb{C} \times \mathbb{R}$ such that

$$(31) \quad \Psi^* \Omega_S = g \Omega_0.$$

Both sides of (31) are closed. Hence

$$0 = d(g \Omega_0) = dg \wedge \Omega_0 = (L_0 g) \nu,$$

so that

$$(32) \quad L_0 g = 0.$$

Choose the function ϕ in (4) with support in a relatively compact neighborhood of the origin in the Cayley chart. The compactly supported one-form $\phi d\bar{z}$ defines a compactly supported one-form β on $S^3 \setminus \{q\}$ by $\Psi^* \beta = \phi d\bar{z}$. The one-form β extends by zero to a one-form on S^3 , which we also denote by β . Set

$$(33) \quad \hat{\Omega} = \Omega_S - d\beta.$$

The form $\widehat{\Omega}$ is globally defined and closed. In the Cayley chart, Theorem 2.1 and (31) give

$$(34) \quad \Psi^* \widehat{\Omega} = g\Omega_0 - d(\phi d\bar{z}) = \iota_{gL_0 + X_\phi} \nu.$$

For $\|\phi\|_{C^2}$ sufficiently small, $\widehat{\Omega}$ is nowhere zero and its kernel defines a strongly pseudoconvex CR structure $\widehat{T}^{0,1}$ on S^3 . Indeed, in the Cayley chart this line field is spanned by

$$gL_0 + X_\phi = g(L_0 + g^{-1}X_\phi),$$

and since g^{-1} and its first derivatives are bounded on the fixed compact support of ϕ , the corresponding CR structure is therefore a C^1 -small perturbation of the Heisenberg CR structure when $\|\phi\|_{C^2}$ is small. Outside the support of ϕ it is the standard spherical CR structure. Thus, by Theorem 3.1, the perturbation can be taken to be arbitrarily small in the C^1 norm.

It remains to prove nonembeddability at p . Let h be a C^1 CR function defined near p , and denote its pullback to the Cayley chart again by h . Then

$$(35) \quad (gL_0 + X_\phi)h = 0.$$

For all sufficiently large j , the solid torus D_j lies in the domain of h . On the set $w^{-1}(G)$ used in the proof of Theorem 4.2, one has $X_\phi = 0$, and therefore $L_0h = 0$ there. In view of (21) and (32), the two functions

$$gzh, \quad gwh$$

are also annihilated by L_0 on $w^{-1}(G)$. Applying Theorem 4.1 to these two functions gives

$$(36) \quad 0 = \int_{D_j} L_0(gzh) dV = \int_{D_j} gzL_0h dV, \quad 0 = \int_{D_j} L_0(gwh) dV = \int_{D_j} gwL_0h dV.$$

On D_j only the summand $\delta\phi_j$ is nonzero, so (35) reads

$$gL_0h = -\delta X_{\phi_j}h.$$

Consequently, (36) implies

$$\int_{D_j} zX_{\phi_j}h dV = 0 \quad \text{and} \quad \int_{D_j} wX_{\phi_j}h dV = 0.$$

The integration-by-parts arguments in the proof of Theorem 5.1 now apply without change and yield

$$\int_{D_j} \phi_j h_t dV = 0 \quad \text{and} \quad \int_{D_j} \phi_j \left(h_z - \frac{i}{2} \bar{z} h_t \right) dV = 0.$$

Dividing by the masses M_j in (26) and letting $j \rightarrow \infty$ gives

$$h_t(0) = h_z(0) = 0.$$

Finally, X_ϕ vanishes at the origin and $g(0) \neq 0$, so (35) also gives $h_{\bar{z}}(0) = 0$. Thus every C^1 CR function near p has zero differential at p . It follows that $\hat{T}^{0,1}$ is not locally embeddable at p , completing the proof. $\square$

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