Obstruction Flat Rigidity of the CR 3-Sphere

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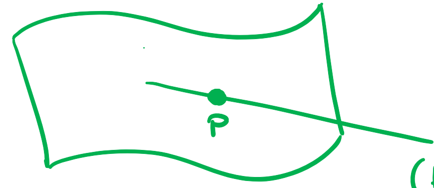
CR 3-Manifolds

$M^3 \subseteq \mathbb{C}^2$ real hypersurface

$\rightsquigarrow$ CR structure (M, H, J)



(H_p, J_p) max. compl. subspace

- (generically) H contact distribution

  $\hookrightarrow H = \ker\theta$, θ 1-form, $\theta \wedge d\theta \neq 0$

- $J : H \to H$, $J^2 = -\mathrm{Id}$.

Notation: $(M, H, J) \longleftrightarrow (M, T^{1,0}) \longleftrightarrow (M, \bar\partial_b)$

$$\mathbb{C}H = \underset{i}{T^{1,0}} \oplus \underset{-i}{T^{0,1}}$$ eigenspaces of J

Often $T^{1,0} = \operatorname{span}\{Z_1\}$ " $\bar\partial_b = Z_{\bar 1} = \overline{Z_1}$ "

! $[T^{1,0}, T^{1,0}] \subseteq T^{1,0}$ trivially.

CR Structure of Unit $S^3 \subseteq \mathbb{C}^2$

$$u = 1 - |z|^2 - |w|^2, \qquad \Theta = i\partial u|_{TS^3}$$

On S^3

- $\Theta = i(z\,d\bar z + w\,d\bar w)|_{TS^3}$ (real)
- $d\Theta = i(dz\wedge d\bar z + dw \wedge d\bar w)$
- $Z_1 = \bar w \partial_z - \bar z \partial_w$
- $Z_{\bar 1} = w\partial_{\bar z} - z\partial_{\bar w}$
- $T = i(z\partial_z + w\partial_w) - i(\bar z \partial_{\bar z} + \bar w \partial_{\bar w})$

(Reeb v.f. of Θ)

$[Z_1, Z_{\bar 1}] = -iT$

$[T, Z_1] = -2i Z_1$

$[T, Z_{\bar 1}] = 2i Z_{\bar 1}$

Abstract Deformations

- Gray's theorem: might as well fix H. $\to$ only vary J.

$|\varphi|^2 < 1$ (pointwise)

$$\rightsquigarrow \quad \hat Z_1 = Z_1 + \underline{\varphi_1{}^{\bar 1}}\, Z_{\bar 1} \qquad \varphi_1{}^{\bar 1} \in C^\infty(S^3, \mathbb{C})$$

CR Obstruction Flatness $(\mathcal{O} \equiv 0)$

Chern-Moser vs Bergman/Szegö

Fefferman ('76, '79) proposed the development of CR invariant theory through the study of the formal asymptotics of the following (biholomorphically invariant) Dirichlet problem:

$$\begin{cases} \mathcal{J}(u) = (-1)^n \det \begin{pmatrix} u & u_{\bar{z}^k} \\ u_{z^j} & u_{z^j\bar{z}^k} \end{pmatrix} = 1 & \text{in } \Omega \subseteq \mathbb{C}^n \\ (u > 0 \text{ in } \Omega) \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

bdd., str. ψ-cvx.

$v = -\log u$ is a Kähler potential for ! complete K-E metric

$\left(\det(v_{z_j\bar{z}_k}) = e^{(n+1)v}\right)$

- Cheng-Yau: $\exists!$ solution $u \in C^\infty(\Omega) \cap C^{n+1}(\bar{\Omega})$
- Lee-Melrose: $u \sim \rho + \underline{\mathcal{O}}\,\rho^{n+2}\log\rho + \dots$

  a local CR invariant

  ρ a Fefferman def. function $\mathcal{J}(\rho) = 1 + O(\rho^{n+1})$
- Graham: $\mathcal{O} \equiv 0 \Rightarrow u \in C^\infty(\bar{\Omega})$

  ← "CR obstruction density" (Many roles!)

*Physics: Bach flat (conf. gravity) & AdS/CFT

Solutions to $\mathcal{O} \equiv 0$?

Cauchy–Kowalewski

Set $\mathcal{O} \equiv 0$ up as a Cauchy problem with appropriate initial data.

* Local:
  - Graham '87: $\exists$ infinitely many (inequivalent) local real analytic hypersurfaces in $\mathbb{C}^n$ $(\forall n \geq 2)$ with $\mathcal{O} \equiv 0$.

* Global (compact):
  - $u = 1 - \|z\|^2$ on the ball, so the CR spheres are obstruction flat (as are locally spherical structures).
  - That is all we know... 

* Rigidity Results:
  - (non-perturbative) C.-Ebenfelt '18: $S^3 \subseteq \mathbb{C}^2$ has no obstruction flat deformations inside $\mathbb{C}^2$.
  - (perturbative) Hirachi-et. al. (in preparation): $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ ...

what about outside?

Hope!

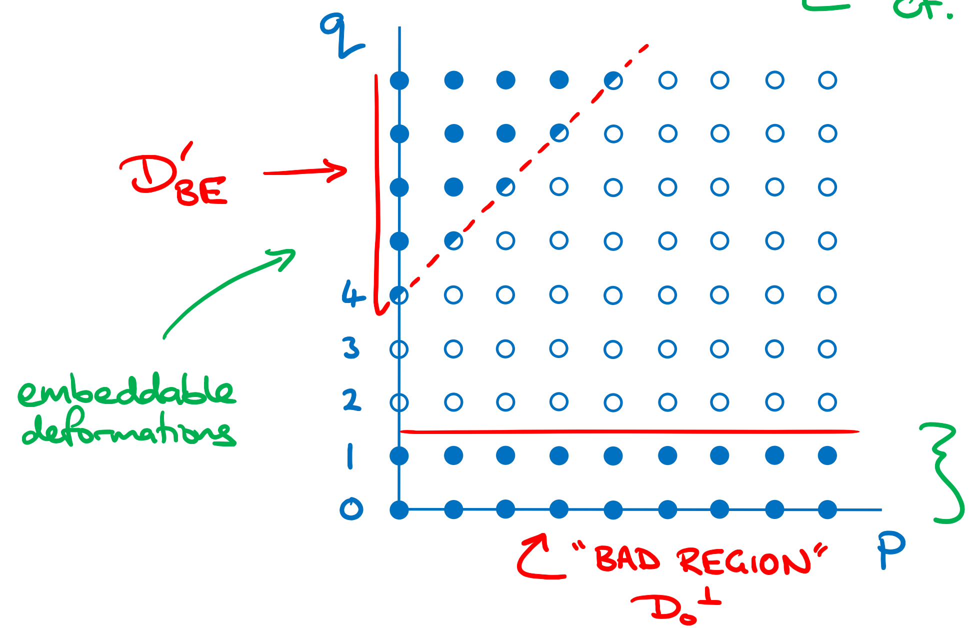
spherical harmonics:

$H_{p,q} \subseteq P_{p,q}$

(e.g., $(z-w)^p(\bar{z}+\bar{w})^q \in H_{p,q}$)

Slice Theorem (C.-Ebenfelt '20):

↖ cf. Burns-Epstein, Bland, Cheng-Lee



Any sufficiently small abstract deformation of the unit $S^3 \subseteq \mathbb{C}^2$ is equivalent to one in this slice (unique mod $\mathrm{Aut}_{CR}(S^3)$).

$\uparrow$ PSU(2,1)

kernel of the linearized obstruction operator $D\Theta$!

infinite dimensional & complementary to the embeddable deformations!

⇝ Deformation Theory! Obstructed?

Dashed!

Theorem (C.-Ebenfelt '21):

There is an open neighborhood $\mathcal{U}$ of the origin in the slice $D'_{BE} \oplus D_0^{\perp}$ such that the CR structure corresponding to $\varphi_1^{\top} \in \mathcal{U}$ is obstruction flat if and only if $\varphi_1^{\top} = 0$.

* Topology: C^3 (or better H^3_{FS})

$\longrightarrow$ There are no solutions of $\mathcal{O} \equiv 0$ near (S^3, H_{std}, J_{std}), embeddable or nonembeddable.

Indication of Proof (by contradiction):

Suppose $\varphi^{(k)} \xrightarrow[H^3_{FS}]{} 0$ solves $\Theta(\varphi^{(k)}) = 0 \quad \forall k \in \mathbb{N}$.

① set $\varphi^{(k)} = \varepsilon_k \hat{\varphi}^{(k)}$ with $\|\hat{\varphi}^{(k)}\|_{H^3_{FS}} = 1$ $\quad (\varepsilon_k \to 0)$

② $\underset{\text{6th order in } \varphi}{\Theta} = \nabla'\nabla' Q_{11} - iA'' Q_{11}$. Show (WORK) $\underbrace{\|\hat{\varphi}^{(k)}_{D'_{BE}}\|_{H^3_{FS}} \to 0}_{\|\hat{\varphi}^{(k)}_{D_0^\perp}\|_{H^3_{FS}} \to 1}$

$\Theta(\varphi^{(k)}) = 0 \rightsquigarrow \underline{\underline{D\Theta}}(\varphi^{(k)}_{D'_{BE}}) = \mathcal{F}(\varphi^{(k)})$

injective on D'_{BE} and inverse gains 6 derivatives in Folland-Stein spaces.

nonlinear terms (involve 6 derivatives but we only control 3!)

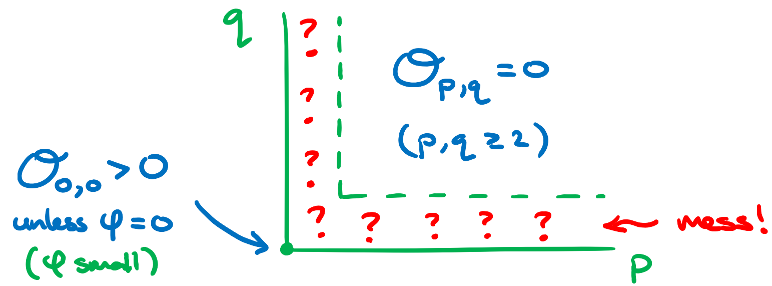
③ $\int_{S^3} \Theta(\varphi^{(k)}) \sim \varepsilon_k^2 \|\hat{\varphi}^{(k)}_{D_0^\perp}\|^2_{H^3_{FS}} + O(\varepsilon_k^{5/2})$

$\hookrightarrow$ So $\quad 0 = \lim_{k\to\infty} \frac{1}{\varepsilon_k^2} \int \Theta(\varphi^{(k)}) = 1 \quad ⚡$ $\quad \square$

* Formally: we can solve to first order, but not to second order as the $iA''Q_{11}$ term "wants to be positive" (on average).

Concluding Remarks:

- One can solve $\Theta \equiv 0 \mod \operatorname{coker}(D\Theta)$, i.e. $\Theta_{p,q} = 0$ for $p,q \geq 2$. Every $\dot{\varphi} \in D_0^{\perp}$ integrates to a family $\varphi(t)$ of solutions of this equation (with $\varphi(0)=0$ & $\frac{d}{dt}\big|_{t=0} \varphi(t) = \dot{\varphi}$).



- Our result proves linearization instability for the conformal Einstein static universe $S^3 \times \mathbb{R}$ in the space of Bach flat metrics. ("Expected." Cf. Moncreif, Fischer-Marsden, and others in the Einstein case.)

Thank You!