Submanifolds in Conformal and CR Manifolds and Applications

by

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A thesis submitted in fulfilment of the requirements
for the degree of *Doctor of Philosophy*
in Mathematics

The University of Auckland
Auckland, 2016

Contents

Abstract

Conformal geometry has its origins in the classical theory of holomorphic plane mappings in complex analysis. The study of conformal geometry in both two and higher dimensions is strongly motivated by physics and by geometric analysis. Closely related is (hypersurface type) CR geometry, which arises in several complex variables analysis as the geometry of real hypersurfaces in complex n-space preserved by ambient biholomorphisms. In this thesis we present work on the calculus and local curvature theory of submanifolds in conformal and (nondegenerate hypersurface type) CR manifolds. The main contribution is the development of a complete local theory for CR embedded submanifolds of CR manifolds, which parallels the standard Ricci calculus treatment of Riemannian submanifold theory. This is based on adapting the well established tractor calculus of conformal hypersurfaces to the more difficult CR setting. We also extend this conformal hypersurface calculus to the higher codimension case and relate it to the work of Burstall and Calderbank. The treatments of conformal and CR embeddings are parallel, and the conformal case serves to illustrate and elucidate the more technical CR case.

Acknowledgments

I would like to start by expressing my gratitude to Professor Rod Gover for his patient guidance and encouragement as my doctoral supervisor. His supervision, as I have both experienced and observed, is excellent in all respects. His knowledge and enthusiasm have been a great source of inspiration. It has been a pleasure to work with him.

I am grateful to Andi Čap and Mike Eastwood for taking interest in my work and providing helpful comments. Thanks also to my examiners for their careful reading of the thesis and further helpful comments.

I would also like to acknowledge Callum Sleigh, who has been a sounding board for ideas. Having your door to knock on made being exiled in the Old Choral Hall both more bearable and more productive.

I am also thankful to have enjoyed the company of my fellow graduate students: Jesse Hart, Yuri Vyatkin, Charles Hadfield, Roberto Panai, Sam Porath, Keegan Flood, Bartek Ertowski, Daniel Snell, Sam Walls, and Stephen Bell.

Finally, to Angela, thank you for your love and support though difficult times. I couldn't have done it without you.

1 Introduction

This thesis presents work on the calculus and local curvature theory of submanifolds in conformal and (nondegenerate hypersurface type) CR manifolds. The main contribution is the development of a complete local theory for CR embedded submanifolds of CR manifolds, which parallels the standard Ricci calculus treatment of Riemannian submanifold theory. Our approach is to generalise the established local theory of conformal hypersurfaces obtained using conformal tractor calculus [7, 21, 90, 128, 141] to the CR setting. This is accomplished using the direct construction of the CR tractor bundle and connection in [79], and turns out to be much more delicate than the conformal case. Though independent, this work is closely related to the approach developed in [144, 54] for the study of rigidity of CR immersions into the sphere. In several ways we go beyond the work of [144, 54], in particular by providing a more conceptual approach and one which easily enables the construction of local invariants and invariant operators. The latter is the test of completeness for any local theory.

Since CR embedded submanifolds of CR manifolds are necessarily of higher real codimension than hypersurfaces, their local theory is more closely related to that of higher codimension submanifolds in conformal manifolds. The conformal standard tractor bundle and connection have already been employed in the study of general conformally embedded submanifolds in [23] (cf. [22]) which gives a unifying approach to the classical theory of submanifolds in the conformal sphere. We independently develop and present the basic calculus of conformally embedded submanifolds in conformal manifolds in the manner of [7, 21, 90, 128, 141]. Our approach is shown to be consistent with the abstract approach of [23] (which uses different terminology and notation), and yields a more direct construction of the basic calculus.

Invariant Theory and Tractor Calculus

In Riemannian geometry the classical Ricci calculus built from the Levi-Civita connection and Riemannian curvature tensor enables one to construct all possible invariant (i.e. well-defined) differential operators and all invariant tensors. This is well understood through Weyl's classical invariant theory of the orthogonal group [145, 5], which relies heavily on the reductive nature of the group. The Levi-Civita calculus extends in a fairly straightforward way to a calculus for invariant differential operators and invariant tensors on a Riemannian submanifold. The key step here is to introduce the second fundamental form via the Gauss formula, relating the ambient and intrinsic Levi-Civita connections; coupling the submanifold Levi-Civita connection with the normal connection induced by the ambient Levi-Civita connection one can invariantly differentiate the second fundamental form. These are the basic ingredients necessary for the local theory of Riemannian submanifolds.

It is well known that conformal and CR geometries are *parabolic geometries*, meaning that they admit an equivalent description in terms of a Cartan geometry of type (G, P) where G is a semisimple Lie group and P is a parabolic subgroup of G [33]. In Riemannian geometry the remaining freedom for coordinate changes between Riemannian normal coordinates at a given point is the orthogonal group (and one can construct invariants from the orthogonal group invariants of the coefficients in the Taylor expansion of the metric). In conformal and CR geometry the corresponding group is the parabolic P, which is *not* reductive. This links the problem of constructing conformal and CR invariants with deep questions in representation theory and invariant theory. This problem was taken up in the CR setting by Fefferman in [63], initiating a programme of *parabolic invariant theory*.

In previous groundbreaking work in CR geometry Fefferman [62] gave an explicit construction of a Lorentzian conformal structure on the trivial circle bundle over a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ (this was later developed in the abstract setting by Lee [103], cf. [60]). Fefferman's approach involved constructing (formally along $\partial\Omega \times \mathbb{C}^*$) a Lorentzian Kähler metric on $\overline{\Omega} \times \mathbb{C}^*$. Motivated by this Fefferman and Graham [64, 66] gave a formal construction of a Ricci flat signature $(p+1, q+1)$ metric extending the metric cone $\mathcal{Q}$ of a signature (p, q) conformal structure. (For even dimensional conformal structures this construction is obstructed at finite order.) This *ambient metric* construction (and the related Poincaré metric construction [64, 66]) represented a significant advance in the invariant theory for conformal structures. In particular, in the case of odd dimensional real analytic conformal manifolds the formal series for the ambient metric converges to a real analytic metric and the local conformal invariants can be obtained from semi-Riemannian invariants. More generally the ambient metric construction gives a framework in which one can apply algebraic results in parabolic invariant theory to the local invariant theory of conformal (and CR) manifolds.

The model parabolic invariant theory problem of Fefferman [63] arising from CR geometry was completely solved in [8], along with a conformal analogue (see [9] for exceptional invariants in the conformal case). Via the ambient metric this gives a complete description of local scalar invariants for odd dimensional conformal manifolds. However for CR manifolds and even dimensional conformal manifolds this gives very restricted results because the ambient metric is obstructed at finite order depending on the dimension. In particular, for conformal manifolds in dimension 4 one obtains essentially no results.

Another well known parabolic geometry is *projective (differential) geometry*. This is the study of manifolds equipped with an equivalence class of affine connections having the same unparametrised geodesics. The work of [8] relied on ideas of Gover from the analogous projective problem [71] (see [72] for exceptional invariants). This work has been applied to give a complete theory of local invariants for projective manifolds in [73] using the *projective tractor calculus* of [7].

Tractor calculi exist for any parabolic geometry [27] and provide a natural analogue of the Riemannian Ricci calculus. (In the conformal and projective cases the main ideas go back to work of Tracey Thomas in the 1920's [134, 135, 136, 137].) The conformal tractor calculus [7, 26] can be used to replace the ambient metric construction in the application of parabolic invariant theory to the construction of local conformal invariants [75]. This leaves only a finitely generated 'window' (in terms of conformal density weight and principal degree) in which not all conformal invariants are known. In low dimensions this 'window' is very small. In $n \geq 6$ dimensions the results of [75] are complemented in low degree by [8]. The tractor calculi for conformal and CR geometry are intimately connected with the ambient metric constructions of Fefferman and Graham [64] and Fefferman [62] respectively [28, 29].

The tractor calculus can also be seen to underlie the general construction of invariant differential operators between irreducible bundles on parabolic geometries, organised into *Bernstein-Gelfand-Gelfand (BGG) sequences* [35, 25]. These sequences extend the (generalised) BGG resolutions of [17, 106] to the curved case (where they are no longer complexes). Many of the ideas of [35, 25] originated in work of Eastwood and Rice in the conformal case [52], and were developed in subsequent work of Baston [11, 12, 13]. The general method for producing invariant differential operators introduced in [35, 25] has come to be known as the 'BGG machinery' [34]. Of particular importance are the first operators arising in these sequences, termed 'first BGG operators'. These are always overdetermined and include many important and well known operators. For example, in conformal geometry these include the conformal-to-Einstein operator, the conformal Killing operator, and the conformal Killing form operator [7, 77, 82], and in projective geometry these include the Killing operator and operators governing the existence of metrics, Ricci-flat metrics, and non-Ricci flat Einstein metrics whose Levi-Civita connection is in the projective class [30, 51].

Parabolic Submanifold Theory

In seeking to develop the local theory of submanifolds in conformal and CR manifolds it is natural to work within the framework of parabolic geometries and the corresponding invariant theory and calculus. Conformal submanifold theory is by now fairly well understood from this point of view [7, 21, 22, 23, 78, 90, 128, 141]. However, submanifold geometry in parabolic geometries more generally is currently not very well developed. Some ideas have been put forward in [22], where submanifolds in the conformal sphere are discussed as a model case, but little is said about the general case beyond raising some technical issues which need to be resolved.

Insight into the submanifolds in parabolic geometries can also be gained from the study of stratifications of parabolic geometries arising from so called 'normal solutions' of first BGG operators (coming from the varying algebraic type of the solution) [31, 32]. In conformal geometry all solutions to the conformal-to-Einstein equation are 'normal', and the zero locus of a solution is the conformal infinity of an Einstein metric defined on the open dense region where the solution is nonzero. In the case of a negative Einstein metric the conformal infinity is forced to be a smooth nondegenerate hypersurface [78, 44]. Not every nondegenerate hypersurface in a conformal manifold arises as the conformal infinity of a negative Einstein metric (this would force it to be umbilic), but it turns out that every nondegenerate conformal hypersurface can be realised as the conformal infinity of a negative constant scalar curvature metric (formally, up to order the dimension of the ambient space) [3, 84]. This leads to an alternative 'holographic' construction of conformal hypersurface invariants from Riemannian invariants of the resulting (formal) metric [84]. Higher codimension submanifolds may arise as the common intersection of the zero loci of a family of solutions to the conformal-to-Einstein equations [105, 4]. This provides a starting point for related developments in the study of higher codimension conformal submanifolds.

Conformally Embedded Submanifolds

Understanding the local curvature theory of submanifolds is basic to the understanding of any geometry. Just as in classical surface theory the intrinsic Gauss curvature is understood as the product of the extrinsic principal curvatures, so in conformal geometry intrinsic invariants may be best understood via some ambient space construction [64, 98]. Further motivation for the study of local theory in conformal submanifolds comes from physics, in particular from string (and brane) theory and the AdS/CFT correspondence [89, 121]. Local invariant theory for conformal hypersurfaces is especially important in geometric analysis because of its role in formulating and studying conformally invariant boundary value problems [38, 19, 20]. In major recent progress in geometric analysis Marques and Neves have solved the Willmore conjecture of [146] stating that the Clifford torus is the unique minimiser of the conformally invariant Willmore energy among immersed tori in the round 3-sphere [112].

In Chapter 3 of this thesis we develop and present the basic calculus of conformally embedded submanifolds in conformal manifolds in the manner of [7, 21, 90, 128, 141]. Though we present some original calculations for the basic invariants this chapter should be seen as mainly expository, and serves the dual purpose of:

(1) showing that the approaches of Gover *et al.* [7, 21, 90, 128, 141] and Burstall and Calderbank [22, 23] are consistent, and

(2) providing a treatment of the tractor calculus and local theory of conformal embeddings which parallels the treatment of CR embeddings in Chapter 5, for ease of reference and comparison.

For expository purposes, we discuss the hypersurface case in §3.1 and §3.2 before moving to the general case in §3.3.

Our approach relies on the construction of the conformal standard tractor bundle and connection of [7]. This is presented in Chapter 2 which is intended to give an accessible exposition of the basic conformal tractor calculus and some of its applications. As in [7] we use a choice of metric in the conformal class to split the standard tractor bundle $\mathcal{T}M$ of $(M, \boldsymbol{c})$ as a direct sum, which after trivialising the density bundles is simply

$$\underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}},$$

where $\underline{\mathbb{R}}$ is the trivial line bundle $M \times \mathbb{R}$. We retain the density line bundles in our presentation for a more direct comparison with the CR case, where the corresponding density bundles play a more important role. Splitting the tractor bundle using a choice of metric $g \in \boldsymbol{c}$ corresponds in the general theory of parabolic geometries to using exact Weyl structures (rather than arbitrary Weyl structures) to reduce the Cartan frame bundle to a reductive structure group, and thereby split the tractor bundle(s) [33]. In conformal geometry a 'Weyl structure' simply corresponds to a choice of Weyl connection for the conformal class, and we are restricting to those Weyl connections which are the Levi-Civita connection of some metric $g \in \boldsymbol{c}$. This is the standard approach in conformal geometry. The invariant constructions of tractor calculus can be done in terms of a choice of metric, with invariance being checked by straightforward calculations. Once the basic tractor calculus is established to be invariant in this way it can be employed to construct tensors and differential operators which are manifestly invariant. The abstract general picture is therefore left in the background.

Given a conformal embedding $\iota : \Sigma \hookrightarrow M$ between conformal manifolds $(\Sigma^m, \boldsymbol{c}_\Sigma)$ of $(M^n, \boldsymbol{c})$ with $n > m \geq 2$ the key first step in the geometric part of the invariant theory is to construct what we term the *standard tractor map*

$$\mathcal{T}\iota : \mathcal{T}\Sigma \to \mathcal{T}M,$$

by analogy with the tangent map $T\iota$ of the embedding. This is accomplished in §§§3.3.2.2 (cf. §§3.2.2) using Lemma 3.3.2:

Lemma. *Given any metric $g_\Sigma \in \boldsymbol{c}_\Sigma$ there exists a metric $g \in \boldsymbol{c}$ extending g_Σ (i.e. such that $g_\Sigma = \iota^* g$) for which the mean curvature vector of Σ vanishes.*

Such ambient metrics are called *minimal scales.* If we use a minimal scale $g \in \boldsymbol{c}$ to split the ambient tractor bundle, and the corresponding metric $g_\Sigma = \iota^* g$ to split the submanifold tractor bundle, then the obvious map

$$(\mathrm{id}, T\iota, \mathrm{id}) : \underline{\mathbb{R}}|_\Sigma \oplus T\Sigma \oplus \underline{\mathbb{R}}|_\Sigma \to \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}$$

corresponds to a map $\mathcal{T}\Sigma \to \mathcal{T}M$ which does not depend on the choice of such $g \in \boldsymbol{c}$. The ambient tractor bundle therefore splits along Σ as an orthogonal direct sum

$$\mathcal{T}M|_\Sigma = \mathcal{T}\Sigma \oplus \mathcal{N}, \tag{$*$}$$

where $\mathcal{N}$ is the *normal tractor bundle.*

The splitting $(*)$ of $\mathcal{T}M|_\Sigma$ allows us to decompose the (normal) tractor connection ∇ on $\mathcal{T}M$ along Σ in §§§3.3.2.7 as

$$\iota^*\nabla = \begin{pmatrix} D + \mathsf{S} & -\mathbb{L}^T \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \quad \text{on} \quad \begin{matrix} \mathcal{T}\Sigma \\ \oplus \\ \mathcal{N} \end{matrix}$$

where D is the (normal) standard tractor connection of Σ, S is the *difference tractor* of §§§3.3.2.3 (cf. §§3.2.3), and $\mathbb{L}$ is the *tractor second fundamental form* of §§§3.3.2.5 (cf. §§3.2.4). This enables us to show, in an extended remark (Remark 3.3.13), that

$$\begin{pmatrix} 0 & -\mathbb{L}^T \\ \mathbb{L} & 0 \end{pmatrix}$$

satisfies the algebraic normalisation condition of [23], verifying consistency with their approach. This also allows us to straightforwardly compute tractor Gauss, Codazzi and Ricci equations (or 'fundamental equations') for the submanifold, §§§3.3.2.8.

With this setup established it becomes fairly straightforward to construct conformal invariants of submanifolds quite generally following the approach of [75, 141]. We discuss this only briefly, §§§3.3.2.9 (cf. §§3.2.6 and §§2.7.7). (See also §6.4 for the case of CR embeddings where invariant theory and the practical construction of invariants is treated in more detail.) Further developments of the local and global invariant theory for conformal submanifolds are work in progress.

CR Embedded Submanifolds in CR Manifolds

The study of local theory for CR embedded submanifolds in CR manifolds is strongly motivated by problems in several complex variables analysis. By developing basic aspects of the local theory of CR embeddings Webster [144] established the rigidity of real codimension 2 CR embeddings into the unit sphere in $\mathbb{C}^n$ for $n \geq 4$. Webster then applied the work of [144] to show that the only proper holomorphic maps $B^{n-1} \to B^n$ which are suitably regular at the boundary are given, up to automorphism of the domain and target, by the linear embedding [143]. This result sparked many subsequent developments in the study and classification of proper holomorphic maps between balls in different dimensions which extend (in a suitable sense) smoothly to the boundary [58, 59, 93, 94, 95, 96]. The basic local theory developed in [144] by Webster was extended to higher codimension CR embeddings by Ebenfelt, Huang, and Zaitsev in [54] (cf. [99, 111]). This is applied in [54] to prove that rigidity holds for any CR immersion of a CR manifold Σ^{2m+1} into the CR sphere $\mathbb{S}^{2n+1}$ provided $d = n - m < m/2$. The authors also discuss the implications of this work for Milnor links of isolated hypersurface singularities. (For further developments regarding CR rigidity phenomena see [55, 10, 97, 53] and the references therein.)

Our work on the local theory of CR embedded submanifolds in CR manifolds is presented in Chapters 4, 5 and 6 which reproduce the preprint [45] (with only minor changes). Our approach relies on the construction of the CR standard tractor bundle and (normal) tractor connection of [79], using the direct sum decomposition of the tractor bundles induced by a choice of pseudohermitian contact form. We work with Tanaka-Webster connections rather than 'Weyl connections' as the former are standard in the CR literature. (The relation of the approach of [79] to the approach of the general theory using Weyl structures is discussed in [33].) Chapter 4 gives a detailed introduction to this work.

The theory we develop is considerably more subtle technically than in the conformal case. There are two main reasons for this. Firstly, in CR geometry the relevant parabolic subgroup P of $\mathrm{SU}(p+1, q+1)$ has nilpotent part P_+ which is no longer Abelian. In practical terms this simply means we have a non-trivial filtration of the tangent budle to deal with; in this case one simply has the usual CR contact distribution. The problems resulting from the filtration of the tangent bundle are resolved without much apparent drama in Chapter 6 because of the convenient fact that admissible ambient contact forms (in the sense of [54]) are equivalently the ambient contact forms for which the CR embedded submanifold has vanishing pseudohermitian mean curvature (see §§6.2.1). In brief admissibility is equivalent to minimality. Admissible ambient contact forms play a role analogous to minimal scales in conformal geometry. The relation between admissibility and minimality is key to establishing the relation between the submanifold and ambient standard tractor bundles of Theorem 6.2.6. The second major technical issue resolved was relating the submanifold and ambient CR density bundles. This is discussed in Chapter 4. These issues are representative of the general case of geometric embeddings between parabolic geometries. Our work therefore gives a template for dealing with other kinds of embeddings.

Our treatment is complete in the sense that we solve the geometric part of the invariant theory, and set up a practical and general construction of invariants. This can be applied, e.g., to produce biholomorphic invariants of Milnor links. Another potential application lies in work on the geometric reflection principle for holomorphic mappings in several complex variables [102]. A natural question which remains is whether one can produce intrinsic CR invariants whose vanishing characterises the embeddability of a given abstract CR manifold into the CR sphere or hyperquadric (cf. [149, 99]). Motivation for this work also included establishing a framework for the study of 'CR infinities' of Sasakian and (η-)Sasaki-Einstein manifolds. This is work in progress.

We refer the reader to Chapter 4 for further introductory details.

Real Hypersurfaces in CR Manifolds

We wish to indicate here that the ideas of Chapters 4–6 can be applied to submanifolds in CR manifolds more generally, and in particular to real hypersurfaces in CR manifolds. Preliminary work on this has been excluded from the thesis because of constraints of time and space. The case where the ambient space is the CR sphere (or Heisenberg group $\mathbb{H}_1$) is already of great interest. The author has observed that on surfaces in 3-dimensional CR manifolds, away from the singular set where the surface is tangent to the contact distribution, there is canonical 'normal tractor' analogous to the normal tractor of a conformal hypersurface [7]. This is defined using the (weighted) normal in the contact direction and the p-mean curvature of [39, 40]. Taking the tractor covariant derivative of this normal tractor tangentially along the surface defines a notion of 'tractor second fundamental form' (or 'tractor shape operator') once again analogous to the case of conformal hypersurfaces (see, e.g., [141]). This provides a starting point for the local CR invariant theory (cf. [42] where pseudohermitian invariant theory is developed in this setting). We aim to apply this to the study of CR invariant boundary problems on domains in 3-dimensional CR manifolds.

This work is naturally related to the study of codimension 2 CR structures [114, 56, 125].

2 Conformal Geometry

Conformal Riemannian geometry is the study of manifolds with a Riemannian metric defined only up to scale at each point. Geometrically this means that angles between curves, but not lengths of curves, are well-defined. One can also define conformal geometry in the semi-Riemannian setting, and we will take this broader point of view. Two semi-Riemannian metrics g and $\hat{g}$ on a smooth manifold are called *conformally equivalent* if there is a positive smooth function Ω such that

$$\hat{g} = \Omega^2 g.$$

In this case we often call $\hat{g}$ a *conformal rescaling* of g. *Conformal geometry* is the study of manifolds equipped with a semi-Riemannian metric up to conformal equivalence, and of their mappings.

In this chapter we give the necessary technical background in conformal geometry for our treatment of submanifolds in Ch. 3. We start with some general background on conformal geometry in two and higher dimensions.

2.1 Conformal Geometry in Two Dimensions

Conformal geometry in two dimensions is intimately connected with single variable complex analysis, and has been a major part of mathematics since the 19th century. A *conformal mapping* between plane domains is one which does not distort the shape of very small figures; more precisely, a conformal mapping is one which preserves the angle between any pair of intersecting curves. Holomorphic mappings between plane domains with nonzero derivative are conformal, and orientation preserving conformal mappings are holomorphic. (Orientation reversing conformal mappings are 'antiholomorphic', i.e. holomorphic in $\bar{z}$ rather than z.) Moving to abstract surfaces we note that a complex structure is therefore equivalent to a conformal structure (and an orientation).

A surface equipped with a complex structure is called a *Riemann surface*. The study of Riemann surfaces has become one of the major areas of modern mathematics, with strong connections to a large number of other parts of mathematics and mathematical physics. In many applications the Riemann surfaces one is interested in are embedded in some higher dimensional manifold, which is often the true object of study.

The real part of a holomorphic function on a plane domain is always a harmonic function. Conversely, on a simply connected domain any harmonic function is the real part of some holomorphic function. This links complex analysis, and therefore conformal geometry, with harmonic analysis and potential theory in the plane. (This can also be seen as the starting point for elliptic regularity theory, since holomorphic functions are necessarily smooth.) With the strong connection between the Cauchy-Riemann equations (governing orientation preserving conformal mappings) and the Laplace equation in two dimensions, it should not be surprising that on an abstract Riemannian surface (M, g) the Laplacian Δ_g is *conformally covariant*, in the sense that if $\hat{g} = e^{2\Upsilon} g$ for some smooth Υ then

$$\Delta_{\hat{g}} = e^{-2\Upsilon} \Delta_g.$$

Potential theory can be used to establish the *Riemann mapping theorem*, a fundamental result in the theory of conformal mappings which states that any proper simply connected open subset U of $\mathbb{C}$ can be mapped conformally onto the unit disc. This is a remarkable theorem when one considers the large class of domains U it covers, including domains with fractal boundary! In the context of Riemann surfaces the Riemann mapping theorem is generalised by the *uniformisation theorem*, which states that the universal cover of any Riemann surface must be conformally equivalent to the Riemann sphere, the complex plane, or the unit disc. One way to prove this theorem is to pick any metric g representing the conformal structure of the given surface M and then solve the equation

$$-\Delta_g \Upsilon + K_g = c e^{2\Upsilon} \tag{2.1.1}$$

for some constant c equal to 1, 0 or -1, where K_g is the Gaussian curvature of g and Δ_g is the (negative spectrum) Laplacian. This allows one to find a metric $\hat{g} = e^{2\Upsilon} g$ on the surface which has constant Gaussian curvature c. The surface $(M, \hat{g})$ then has universal cover isometric to the round sphere, Euclidean space, or hyperbolic space depending on whether c equals 1, 0 or -1, and the conformal structures of these correspond to the Riemann sphere, the complex plane, and the disc respectively.

2.2 Conformal Geometry in Higher Dimensions

Conformal geometry in higher dimensions is no longer closely connected with complex analysis, and exhibits a rigidity unlike the two dimensional case. The strong connection between Riemannian signature conformal geometry and elliptic PDE continues however. In particular, although the Laplace operator is no longer conformally covariant in $n \geq 3$ dimensions, the operator

$$L_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g \tag{2.2.1}$$

is conformally covariant, where R_g is the scalar curvature of g. The operator L_g is known as the *conformal Laplacian*. Many problems in Riemannian geometry involve conformally rescaling the metric. An analogue of (2.1.1) in higher dimensions is the *Yamabe problem* of finding a metric in the conformal class or a (compact) Riemannian manifold with constant scalar curvature. This amounts to solving the PDE

$$L_g u = c u^{\frac{n+2}{n-2}} \tag{2.2.2}$$

for some real constant c, which is a 'nonlinear eigenvalue problem' with the critical Sobolev exponent $\frac{n+2}{n-2}$. This problem was solved through the successive works of Yamabe, Trudinger, Aubin, and Schoen [148, 139, 6, 124] and represents an important development in nonlinear elliptic PDE theory.

Conformal invariance has also proved to be a deeply important phenomenon in physics. Indeed, the operator L_g seems first have appeared in the Lorentzian geometric setting of spacetime as the *conformal wave operator* [46, 140]. Conformal invariance is something of a governing principle for the operators appearing in fundamental (particle) physics. Consider for example the Dirac, Maxwell and Weyl operators, pertaining to the free field theories for the electron, photon, and neutrino respectively. Each of these operators is conformally covariant (and hence conformally invariant when interpreted correctly). Conformal geometry also plays an important role in general relativity. The conformal structure of spacetime precisely encodes the *causal structure*, i.e. the relation on spacetime which says which events may be influenced by which other events. This is because light paths (null geodesics) depend only on the conformal structure, and collectively they determine the both the conformal and causal structure via the 'light cone' at each event. The notion of 'conformal infinity' [117, 68] has proven to be important in general relativity for the study of isolated systems, allowing for rigorous definitions of mass and angular momentum, and for the study of gravitational radiation emitted by isolated systems (recently detected for the first time [1]). Moreover, since Einstein's field equations specify the Ricci curvature of spacetime in terms of the stress-energy tensor of the matter content, it is only the conformal (Weyl) curvature which propagates in empty space. Naturally conformal symmetry plays an important role in many of the theories which seek to combine high energy particle physics and general relativity, notably in Penrose's twistor theory [118] and in string theory via, e.g., the AdS/CFT correspondence (or 'gauge/gravity duality') of Maldacena [110, 147].

The importance of conformally invariant (linear) differential operators in physics has motivated considerable study of the 'spectrum' of such operators available on conformal manifolds. On Euclidean (or pseudo-Euclidean) space classifying the conformally invariant operators is an algebraic problem. This was solved explicitly for 4-dimensional Minkowski space in [52], the necessary representation theory for treating the general case being already well established [106, 15, 16]; in [52] Eastwood and Rice then use the *conformal Cartan* (or 'local twistor', or 'tractor') *connection* to show that, in almost all cases, corresponding conformally invariant differential operators exist on any conformal 4-manifold. This work is generalised to conformal manifolds of dimension $n \geq 3$ in [11, 12]. Exceptional cases include the 'conformal powers of the Laplacian' which exist on conformally flat manifolds, but in even dimensions do not exist on conformally curved manifolds beyond the order of the dimension [87, 80]. Closely related to the problem of constructing invariant operators is the problem of constructing conformally invariant scalar and tensor quantities. Important tools for the study of conformal invariants are the *ambient metric* and *Poincaré* (or 'Poincaré-Einstein') *metric* constructions of [64, 66]. A Poincaré metric for a conformal manifold is a negative Einstein metric with the given manifold as conformal infinity; this is the basic situation to which the AdS/CFT correspondence applies [2, 91], generating a lot of interest in these metrics in their own right (see, e.g., [65, 78, 88, 104, 122]). Another key tool for the construction of conformal invariants is the *conformal tractor calculus* [7, 75] described in more detail in §2.7. The conformal tractor calculus is closely related to the Ricci calculus of the ambient metric [28], and avoids the technical issues occurring with the ambient metric construction in even dimensions. The tractor calculus and the associated 'Bernstein-Gelfand-Gelfand machinery' of [35, 25] (which generalise [52, 11, 12]) are intimately connected with the study of natural overdetermined PDE in parabolic (Cartan) geometry, and in particular in conformal geometry. For example in conformal geometry the conformal-to-Einstein equation and the conformal Killing equation can be *prolonged* (i.e. written as a closed first order system) in terms of the normal tractor connection on the standard tractor bundle and the adjoint tractor bundle respectively [7, 77].

2.3 Conformal Manifolds

A conformal manifold $(M, \boldsymbol{c})$ is a smooth $n \geq 2$ dimensional manifold M equipped with a conformal equivalence class $\boldsymbol{c}$ of semi-Riemannian metrics (necessarily of fixed signature). One may alternatively describe the conformal structure on M in terms of the *bundle of metrics* $\mathcal{Q}$, a ray subbundle of S^2T^*M (with nondegenerate sections). From this point of view $\boldsymbol{c} = \Gamma(\mathcal{Q})$, the space of smooth global sections of $\mathcal{Q}$. A third approach is to think of a signature (p, q) conformal structure on M^n as a reduction $\mathcal{G}_0$ of the frame bundle $\mathcal{F}M$ to the structure group

$$G_0 = \mathrm{CO}(p, q) = \{A \in GL(n, \mathbb{R}) \ : \ A^T I_{p,q} A = \lambda I_{p,q}, \text{ for some } \lambda \in \mathbb{R}_+\}.$$

On a conformal manifold $(M, \boldsymbol{c})$ a frame for T_xM is *conformally orthonormal* if it is orthonormal for some choice of metric in $\mathcal{Q}_x$. The bundle $\mathcal{G}_0$ is simply the bundle of conformally orthonormal frames for $(M, \boldsymbol{c})$.

A smooth map $f : M_0 \to M$ between conformal manifolds $(M_0, \boldsymbol{c}_0)$ and $(M, \boldsymbol{c})$ is called *conformal* if one (equivalently any) metric $g \in \boldsymbol{c}$ pulls back to a metric $f^*g \in \boldsymbol{c}_0$.

2.3.1 Conformal Densities

Density bundles on a smooth n-manifold M are roots and powers of the oriented line bundle $\otimes^2\Lambda^nT^*M$. Conformal density bundles arise in conformal geometry as a kind of bookkeeping tool for dimensional analysis. If we rescale a metric g by Ω^2 to give $\hat{g} = \Omega^2 g$, we rescale 'lengths' by a factor Ω and 'volumes' by a factor of Ω^n. Informally, on a conformal manifold $(M, \boldsymbol{c})$ a section of the conformal density bundle $\mathcal{E}[w]$ is simply a smooth function, given with respect to some metric $g \in \boldsymbol{c}$, that transforms by a factor of Ω^w to give $\hat{f} = \Omega^w f$ when g is rescaled to $\hat{g} = \Omega^2 g$.

It is conventional to think of the ray bundle $\mathcal{Q} \to M$ as a principal $\mathbb{R}_+$-bundle with principal action given by $r^s(g_x) = s^2 g_x$ for $s \in \mathbb{R}_+$ and $g_x \in \mathcal{Q}_x$. The *conformal density bundle of weight* w is the associated bundle

$$\mathcal{E}[w] = \mathcal{Q} \times_{\mathbb{R}_+, \rho_w} \mathbb{R}$$

where the $\mathbb{R}_+$-action on $\mathbb{R}$ is given by $\rho_w(s)t = s^{-w}t$. This means that the total space $\mathcal{E}[w]$ is the quotient of $\mathcal{Q} \times \mathbb{R}$ by the equivalence relation $(g_x, t) \sim (s^2 g_x, s^w t)$ for any $s \in \mathbb{R}_+$, and the bundle projection $\mathcal{E}[w] \to M$ is $[(g_x, t)] \mapsto x$. A choice of metric $g \in \boldsymbol{c}$ trivialises each of the density bundles $\mathcal{E}[w]$, and the weight w tells us how the different trivialisations for different metrics in $\boldsymbol{c}$ are related. Of course, $\mathcal{E}[0]$ is the trivial line bundle.

On an oriented conformal n-manifold there is a well-defined top form $\boldsymbol{\epsilon}$ taking values in the line bundle $\mathcal{E}[n]$. Trivialising $\mathcal{E}[n]$ with a metric $g \in \boldsymbol{c}$, the top form $\boldsymbol{\epsilon}$ is given by the Riemannian volume form ϵ of g. Since $\hat{g} = \Omega^2 g$ implies $\hat{\epsilon} = \Omega^n \epsilon$ we see that $\boldsymbol{\epsilon}$ is well-defined. The $\mathcal{E}[n]$-valued top form $\boldsymbol{\epsilon}$ defines an isomorphism

$$\Lambda^n TM \to \mathcal{E}[n],$$

and dually $\Lambda^n T^*M \cong \mathcal{E}[-n]$. In the case of non-oriented manifolds we do not have the above displayed isomorphism, but we do have $\otimes^2 \Lambda^n TM \cong \mathcal{E}[2n]$ and $\otimes^2 \Lambda^n T^*M \cong \mathcal{E}[-2n]$. Thus conformal densities may be identified with the usual density bundles on smooth manifolds.

Conformal density bundles provide us with the right way to think about conformally covariant operators, namely as conformally invariant operators between weighted bundles. For example, the conformal Laplacian $L = -\Delta + \frac{n-2}{4(n-1)} R$ has the conformal covariance property

$$\hat{L} \circ \Omega^{1-\frac{n}{2}} = \Omega^{-1-\frac{n}{2}} \circ L$$

when $\hat{g} = \Omega^2 g$, and therefore L can be thought of as a conformally invariant operator

$$L : \mathcal{E}[1 - \frac{n}{2}] \to \mathcal{E}[-1 - \frac{n}{2}].$$

Conformal densities arise naturally in the study of conformally invariant operators and conformal invariants more generally.

Remark 2.3.1. Note that one can lift the $\mathbb{R}_+$-action ρ_w on $\mathbb{R}$ to $G_0 = \mathrm{CO}(p, q) = \mathbb{R}_+ \mathrm{SO}(p, q)$ by making $\mathrm{SO}(p, q)$ act trivially. This allows one to think of the density bundles as associated bundles to the conformal orthonormal frame bundle $\mathcal{G}_0$, which is the point of view taken in parabolic theory [33]. This point of view makes it clear that the Levi-Civita connection of each metric $g \in \boldsymbol{c}$ acts naturally on conformal densities (since the Levi-Civita connection may be thought of as a principal connection on $\mathcal{G}_0$). This point of view also clarifies the relation between the conformal density bundles to the usual density bundles (which are associated to the linear frame bundle by powers of the determinant representation). ∎

2.3.2 Densities and Scales

On any conformal manifold $(M, \boldsymbol{c})$ there is a (tautological) $\mathcal{E}[2]$-valued nondegenerate bilinear form $\boldsymbol{g}$, given with respect to the trivialisation of $\mathcal{E}[2]$ coming from $g \in \boldsymbol{c}$ by g itself. We call $\boldsymbol{g}$ the *conformal metric* of $(M, \boldsymbol{c})$. If σ is a nowhere vanishing section of the line bundle $\mathcal{E}[1]$ then

$$g = \sigma^{-2}\boldsymbol{g}$$

is a metric in $\boldsymbol{c}$. This gives rise to a 1-1 correspondence between positive sections of the (naturally oriented) line bundle $\mathcal{E}[1]$ and metrics in $\boldsymbol{c}$. We refer to $\sigma > 0$ as the *scale* corresponding to g if $g = \sigma^{-2}\boldsymbol{g}$. It is common to refer to a choice of positive section σ of $\mathcal{E}[1]$, or correspondingly a metric $g \in \boldsymbol{c}$, as a *choice of scale* for the manifold $(M, \boldsymbol{c})$.

2.4 Ricci Calculus and Conformal Rescalings

Fixing a scale g on a conformal manifold allows one to compute in terms of the associated *Ricci calculus* (Levi-Civita connection, Riemannian curvature, etc.). One then must consider how the resulting expression transforms under conformal rescalings. Proceeding this way it is difficult to produce more than a few basic conformal invariants and invariant operators. However, the Ricci calculus can also be used to explicitly construct the natural conformally invariant calculus on conformal manifolds, namely the *conformal tractor calculus.* Here we present some of the necessary background for this construction.

2.4.1 Abstract Index Notation

In the following, when convenient, we will make use of abstract index notation for tensor calculus (formalised in [119]). We introduce the alternate notation $\mathcal{E}^a$ for the tangent bundle of a given smooth manifold M, allowing for the use of abstract indices from the beginning of the (lower case) Latin alphabet. We also denote the cotangent bundle by $\mathcal{E}_a$. Correspondingly we may write a (tangent) vector field V on M as V^a (or V^b or V^c) and a 1-form ω as ω_a (or ω_d). We denote tensor products of the tangent and cotangent bundles (i.e. tensor bundles) by appending appropriate indices to the symbol $\mathcal{E}$. For example, $\otimes^2 T^*M$ is denoted by $\mathcal{E}_{ab}$ and $(\otimes^2 T^*M) \otimes TM$ by $\mathcal{E}_{ab}{}^c$. The symbols for elements (or sections) of these bundles are also appended with the corresponding indices, thus $\Gamma_{ab}{}^c$ denotes an element (or section) of $\mathcal{E}_{ab}{}^c$. Tensor products are indicated by concatenation (with indices made distinct), so that $V \otimes \omega$ is written as $V^a \omega_b$ and $\omega \otimes \Gamma$ is written as $\omega_a \Gamma_{bc}{}^d$. Tensor contractions are then indicated by repeated indices, so that $\omega(V)$ is written as $\omega_a V^a$ (or as $V^a \omega_a$) and $\mathrm{cont}(\omega \otimes \Gamma)$ as $\omega_a \Gamma_{bc}{}^a$. Symmetrisation over a set of covariant or contravariant indices is denoted by enclosing them with round brackets, so that, e.g.,

$$T_{(ab)} := \frac{1}{2}\left(T_{ab} + T_{ba}\right).$$

Similarly antisymmetrisation is indicated by square brackets, so that, e.g.,

$$T_{[ab]} := \frac{1}{2}\left(T_{ab} - T_{ba}\right).$$

The purpose of the indices is to indicate the type of the tensor and to make contractions (and symmetrisations) more explicit. Working with abstract indices also helps to eliminate extraneous vector fields which appear in standard tensorial expressions, thus clarifying their content.

If ∇ is an affine connection on our manifold M and $V \in \mathfrak{X}(M)$ then the endomorphism ∇V of TM is written using abstract indices as $\nabla_a V^b$, so that $\nabla_V W$ would be written as $V^a \nabla_a W^b$. More generally one writes the covariant derivative of $T^{b\cdots c}{}_{d\cdots e}$ as $\nabla_a T^{b\cdots c}{}_{d\cdots e}$. We may also occasionally refer to the connection ∇ as ∇_a.

On a conformal manifold $(M, \boldsymbol{c})$ we indicate the tensor product of some (unweighted) vector bundle $\mathcal{V} \to M$ with the density bundle $\mathcal{E}[w]$ by appending $[w]$, i.e. $\mathcal{V}[w] = \mathcal{V} \otimes \mathcal{E}[w]$. The conformal metric $\boldsymbol{g}$ is a section of $\mathcal{E}_{ab}[2]$ and is commonly written as $\boldsymbol{g}_{ab}$. The 'inverse' $\boldsymbol{g}^{ab}$ of $\boldsymbol{g}_{ab}$ is a section of $\mathcal{E}^{ab}[-2]$ and satisfies $\boldsymbol{g}^{ab}\boldsymbol{g}_{bc} = \delta^b_c$. We will use $\boldsymbol{g}_{ab}$ and $\boldsymbol{g}^{ab}$ to raise and lower indices, thus identifying $\mathcal{E}^a[w]$ with $\mathcal{E}_b[w+2]$. Note that if we choose a scale $g \in \boldsymbol{c}$ and use it to trivialise the density bundles then this identification reduces to the usual isomorphism $TM \cong T^*M$ induced by g. Raising and lowering indices with $\boldsymbol{g}_{ab}$ has the advantage of being scale independent (i.e. conformally invariant). We denote the trace-free symmetric part of a covariant 2-tensor T_{ab} by $T_{(ab)_0}$.

2.4.2 The Levi-Civita Connection on Densities

Let $(M, \boldsymbol{c})$ be a conformal n-manifold, and fix a metric $g \in \boldsymbol{c}$. The Levi-Civita connection ∇ of g acts naturally on sections of $\otimes^2 \Lambda^n TM$. This determines an action of the Levi-Civita connection on sections of the conformal density bundles such that the identification of $\otimes^2 \Lambda^n TM$ with $\mathcal{E}[2n]$ is parallel. If one trivialises the conformal density bundles using the metric g then this action of the Levi-Civita connection on sections (which become functions) is simply given by the exterior derivative. Thus if f is a section of $\mathcal{E}[w]$ then

$$\nabla f \;=\; \sigma^w \,\mathrm{d}\left(\sigma^{-w} f\right) \tag{2.4.1}$$

where σ is the scale corresponding to $g \in \boldsymbol{c}$. (The trivialisation of $\mathcal{E}[w]$ with respect to g is given explicitly by $\mathcal{E}[w] \ni \tau \mapsto \sigma^{-w}\tau \in \mathcal{E}[0]$.)

Notice that the weight 1 density σ corresponding to g is parallel with respect to the Levi-Civita connection of g. (Under the trivialisation of $\mathcal{E}[1]$ induced by g, σ corresponds to the constant function 1.) As a consequence of this we observe that

$$\nabla \boldsymbol{g} = 0 \tag{2.4.2}$$

since $\boldsymbol{g}$ can be expressed as $\sigma^2 g$. Thus raising and lowering indices with the conformal metric commutes with covariant differentiation with respect to the Levi-Civita connection of any metric in the conformal class. Note that since σ^w is a global parallel section of $\mathcal{E}[w]$ for each w the conformal density bundles are all flat for the Levi-Civita connection of g.

2.4.3 Riemannian Curvature

Let $(M^n, \boldsymbol{c})$ be a conformal manifold of dimension $n \geq 2$. Fix a metric $g \in \boldsymbol{c}$ and let ∇ denote its Levi-Civita connection. We define the *Riemannian curvature tensor* $R_{ab}{}^c{}_d$ by the *Ricci identity*

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R_{ab}{}^c{}_d V^d \tag{2.4.3}$$

for all sections V^c of $\mathcal{E}^c = TM$. The *Ricci tensor* is

$$R_{ab} = R_{ca}{}^c{}_b \tag{2.4.4}$$

and the *scalar curvature* is $R = \boldsymbol{g}^{ab} R_{ab}$. (Since we have fixed a Riemannian metric $g \in \boldsymbol{c}$, it doesn't really matter whether we use $\boldsymbol{g}$ or g to raise and lower indices, however sticking with the conformal metric simplifies the conformal transformation laws of §§2.4.4.) For $n \geq 3$ the Riemannian curvature tensor naturally decomposes as

$$R_{abcd} = W_{abcd} + P_{ac}\boldsymbol{g}_{bd} - P_{bc}\boldsymbol{g}_{ad} - P_{ad}\boldsymbol{g}_{bc} + P_{bd}\boldsymbol{g}_{ac} \tag{2.4.5}$$

where the totally trace free tensor W_{abcd} is the *Weyl curvature* and

$$P_{ab} = \frac{1}{n-2}\left(R_{ab} - \frac{R}{2(n-1)}\boldsymbol{g}_{ab}\right) \tag{2.4.6}$$

is the *Schouten tensor*. We denote the trace of the Schouten tensor by $P = \boldsymbol{g}^{ab} P_{ab}$. In 3-dimensions the Weyl curvature necessarily vanishes. For $n = 2$ we simply have

$$R_{abcd} = K\left(\boldsymbol{g}_{ac}\boldsymbol{g}_{bd} - \boldsymbol{g}_{bc}\boldsymbol{g}_{ad}\right) \tag{2.4.7}$$

where $K = \frac{1}{2}R$ is the Gauss curvature (as weight -2 density).

2.4.4 Conformal Transformations

Let $(M^n, \boldsymbol{c})$ be a conformal manifold. If we conformally rescale $g \in \boldsymbol{c}$ to $\hat{g} = \Omega^2 g$ then the respective Levi-Civita connections satisfy the transformation law

$$\hat{\nabla}_a V^b = \nabla_a V^b + \Upsilon_a V^b - V_a \Upsilon^b + \Upsilon_c V^c \delta_a^b \tag{2.4.8}$$

for any vector field V^b, where $\Upsilon_a = \Omega^{-1}\nabla_a \Omega$. (This can be easily seen from the Koszul formula, or the local coordinate expression for the Christoffel symbols.) From this one can compute the transformation law for the Riemannian curvature tensor. One obtains

$$\hat{R}_{abcd} = R_{abcd} - \Xi_{ac}\boldsymbol{g}_{bd} + \Xi_{bc}\boldsymbol{g}_{ad} + \Xi_{ad}\boldsymbol{g}_{bc} - \Xi_{bd}\boldsymbol{g}_{ac} \tag{2.4.9}$$

where

$$\Xi_{ab} = \nabla_a \Upsilon_b - \Upsilon_a \Upsilon_b + \frac{1}{2}\Upsilon_c \Upsilon^c \boldsymbol{g}_{ab}.$$

From (2.4.5) when $n \geq 3$ one therefore has that

$$\hat{W}_{abcd} = W_{abcd} \tag{2.4.10}$$

and

$$\hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} \Upsilon_c \Upsilon^c \boldsymbol{g}_{ab}. \tag{2.4.11}$$

Note that we have lowered the indices on the respective Weyl tensors using the conformal metric, so that $\hat{W}_{abcd} = W_{abcd}$ only holds when the Weyl tensor W_{abcd} is thought of as a section of $\mathcal{E}_{abcd}[2]$. Tracing (2.4.11) yields

$$\hat{P} = P - \nabla_a \Upsilon^a + \left(1 - \frac{n}{2}\right) \Upsilon_a \Upsilon^a. \tag{2.4.12}$$

From these one can easily obtain the transformation laws for the Ricci and scalar curvatures in $n \geq 3$ dimensions. (Note $R_{ab} = (n-2)P_{ab} + P\boldsymbol{g}_{ab}$.) When $n = 2$ we obtain the Gauss curvature transformation law by tracing (2.4.9)

$$\hat{K} = K - \nabla_a \Upsilon^a. \tag{2.4.13}$$

Remark 2.4.1. If we treat the Gauss curvature as a function (rather than as a conformal weight -2 density) and write Ω as e^{Υ} so that $\Upsilon_a = \nabla_a \Upsilon$ then the transformation law (2.4.13) becomes the more familiar

$$K_{\hat{g}} = e^{-2\Upsilon} \left(K_g - \Delta_g \Upsilon\right), \tag{2.4.14}$$

cf. (2.1.1). Treating the curvature tensors/scalars as carrying natural density weights has the effect of eliminating overall rescaling factors in the transformation laws. ∎

It will also be useful to have the transformation law for the Levi-Civita connection on 1-forms

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b - \Upsilon_a \omega_b - \omega_a \Upsilon_b + \Upsilon^c \omega_c \boldsymbol{g}_{ab} \tag{2.4.15}$$

obtained from (2.4.8) by duality, or by lowering indices using g (noting $\hat{\nabla} g \neq 0$). We will also need the transformation law for the Levi-Civita connection on conformal weight w densities

$$\hat{\nabla}_a f = \nabla_a f + w \Upsilon_a f \tag{2.4.16}$$

which follows directly from (2.4.1). Combining (2.4.8) and (2.4.15) with (2.4.16) we have

$$\hat{\nabla}_a V^b = \nabla_a V^b + (w+1)\Upsilon_a V^b - V_a \Upsilon^b + \Upsilon_c V^c \delta_a^b \tag{2.4.17}$$

and

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b + (w-1)\Upsilon_a \omega_b - \omega_a \Upsilon_b + \Upsilon^c \omega_c \boldsymbol{g}_{ab} \tag{2.4.18}$$

where V^b and ω_b are sections of $\mathcal{E}^b[w]$ and $\mathcal{E}_b[w]$ respectively.

2.5 The Conformal Sphere

Liouville's theorem in conformal geometry states that any conformal mapping between connected open sets in $\mathbb{R}^n$ for $n \geq 3$ is a *Möbius transformation*, i.e. is given by the composition of translations, orthogonal transformations, dilations, and inversions in spheres [109, 108]. Inversion in the unit sphere is the map

$$x \mapsto \frac{x}{||x||^2}$$

which sends the origin in $\mathbb{R}^n$ to the point at infinity. This can be formalised by conformally compactifying Euclidean space using inverse stereographic projection $\mathbb{R}^n \hookrightarrow \mathbb{S}^n$. We therefore think of conformal $\mathbb{R}^n$ as a subspace of the conformal sphere on which the Möbius transformations become globally defined. For this reason we refer to the sphere $\mathbb{S}^n$ with its standard conformal structure $\boldsymbol{c} = [g_{\text{round}}]$ as the *flat model space* for conformal geometry in dimensions $n \geq 3$.

The group of Möbius transformations $\text{Möb}(n)$ is naturally isomorphic to $\text{PO}(n+1,1)$, the quotient of $\text{O}(n+1,1)$ by its (finite) center. The group $\text{PO}(n+1,1)$ acts on the n-sphere via the following construction: Let $\mathbb{R}^{n+1,1}$ denote $\mathbb{R}^{n+2}$ equipped with the signature $(n+1,1)$ inner product $\langle \cdot , \cdot \rangle$ represented in the standard coordinate basis by $\text{diag}(1,\ldots,1,-1)$. We identify the sphere $\mathbb{S}^n$ with the space of null (i.e. isotropic) lines in the projectivisation of $\mathbb{R}^{n+1,1}$. The action of $\text{O}(n+1,1) = \text{O}(\langle \cdot , \cdot \rangle)$ on $\mathbb{R}^{n+1,1}$ preserves the space of isotropic lines (through the origin) so that $\text{O}(n+1,1)$ acts on $\mathbb{S}^n$. The center of $\text{O}(n+1,1)$ acts trivially on the space of lines through the origin, so the action descends to $\text{PO}(n+1,1)$. It is easy to see that $\text{PO}(n+1,1)$ acts by conformal diffeomorphisms and hence by Möbius transformations. Writing the action explicitly one can see that $\text{PO}(n+1,1) = \text{Möb}(n)$.

By the construction above $\text{PO}(3,1)$ acts conformally on the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$. Up to complex conjugation the action of an element of $\text{PO}(3,1)$ on the sphere is given by a (holomorphic) Möbius transformation

$$z \mapsto \frac{az+b}{cz+d} \tag{2.5.1}$$

where a, b, c, and d are complex numbers with $ad - bc \neq 0$. However, the Liouville theorem does not hold in 2 dimensions; biholomorphic maps between open sets in the plane form an infinite dimensional pseudogroup, whereas the group of Möbius transformations is 6-dimensional.

One way to obtain a Liouville-type theorem in 2 dimensions is to endow the Riemann sphere with additional structure. The space of circles in $\mathbb{S}^2$ maps under stereographic projection to the space of circles and lines in $\mathbb{C}$. Specifying the space of (standard) circles gives additional structure to the Riemann sphere; the 2-sphere with this structure is known as the *Möbius sphere*. Realising the 2-sphere as the *celestial sphere* (space of null lines through the origin in $\mathbb{R}^{3,1}$) naturally endows it with the structure of the Möbius sphere (circles are defined by intersecting the light cone in $\mathbb{R}^{3,1}$ with transverse 3-dimensional subspaces). It is well known that the only (orientation preserving) conformal mappings between plane domains which preserve the space of lines and circles are the Möbius transformations (2.5.1), so that a Liouville-type theorem holds for domains in the Möbius sphere.

2.6 2D Möbius Structures

Möbius structures can be defined on conformal surfaces more generally. We define a *Möbius structure* ([24], cf. [123]) on a conformal surface $(M, \boldsymbol{c})$ to be a second order linear differential operator

$$\mathcal{D}_{ab} : \mathcal{E}[1] \to \mathcal{E}_{(ab)_0}[1]$$

with the property that $\mathcal{D}_{ab} - \nabla_{(a}\nabla_{b)_0}$ is of order zero, where ∇ is the Levi-Civita connection of any metric in $\boldsymbol{c}$. Given a metric $g \in \boldsymbol{c}$ on a Möbius surface $(M, \boldsymbol{c}, \mathcal{D})$ we may define a trace-free symmetric 2-tensor $\mathcal{P}_{ab}$ by

$$\mathcal{P}_{ab}\sigma = \left(\mathcal{D}_{ab} - \nabla_{(a}\nabla_{b)_0}\right)\sigma \tag{2.6.1}$$

for any section σ of $\mathcal{E}[1]$. One then computes using (2.4.16) and (2.4.18) that under the conformal rescaling $\hat{g} = \Omega^2 g$

$$\hat{\mathcal{P}}_{ab} = \mathcal{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b + \frac{1}{2}(\nabla_c\Upsilon^c)\boldsymbol{g}_{ab} - \frac{1}{2}\Upsilon_c\Upsilon^c\boldsymbol{g}_{ab} \tag{2.6.2}$$

where $\Upsilon_a = \Omega^{-1}\nabla_a\Omega$. We define the symmetric 2-tensor

$$P_{ab} = \mathcal{P}_{ab} + \frac{1}{2}K\boldsymbol{g}_{ab}. \tag{2.6.3}$$

Recalling (2.4.13), i.e. $\hat{K} = K - \nabla_a\Upsilon^a$, we get that

$$\hat{P}_{ab} = P_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon_c\Upsilon^c\boldsymbol{g}_{ab}, \tag{2.6.4}$$

cf. (2.4.11). We refer to P_{ab} as the *Rho-tensor* of g (the Schouten tensor in higher dimensions may also be called the Rho-tensor). A Möbius structure on a conformal surface $(M, \boldsymbol{c})$ may alternatively be described as an assignment of a symmetric Rho-tensor P_{ab} with trace K to each metric $g \in \boldsymbol{c}$, satisfying (2.6.4), since

$$\mathcal{D}_{ab} = \nabla_{(a}\nabla_{b)_0} + P_{(ab)_0}. \tag{2.6.5}$$

Remark 2.6.1. Prescribing the trace ${P_a}^a$ of the Rho-tensor to be K allows us to write

$$R_{abcd} = P_{ac}\boldsymbol{g}_{bd} - P_{bc}\boldsymbol{g}_{ad} - P_{ad}\boldsymbol{g}_{bc} + P_{bd}\boldsymbol{g}_{ac}. \tag{2.6.6}$$

This is important in that it ensures the tractor connection constructed from the Möbius structure, as in §§2.7.4 below, is *normal* (in the sense of [27]). In [23] a more general notion of Möbius structure on conformal 2-manifolds is introduced, which amounts to prescribing a 'Rho-tensor' P_{ab} to each $g \in \boldsymbol{c}$ transforming according to (2.6.4) but without the condition that ${P_a}^a = K$; such a Möbius structure gives rise (or corresponds) to a tractor connection which is normal if and only if ${P_a}^a = K$ for some (equivalently any) metric $g \in \boldsymbol{c}$. ∎

The term 'Möbius structure' in conformal geometry often refers to a manifold with atlas whose transition functions are given by Möbius transformations. Such an atlas naturally endows the manifold with a conformal structure; in 2-dimensions one can further define a Möbius structure $\mathcal{D}_{ab}$ by defining P_{ab} to be zero for the flat metric corresponding to each chart. In higher dimensions these are conformally flat manifolds, and in 2 dimensions these correspond to what we will call *(locally) flat Möbius structures*, meaning those Möbius structures for which the conformally invariant tensor $C_{abc} = 2\nabla_{[a}P_{b]c}$ is identically zero. (One can easily verify these claims using the tractor calculus of §2.7.) Clearly the Möbius sphere defined in §2.5 has a natural atlas whose transition functions are Möbius transformations. The Möbius sphere is the *(globally) flat model space* for

Möbius surfaces.

We will use the term *Möbius conformal sphere* in $n \geq 2$ dimensions to refer to the usual conformal sphere in the case $n \geq 3$ and to the Möbius sphere in the case $n = 2$.

2.7 The Tractor Calculus

The (Möbius) conformal sphere arises naturally as the space of null lines in the projectivisation of $\mathbb{R}^{n+1,1}$. It therefore comes naturally equipped with a flat vector bundle of rank $n + 2$, the *standard tractor bundle* $\mathcal{T}\mathbb{S}^n$. A standard tractor at a point $x \in \mathbb{S}^n$ corresponding to a null line $\ell \in \mathbb{P}(\mathbb{R}^{n+1,1})$ is a constant vector field in $\mathbb{R}^{n+1,1}$ along ℓ. Parallel

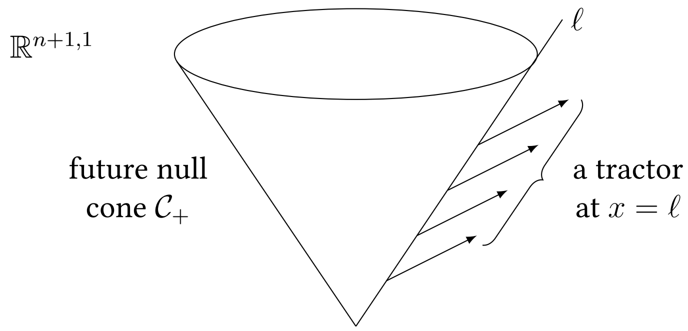

Figure 2.7.1: An element of $\mathcal{T}_x\mathbb{S}^n$ is a homogeneous of degree zero vector field along the line $\ell \in \mathbb{P}(\mathbb{R}^{n+1,1})$ corresponding to x.

transport in $\mathcal{T}\mathbb{S}^n$ is simply given by the affine space structure of $\mathbb{R}^{n+1,1}$. This observation is the starting point for the development of the *conformal tractor calculus*, which originated in the work of Tracey Thomas [135, 136, 137] and in the later independent 'rediscovery' of Bailey, Eastwood and Gover [7]. The tractor calculus is closely connected to the canonical conformal Cartan connection of [36] (for expositions see [100, 126, 33]), being the natural calculus on associated bundles (see [26, 27], cf. [69]).

Here we present the basics of the tractor calculus for (Möbius) conformal manifolds, following the style of [7, 50, 44].

2.7.1 The Standard Tractor Bundle

Let $(M, \boldsymbol{c})$ be a conformal n-manifold, $n \geq 3$. The *standard tractor bundle* of $(M, \boldsymbol{c})$ is a rank $n+2$ vector bundle $\mathcal{T}$ on M, also denoted $\mathcal{E}^A$ (we use capital Latin abstract indices from the start of the alphabet); given a choice of metric $g \in \boldsymbol{c}$ it may be identified with the direct sum bundle

$$[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]; \tag{2.7.1}$$

we write $v^A \stackrel{g}{=} (\sigma, \mu^a, \rho)$,

$$v^A \stackrel{\theta}{=} \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}, \quad \text{or} \quad [v^A]_g = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}$$

if an element or section of $\mathcal{E}^A$ is represented by (σ, μ^a, ρ) with respect to this identification; the identifications given by two metrics g and $\hat{g} = \Omega^2 g$ are related by the transformation law

$$[\mathcal{E}^A]_g \ni \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^a & \delta^a_b & 0 \\ -\frac{1}{2}\Upsilon^b\Upsilon_b & -\Upsilon_b & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^b \\ \rho \end{pmatrix} \in [\mathcal{E}^A]_{\hat{g}} \tag{2.7.2}$$

where $\Upsilon_a = \Omega^{-1}\nabla_a\Omega$. It is easy to check that $\sim$ is an equivalence relation on the disjoint union of the spaces $[\mathcal{E}^A]_g$, so that $\mathcal{E}^A$ is well-defined as the quotient of the disjoint union of the $[\mathcal{E}^A]_g$ over $g \in \boldsymbol{c}$ by (2.7.2).

On a 2-dimensional conformal manifold we may also define the bundle $\mathcal{E}^A$ as above. However this bundle is not a 'tractor bundle' in the sense that $\mathcal{E}^A$ does not carry a canonical 'tractor connection'. (The term 'tractor bundle' indicates a bundle with specific additional structure [26, 27], much like the notion of a 'G-bundle'.) If the conformal structure is supplemented with a Möbius structure then there is a canonical 'tractor connection' (see (2.7.7)) and in this case we refer to $\mathcal{E}^A$ as the standard tractor bundle.

Remark 2.7.1. The term 'standard' in 'standard tractor bundle' refers not to canonicity, but to the standard representation $\mathbb{R}^{n+2}$ (also denoted $\mathbb{R}^{p+1,q+1}$) of the conformal group $G = \mathrm{O}(p+1, q+1)$ to which the standard tractor bundle corresponds; $\mathcal{T}$ is the vector bundle induced from the Cartan frame bundle by the representation $\mathbb{R}^{n+2}$ (see [26]). The representation $\Lambda^2(\mathbb{R}^{n+2})^*$ of G can be identified with the adjoint representation $\mathfrak{g}$ and the corresponding induced bundle $\mathcal{E}_{[AB]}$ is called the *adjoint tractor bundle*. The standard tractor bundle is fundamental in that any irreducible representation of G can be obtained as an irreducible subspace of $\otimes^k\mathbb{R}^{n+2}$ for some k by imposing tensor symmetries. ∎

2.7.2 Splitting Tractors

Here we introduce some convenient notation for working with tractors.

From (2.7.2) it is clear that there is an invariant inclusion of $\mathcal{E}[-1]$ into $\mathcal{E}^A$ given with respect to any $g \in \boldsymbol{c}$ by the map

$$\rho \mapsto \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}.$$

Correspondingly there is an invariant section X^A of $\mathcal{E}^A[1]$ such that the above displayed map is given by $\rho \mapsto \rho X^A$. The weight 1 *canonical tractor* X^A can be written as

$$X^A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

with respect to any choice of metric g. Note that X^A dually gives a map $\mathcal{E}_A \to \mathcal{E}[1]$ sending v_A to $X^A v_A$.

Given a fixed metric g, we also get the corresponding *splitting tractors* of [81]

$$Z_b^A \stackrel{g}{=} \begin{pmatrix} 0 \\ \delta_b^a \\ 0 \end{pmatrix} \quad \text{and} \quad Y^A \stackrel{g}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which both have weight -1. A standard tractor $v^A \stackrel{g}{=} (\sigma, \mu^a, \rho)$ may instead be written as $v^A = \sigma Y^A + Z_b^A \mu^b + \rho X^A$ where we understand that Y^A and Z_b^A are defined in terms of the splitting induced by g. If $\hat{g} = \Omega^2 g$ then by (5.3.3) we have

$$\hat{Z}_b^A = Z_b^A + \Upsilon_b X^A, \tag{2.7.3}$$

$$\hat{Y}^A = Y^A - \Upsilon^b Z_b^A - \frac{1}{2}\Upsilon_b \Upsilon^b X^A. \tag{2.7.4}$$

2.7.3 The Tractor Metric

The standard tractor bundle $\mathcal{E}^A$ carries a canonical tractor metric g_{AB}. If $v^A \stackrel{g}{=} (\sigma, \mu^a, \rho)$ then

$$g_{AB}v^Av^B = \begin{pmatrix} \sigma & \mu^a & \rho \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^b \\ \rho \end{pmatrix} = 2\sigma\rho + \mu_a\mu^a. \tag{2.7.5}$$

We use the tractor metric g_{AB} to raise and lower tractor indices, identifying the standard tractor bundle $\mathcal{E}^A$ with its dual $\mathcal{E}_A$. The various contractions of the splitting tractors (for a given $g \in \boldsymbol{c}$) using the tractor metric are described by the table:

	Y^A	Z^A_a	X^A
Y_A	0	0	1
Z_{Ab}	0	$\boldsymbol{g}_{ab}$	0
X_A	1	0	0

(2.7.6)

2.7.4 The Tractor Connection

Let $(M, \boldsymbol{c})$ be a conformal manifold of dimension $n \geq 3$. In terms of a metric $g \in \boldsymbol{c}$ the *(normal) standard tractor connection* $\nabla^{\mathcal{T}}$ (or simply ∇) is defined by the following formula

$$\nabla_a \begin{pmatrix} \sigma \\ \mu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu^b + P_a{}^b\sigma + \delta^b_a\rho \\ \nabla_a\rho - P_{ab}\mu^b \end{pmatrix}. \tag{2.7.7}$$

If a section of $\mathcal{E}^A$ is given by $[v^A]_g = (\sigma, \mu^a, \rho)$ then if $\hat{g} = \Omega^2 g$ by (2.7.2) we have $[v^A]_{\hat{g}} = (\hat{\sigma}, \hat{\mu}^a, \hat{\rho})$ where

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}^a \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a\sigma \\ \rho - \Upsilon_a\mu^a - \frac{1}{2}\Upsilon_a\Upsilon^a\sigma \end{pmatrix} \tag{2.7.8}$$

with $\Upsilon_a = \nabla_a \log \Omega$. To see that the tractor connection ∇ is well-defined, we just need to check that

$$\begin{pmatrix} \hat{\nabla}_a \hat{\sigma} - \hat{\mu}_a \\ \hat{\nabla}_a \hat{\mu}^b + \hat{P}_a{}^b \hat{\sigma} + \delta_a^b \hat{\rho} \\ \hat{\nabla}_a \hat{\rho} - \hat{P}_{ab} \hat{\mu}^b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^b & \delta_c^b & 0 \\ -\frac{1}{2}\Upsilon^c \Upsilon_c & -\Upsilon_c & 1 \end{pmatrix} \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu^c + P_a{}^c \sigma + \delta_a^c \rho \\ \nabla_a \rho - P_{ac} \mu^c \end{pmatrix} \tag{2.7.9}$$

using the transformation laws (2.4.16) and (2.4.17) for the Levi-Civita connection and the transformation law (2.4.11) for the Schouten tensor. (The explicit calculation can be found in [50].)

Since the conformal transformation law (2.6.4) for the Rho-tensor of a 2-dimensional conformal Möbius structure is formally the same as the conformal transformation law (2.4.11) for the Schouten tensor in higher dimensions, the tractor connection (2.7.7) is also well-defined for 2-dimensional Möbius structures with P_{ab} given by the Rho-tensor of g.

It is often useful to differentiate expressions such as $\eta_a Z_A^a Y^B$ using the Leibniz rule, where Z_a^A and Y^A are splitting tractors determined by $g \in \boldsymbol{c}$. To do this we couple the tractor connection ∇ with the Levi-Civita connection of g. By differentiating the expression $\sigma Y^B + \mu^b Z_b^B + \rho Y^B$ using the Leibniz rule and comparing with (2.7.7) one obtains that

$$\nabla_a X^B = Z_a^B, \tag{2.7.10}$$

$$\nabla_a Z_b^B = -P_{ab} X^B - \boldsymbol{g}_{ab} Y^B, \tag{2.7.11}$$

and

$$\nabla_a Y^B = P_{ab} Z^{Bb}. \tag{2.7.12}$$

Noting that $g_{AB} = 2X_{(A}Y_{B)} + \boldsymbol{g}_{ab} Z_A^a Z_B^b$ these formulae give an easy way to check that

$$\nabla_a g_{BC} = 0. \tag{2.7.13}$$

2.7.5 The Tractor Curvature

Coupling the tractor connection with any torsion free affine connection the *tractor curvature* $\kappa_{ab}{}^C{}_D$ satisfies the Ricci-type identity

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v^C = \kappa_{ab}{}^C{}_D v^D. \tag{2.7.14}$$

Coupling the tractor connection with the Levi-Civita connection of some metric $g \in \boldsymbol{c}$ one may straightforwardly calculate (using (2.4.5)) that

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{pmatrix} \sigma \\ \mu^c \\ \rho \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ C_{ab}{}^c & W_{ab}{}^c{}_d & 0 \\ 0 & -C_{abd} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^d \\ \rho \end{pmatrix} \tag{2.7.15}$$

where $C_{abc} = 2\nabla_{[a} P_{b]c}$ is the *Cotton tensor* of g. In three dimensions the Weyl curvature vanishes and the Cotton tensor is conformally invariant. In the case of a 2-dimensional Möbius structure the same result (2.7.15) holds with $W_{ab}{}^c{}_d$ replaced by zero (because of (2.6.6)), and with $C_{abc} = 2\nabla_{[a} P_{b]c}$ defined in terms of the Rho-tensor of g.

Remark 2.7.2. From this one can easily see that, starting with the (Möbius) conformal sphere, the construction of §§2.7.1–§§2.7.4 yields a globally flat tractor bundle; this is consistent with our discussion of tractors on the model in our introduction to §2.7. One can then reinterpret objects such as the tractor metric g_{AB} and canonical tractor X^A. It is easy to see that in the model case the tractor metric corresponds to the constant metric coming from $\mathbb{R}^{n+1,1}$. Also, in the model case a point $x \in \mathbb{S}^n$ defines a line $\ell \subset \mathcal{T}_x\mathbb{S}^n \cong \mathbb{R}^{n+1,1}$, giving rise to a canonical line subbundle $\mathcal{T}^0\mathbb{S}^n$ of $\mathcal{T}\mathbb{S}^n$. By considering sections of the future null cone $\mathcal{C}_+$ and the induced metrics on them (noting §§2.3.2) one may naturally identify $\mathcal{T}^0\mathbb{S}^n$ with $\mathcal{E}[-1]$, and this gives rise to the canonical inclusion $\mathcal{E}[-1] \hookrightarrow \mathcal{E}^A$. On the model sphere sections of the density bundle $\mathcal{E}[w]$ can be identified with functions on the null cone $\mathcal{C} \subset \mathbb{R}^{n+1,1}$ which are homogeneous of degree w. The canonical tractor X^A can therefore be identified with the Euler vector field of $\mathbb{R}^{n+1,1}$ restricted to $\mathcal{C}$. For more details on the correspondence between the two constructions in the model case see [44]. ∎

2.7.6 Invariant Tractor Operators

Differential splitting operators play an important role in tractor calculus, e.g., in the 'BGG machinery' of [35, 25] (see [127] for a treatment of the conformal case using the tractor formalism as presented here). Here we present some of the most basic and important families of such operators. The following applies in the usual setting of conformal manifolds with dimension $n \geq 3$, but note that one also obtains corresponding operators, *mutatis mutandis*, in the case of 2-dimensional Möbius structures.

2.7.6.1 The tractor *D*-operator(s)

Let $\mathcal{E}^\Phi$ denote any tensor product of copies of $\mathcal{E}^A$ and $\mathcal{E}_A$ (we refer to such bundles, and any subbundles obtained by imposing tensor symmetries, as *tractor bundles*).

Definition 2.7.3. On a conformal n-manifold $(M, \boldsymbol{c})$ the *tractor D-operator* of [136, 7]

$$\mathsf{D}_A : \mathcal{E}^\Phi[w] \to \mathcal{E}_A \otimes \mathcal{E}^\Phi[w-1] \tag{2.7.16}$$

is defined by

$$\mathsf{D}_A f^\Phi = w(n+2w-2)Y_A f^\Phi + (n+2w-2)Z_A^a \nabla_a f^\Phi - X_A(\Delta + wP)f^\Phi \tag{2.7.17}$$

where we calculate with respect to some metric $g \in \boldsymbol{c}$, so ∇ denotes the tractor connection coupled with the Levi-Civita connection and $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b$.

Using the transformation laws of §§2.4.4 one can easily check directly that D_A does not depend on the choice of metric used to define it (an explicit calculation can be found in [50]). A key property of the tractor D-operator is that it can be iterated to produce higher order invariants associated to a given weighted tractor. Note that the tractor D-operator fails to be a splitting operator (i.e. to have a linear bundle map as left inverse) at the critical weights $w = 0$ and $w = 1 - \frac{n}{2}$. At the critical weight $w = 1 - \frac{n}{2}$ we have

$$\mathsf{D}_A f^\Phi = X_A L f^\Phi \tag{2.7.18}$$

where L is the (tractor twisted) conformal Laplacian

$$L = -\Delta + \frac{n-2}{4(n-1)}R. \tag{2.7.19}$$

The tractor D-operator can be used to produce further invariant operators. In particular, we note that if f is section of $\mathcal{E}[2 - \frac{n}{2}]$ then

$$L\mathsf{D}_A = X_A P_4 f \tag{2.7.20}$$

where P_4 is the *Paneitz operator* [116]

$$P_4 f = \Delta^2 f + \nabla_a \left[\left(4P^{ab} - (n-2)P\boldsymbol{g}^{ab} \right) \nabla_b f \right] + \frac{(n-4)}{4} \left[-2\Delta P - 4P_{ab}P^{ab} + nP^2 \right] f. \tag{2.7.21}$$

Continuing along these lines one may use the tractor D-operator to produce explicit (Ricci calculus) formulae [81] for the 'conformally invariant powers of the Laplacian' (or 'GJMS operators') P_{2k} of [87].

Related to the tractor D-operator is the g dependent operator $\tilde{\mathsf{D}}_A$ given by $wf^\Phi Y_A + (\nabla_a f^\Phi) Z_A^a$ on a section f^Φ of $\mathcal{E}^\Phi[w]$. The operator

$$\mathbb{D}_{AB} f^\Phi = 2X_{[A} \tilde{\mathsf{D}}_{B]} f^\Phi \tag{2.7.22}$$

does not depend on the choice of metric $g \in \boldsymbol{c}$ and is known as the *double-D-operator*. The double-D-operator is closely related to the 'fundamental D-operator' of [27]. The relationships between the various 'D-operators' are expounded in [26]. In particular, the tractor D-operator can be derived from the double-D-operator by noting [74] that

$$g^{AD} \mathbb{D}_{A(B} \mathbb{D}_{C)_0 D} f^\Phi = X_{(B} \mathsf{D}_{C)_0} f^\Phi \tag{2.7.23}$$

for any weighted tractor field f^Φ and that the map $v_A \mapsto X_{(A} v_{B)_0}$ is injective.

2.7.6.2 Middle operators

Here we present some members of a class of important conformally invariant first order operators, used by Eastwood to produce conformally invariant operators by 'curved translation' (see, e.g., [50]).

Definition 2.7.4. The *middle operator* acting on weighted covector fields is the operator $\mathsf{M}_A^a : \mathcal{E}_a[w] \to \mathcal{E}_A[w-1]$ given with respect to a choice of $g \in \boldsymbol{c}$ by

$$\mathsf{M}_A^a \tau_a = (n + w - 2) Z_A^a \tau_a - X_A \nabla^a \tau_a. \tag{2.7.24}$$

To see that the operator defined by (2.7.24) is well-defined one simply observes that (2.4.18) implies (when $\hat{g} = \Omega^2 g$ and $\Upsilon_a = \Omega^{-1} \nabla_a \Omega$)

$$\hat{\nabla}^a \tau_a = \nabla^a \tau_a + (n + w - 2) \Upsilon^a \tau_a \tag{2.7.25}$$

for τ_a of weight w, whereas from (2.7.3) we have that $\hat{Z}_A^a = Z_A^a + \Upsilon^a X_A$. The operator M_A^a is a splitting operator except when $w = 2 - n$, in which case $\tau_a \mapsto \nabla^a \tau_a$ is an invariant operator.

One can similarly define a middle operator on weighted differential forms [127]. In particular, on weighted 2-forms we may define the operator M_{AB}^{ab} given by

$$\mathsf{M}_{AB}^{ab}\tau_{ab} = (n + w - 4) Z_A^a Z_B^b \tau_{ab} - 2X_{[A} Z_{B]}^b \nabla^a \tau_{ab}. \tag{2.7.26}$$

Again it is straightforward to verify the invariance of this operator. By considering simple sections $\omega_{[b}\eta_{c]}$ of $\mathcal{E}_{[bc]}[w]$ where ω_b has weight 0 and η_c has weight w one obtains from (2.4.18) and the Leibniz rule the transformation law

$$\hat{\nabla}_a \tau_{bc} = \nabla_a \tau_{bc} + (w - 1)\Upsilon_a \tau_{bc} + 2\tau_{a[b}\Upsilon_{c]} - 2\Upsilon^d \boldsymbol{g}_{a[b}\tau_{c]d}. \tag{2.7.27}$$

Tracing one obtains that

$$\hat{\nabla}^a \tau_{ab} = \nabla^a \tau_{ab} + (n + w - 4)\Upsilon^a \tau_{ab} \tag{2.7.28}$$

for τ_{ab} a 2-form of weight w. Noting that $Z_A^a Z_B^b \tau_{ab} = Z_{[A}^a Z_{B]}^b \tau_{ab}$ since $\tau_{ab} = \tau_{[ab]}$, the invariance of M_{AB}^{ab} then follows immediately from (2.7.3).

2.7.7 The Curvature Tractor

The middle operator M_{AB}^{ab} is *strongly invariant*, meaning that it can be applied to weighted tractor valued 2-forms by coupling the Levi-Civita connection appearing in the definition of M_{AB}^{ab} with the tractor connection. Applying M_{AB}^{ab} to the tractor curvature κ_{abCD} yields the *curvature tractor* (or 'W-tractor') of [75] (cf. [127])

$$W_{ABCD} = \mathsf{M}_{AB}^{ab}\kappa_{abCD}. \tag{2.7.29}$$

Computing W_{ABCD} explicitly with respect to a metric $g \in \boldsymbol{c}$ one obtains

$$\begin{aligned} W_{ABCD} = (n-4)\big(&Z_A{}^a Z_B{}^b Z_C{}^c Z_D{}^d W_{abcd} - 2Z_A{}^a Z_B{}^b X_{[C} Z_{D]}{}^d C_{abd} \\ &- 2X_{[A} Z_{B]}{}^b Z_C{}^c Z_D{}^d C_{cdb}\big) + 4X_{[A} Z_{B]}{}^b X_{[C} Z_{D]}{}^d B_{db} \end{aligned} \tag{2.7.30}$$

where B_{db} is the *Bach tensor*

$$B_{ab} = \nabla^c C_{cba} + P^{dc} W_{dacb}. \tag{2.7.31}$$

Remark 2.7.5. (1) In [75], motivated by consideration of the action of the commutator $[\mathsf{D}_A, \mathsf{D}_B]$ on standard tractors, the curvature tractor W_{ABCD} was defined as

$$\frac{3}{n-2}\mathsf{D}^E \left(X_{[E} Z_A^a Z_{B]}^b \kappa_{abCD} \right).$$

Computing this explicitly with respect to $g \in \boldsymbol{c}$ yields (2.7.30) so the two definitions are equivalent. Note that by (2.7.3) $X_{[E} Z_A^a Z_{B]}^b$ is an invariant object, so that this is also a manifestly invariant definition.

(2) The curvature tractor has Weyl tensor symmetries. In [81] it is shown to be straightforwardly related to the curvature of the (Ricci flat) ambient metric; the exception to this relation is $n = 4$ where W_{ABCD} (being equivalent to the Bach tensor) is instead the obstruction which must vanish for the ambient metric to exist (formally) to all orders. ∎

The curvature tractor and tractor D-operator form the basis for a straightforward construction of all (weighted) scalar conformal invariants in odd dimensions [75] (there is also a closely related approach [8] using the ambient metric). Essentially, one imitates the Riemannian case where a generating set of scalar invariants can be constructed by tensoring together covariant derivatives of the Riemannian curvature tensor and making complete contractions; in the conformal case one simply replaces R_{abcd} with W_{ABCD} (or $X_{[E} Z_A^a Z_{B]}^b \kappa_{abCD}$) and the covariant derivative ∇_a with D_A (or $\mathbb{D}_{AA'}$). In fact a slight improvement on this is required, which involves (roughly speaking) the cancellation of X_A's (see [75]); this idea is illustrated for the case of invariant operators in the derivations of the tractor D-operator from the double-D-operator (2.7.23), the conformal Laplacian from the tractor D-operator (2.7.18), and of the Paneitz operator from the tractor D-operator and the conformal Laplacian (2.7.20). The even dimensional case is much more subtle, and is far from being completely resolved (partial results can be found in [8, 75]). To produce (weighted) tensor invariants one starts by constructing weighted tractor invariants in the obvious way as before (now making incomplete contractions); one then uses the natural bundle projections arising from the composition series structure of $\mathcal{E}^A$, for example,

$$X_A : \mathcal{E}^A[w] \to \mathcal{E}[w+1] \quad \text{and} \quad X^{[A} Z_b^{B]} : \mathcal{E}_{[AB]}[w] \to \mathcal{E}_b[w+2].$$

Such a projection may be identically zero in which case another invariant projection is defined, for example, if $X_A \mathcal{I}^A = 0$ then $Z_A^a \mathcal{I}^A$ is invariant (cf. §§5.3.9).

3 The Geometry of Conformal Embeddings

We now turn to the consideration of (nondegenerate) submanifolds in conformal manifolds. We consider the induced conformal structure on such submanifolds as intrinsic, so that we may equivalently consider a conformal embedding $\iota : \Sigma \hookrightarrow M$ between conformal manifolds $(\Sigma^m, \boldsymbol{c}_\Sigma)$ and $(M^n, \boldsymbol{c})$ with $n > m \geq 2$. A choice of metric $g \in \boldsymbol{c}$ determines a metric $g_\Sigma \in \boldsymbol{c}_\Sigma$ by pullback. In seeking to construct local invariants and invariant operators on conformally embedded submanifolds it is natural to compute with respect to the pair of metrics $(g, \iota^* g)$ and look for constructions which are invariant under conformal rescalings of g. However, one can only get so far with this naïve approach. Here we present the basics of the local theory of conformal hypersurfaces in Riemannian manifolds as developed in [7, 21, 90, 128, 141] using tractor calculus. We then extend this theory to the case of arbitrary codimension nondegenerate submanifolds in conformal manifolds. This is consistent with work of Burstall and Calderbank [22, 23].

3.1 Conformal Hypersurfaces and Scales

Here we recall some basic facts concerning the geometry of hypersurfaces in Riemannian manifolds, from the point of view of conformal geometry.

3.1.1 Notation

We consider a conformal embedding $\iota : \Sigma \hookrightarrow M$ between Riemannian signature conformal manifolds $(\Sigma^m, \boldsymbol{c}_\Sigma)$ and $(M^n, \boldsymbol{c})$, now with $n - 1 = m \geq 2$. We retain the usual abstract index notation $\mathcal{E}^a$ for the tangent bundle of M, and use lower case Latin abstract indices from the later part of the alphabet (i, j, k, l, etc.) for Σ. So $T\Sigma$ is alternatively denoted by $\mathcal{E}^i$, and V^i denotes a submanifold tangent vector or vector field. We denote the weight w submanifold density bundle by $\mathcal{E}_\Sigma[w]$ and the ambient density bundle by $\mathcal{E}[w]$. We denote the conformal metric of $(\Sigma, \boldsymbol{c}_\Sigma)$ by $\boldsymbol{g}_{ij}$ and its inverse by $\boldsymbol{g}^{ij}$.

We identify Σ with its image in M and write $\mathcal{E}^a|_\Sigma \to \Sigma$ for the restriction of $\mathcal{E}^a \to M$ to fibers over Σ (i.e. for the pullback bundle $\iota^* TM$). The submanifold density bundle $\mathcal{E}_\Sigma[w]$ can be identified with the restriction $\mathcal{E}[w]|_\Sigma$ of the ambient density bundle $\mathcal{E}[w]$ to Σ, since choosing any ambient metric $g \in \boldsymbol{c}$ trivialises each of these bundles and the resulting identification is clearly metric independent.

We denote the tangent map $T\iota : T\Sigma \to TM$, considered as a section of $T^*\Sigma \otimes TM|_\Sigma$, by Π^a_i. So if $U = T\iota(V)$ then $U^a = \Pi^a_i V^i$. We define the section Π^i_a of $T^*M|_\Sigma \otimes T\Sigma$ to be the map $TM|_\Sigma \to T\Sigma$ given by orthogonal projection with respect to the conformal metric (or equivalently any metric in $\boldsymbol{c}$). Clearly $\Pi^i_a \Pi^a_j = \delta^i_j$, where δ^i_j is the (abstract) Kronecker delta: $\delta^i_j V^j = V^i$. The composition $\Pi^a_i \Pi^i_b$ gives the orthogonal projection $TM|_\Sigma \to T\iota(T\Sigma)$, which we also denote by Π^a_b. Note also that

$$\boldsymbol{g}_{ij} = \Pi^a_i \Pi^b_j \boldsymbol{g}_{ab} \tag{3.1.1}$$

along the submanifold Σ.

3.1.2 The Unit Normal Field

Given a choice of metric $g \in \boldsymbol{c}$ and a (possibly local) orientation of the normal bundle, one may talk about the unit normal field. However, in conformal geometry it is more natural to work with the corresponding weight -1 normal field which is normed by the conformal metric (and hence conformally invariant). We assume a fixed orientation of the normal bundle, working locally if necessary.

Definition 3.1.1. By the *(weighted) unit normal field* of a conformal hypersurface we mean the section N^a of $\mathcal{E}^a[-1]|_\Sigma$ satisfying $\boldsymbol{g}_{ab} N^a N^b = 1$ and compatible with the orientation. We refer to the corresponding section $N_a = \boldsymbol{g}_{ab} N^b$ of $\mathcal{E}_a[1]|_\Sigma$ as the *(weighted) unit conormal field.*

It is easy to see that

$$\delta^a_b = \Pi^a_b + N^a N_b \quad \text{and} \quad \boldsymbol{g}_{ab} = \boldsymbol{g}_{ij} \Pi^i_a \Pi^j_b + N_a N_b \tag{3.1.2}$$

along Σ.

3.1.3 Tangential Derivatives

Let g be a metric in $\boldsymbol{c}$, and let ∇ denote its Levi-Civita connection. Since a connection can be thought of as differentiating vector fields along curves, by restricting to curves in Σ the connection ∇ induces a connection on $TM|_\Sigma \to \Sigma$. We will refer to this connection as the *pullback connection*, denoted $\iota^*\nabla$. When using abstract index notation we distinguish between ∇ and $\iota^*\nabla$ by denoting them ∇_a and ∇_i respectively. One may equivalently define ∇_i to act on a section V^a of $\mathcal{E}^a|_\Sigma$ by

$$\nabla_i V^b = \Pi_i^a \nabla_a \tilde{V}^b \tag{3.1.3}$$

along Σ, where $\tilde{V}^b$ is any extension of V^b to M.

3.1.4 The Submanifold Levi-Civita Connection

If $g \in \boldsymbol{c}$ then the Levi-Civita connection ∇ of g induces the Levi-Civita connection D of $g_\Sigma = \iota^* g$ by [101]

$$D_i V^j = \Pi_b^j \nabla_i V^b \tag{3.1.4}$$

where $V^b = \Pi_k^b V^k$. Dually, on a 1-form ω_j we have

$$D_i \omega_j = \Pi_j^b \nabla_i \omega_b \tag{3.1.5}$$

where $\omega_b = \Pi_b^k \omega_k$.

3.1.5 The Second Fundamental Form

Here we establish our convention for the (Riemannian) second fundamental form.

Definition 3.1.2. Given a metric $g \in \boldsymbol{c}$ we define the *second fundamental form II* by the *Gauss formula*

$$\nabla_X Y = D_X Y + I\!I(X,Y)N \tag{3.1.6}$$

for all $X, Y \in \mathfrak{X}(\Sigma)$, where ∇ and D are the Levi-Civita connections of g and $\iota^* g$ respectively and N is the (weighted) unit normal field. (As usual we are implicitly using the pullback connection $\iota^*\nabla$ on the left hand side.)

This defines a bilinear form II, which is of conformal weight 1 since N has conformal weight -1. (So trivialising the density bundles using g one gets the usual second fundamental form, with our sign convention.) It is easy to see that the bilinear form II is symmetric since both ∇ and D are torsion free.

In terms of abstract index notation, the Gauss formula can be written as

$$\nabla_i V^c = \Pi^c_j D_i V^j + II_{ij} N^c V^j$$

were $V^c = \Pi^c_k V^k$. Contracting both sides of the above with N_c and using the Leibniz rule (noting that $N_c V^c = 0$) yields that $II_{ij} V^j = -V^j \Pi^c_j \nabla_i N_c$ for all sections V^j of $\mathcal{E}^j$. Thus

$$II_{ij} = -\Pi^b_j \nabla_i N_b. \tag{3.1.7}$$

3.1.5.1 Conformal Transformations

From the transformation law (2.4.18) for the ambient Levi-Civita connection we have

$$\hat{\nabla}_i N_b = \nabla_i N_b + \Pi^a_i \Upsilon^c N_c \boldsymbol{g}_{ab}$$

since the unit conormal has weight 1 and $\Pi^a_i N_a = 0$. Thus from (3.1.7) we have the conformal transformation law

$$\hat{II}_{ij} = II_{ij} - \Upsilon_c N^c \boldsymbol{g}_{ij} \tag{3.1.8}$$

for the second fundamental form under the conformal rescaling $\hat{g} = \Omega^2 g$, with $\Upsilon_a = \nabla_a \log \Omega$. Since the transformation law is by trace only we see that the trace free part $II_{(ij)_0}$ of the second fundamental form is conformally invariant. Denoting $II_{(ij)_0}$ by $\mathring{II}_{ij}$ we have

$$II_{ij} = \mathring{II}_{ij} + H \boldsymbol{g}_{ij} \tag{3.1.9}$$

where $H = \frac{1}{m} \boldsymbol{g}^{ij} II_{ij}$ is the *mean curvature*. Observe that the mean curvature has conformal weight -1. The mean curvature transforms according to

$$\hat{H} = H - \Upsilon_a N^a. \tag{3.1.10}$$

3.1.6 The Gauss and Codazzi-Mainardi Equations

Let us recall the well known Gauss and Codazzi-Mainardi equations which express the components of the ambient curvature along the hypersurface in terms of the intrinsic curvature and the second fundamental form. See, e.g., [43, 115] for derivations, cf. §§6.1.7.

Fix a metric $g \in \boldsymbol{c}$. We denote the full projection $\Pi_i^a \Pi_j^b \Pi_k^c \Pi_l^d R_{abcd}$ of the ambient Riemannian curvature tensor along Σ by R_{ijkl}. The *Gauss equation* for the hypersurface Σ is

$$R_{ijkl} = r_{ijkl} + II_{il} II_{jk} - II_{ik} II_{jl} \tag{3.1.11}$$

where r_{ijlk} is the Riemannian curvature tensor of $g_\Sigma = \iota^* g$. Denoting $\Pi_i^a \Pi_j^b \Pi_k^c R_{abcd} N^d$ by R_{ijlN} the *Codazzi-Mainardi equation* is

$$R_{ijkN} = -2 D_{[i} II_{j]k} \tag{3.1.12}$$

where D is the Levi-Civita connection of g_Σ.

3.1.7 Minimal Scales

Here we present a useful technical lemma which was observed in, e.g., [21, 78].

Lemma 3.1.3. *Given any metric $g_\Sigma \in \boldsymbol{c}_\Sigma$ there exists a metric $g \in \boldsymbol{c}$ extending g_Σ (i.e. such that $g_\Sigma = \iota^* g$) for which Σ has vanishing mean curvature.*

Proof. Fix $g_\Sigma \in \boldsymbol{c}_\Sigma$ and let $g \in \boldsymbol{c}$ be any metric with $g_\Sigma = \iota^* g$ with corresponding scale σ. The mean curvature H of Σ has conformal density weight -1, so the usual (Riemannian) mean curvature *function* is σH. Similarly, the usual (Riemannian) unit conormal is $\sigma^{-1} N_a$. Consider a smooth function Υ on M which, in a neighbourhood of Σ, is given by $s\sigma H$ where s is a normalised defining function for Σ (meaning $s|_\Sigma \equiv 0$ and $\nabla_a s|_\Sigma = \sigma^{-1} N_a$). Then the metric $\hat{g} = e^{2\Upsilon} g$ satisfies $\iota^* \hat{g} = g_\Sigma$ and has mean curvature

$$\hat{H} = H - \Upsilon_c N^c \tag{3.1.13}$$

where $\Upsilon_c = \nabla_c \Upsilon$ by (3.1.10). Computing that $\nabla_c \Upsilon = H N_c$ along Σ (since $\nabla_a \sigma = 0$ and $s|_\Sigma \equiv 0$) we obtain $\hat{H} = 0$. □

Definition 3.1.4. We refer to a scale σ (or the corresponding metric $g \in \boldsymbol{c}$) for which $H \equiv 0$ as a *minimal scale.*

Notice that if g and $\hat{g} = \Omega^2 g$ both correspond to minimal scales then $\Upsilon_a = \nabla_a \log \Omega$ satisfies

$$\Upsilon_a N^a = 0. \tag{3.1.14}$$

3.1.8 The Natural Möbius Structure on 2D Conformal Hypersurfaces

A conformal embedding of a conformal surface $(\Sigma^2, \boldsymbol{c}_\Sigma)$ into a conformal 3-manifold $(M^3, \boldsymbol{c})$ induces a natural Möbius structure on Σ. The notion we present here is termed the 'induced conformal Möbius structure' in [23].

Definition 3.1.5. The *(normal) induced Möbius structure* on a conformally embedded surface is defined by associating to each metric $g_\Sigma \in \boldsymbol{c}_\Sigma$ the Rho-tensor

$$p_{ij} = P_{(ij)_0} + \frac{1}{2} K \boldsymbol{g}_{ij} \tag{3.1.15}$$

where P_{ij} is the projection $\Pi_i^a \Pi_j^b P_{ab}$ of the Schouten tensor of any minimal scale $g \in \boldsymbol{c}$ which extends g_Σ.

To see that this gives a well-defined Möbius structure consider any two minimal scales $g, \hat{g} \in \boldsymbol{c}$ for Σ, with $\hat{g} = \Omega^2 g$. By (2.4.11) along with (3.1.5) and (3.1.14) we have

$$\begin{aligned} \hat{P}_{ij} &= P_{ij} - \Pi_i^a \Pi_j^b \nabla_a \Upsilon_b + \Upsilon_i \Upsilon_j - \Upsilon_c \Upsilon^c \boldsymbol{g}_{ij} \\ &= P_{ij} - D_i \Upsilon_j + \Upsilon_i \Upsilon_j - \Upsilon_k \Upsilon^k \boldsymbol{g}_{ij} \end{aligned} \tag{3.1.16}$$

where $\Upsilon_a = \Omega^{-1} \nabla_a \Omega$, $\Upsilon_i := \Omega^{-1} D_i \Omega = \Pi_i^a \Upsilon_a$, and D is the Levi-Civita connection of $g_\Sigma = \iota^* g$. Thus if $\hat{g_\Sigma} = g_\Sigma$ (i.e. $\Omega|_\Sigma \equiv 1$) then $\Upsilon_i = 0$ and $\hat{P}_{ij} = P_{ij}$, so P_{ij} does not depend on the choice of minimal scale $g \in \boldsymbol{c}$ extending g_Σ. By tracing (3.1.16) we see that $P_i{}^i$ has the same conformal transformation law as the Gauss curvature K (their difference is a weight -2 scalar invariant). The computation therefore shows that the Rho tensor p_{ij} transforms in the appropriate way under conformal rescalings of the submanifold metric.

Remark 3.1.6. By (3.1.16) the assignment to each metric $g \in \boldsymbol{c}$ of the 'Rho-tensor' $\check{p}_{ij} = \Pi^a_i \Pi^b_j P_{ab}$, where P_{ab} is the Schouten tensor of any minimal scale $g \in \boldsymbol{c}$ extending g_Σ, gives rise to a Möbius structure in the more general sense of [23] (cf. Remark 2.6.1). In [23] this is termed the 'induced Möbius structure'. This Möbius structure (and the corresponding tractor connection) was used in the earlier work of [21]. Note that if $g \in \boldsymbol{c}$ is an arbitrary metric extending g_Σ then $\check{p}_{ij} = \Pi^a_i \Pi^b_j P_{ab} + H\mathring{I\!I}_{ij} + \frac{1}{2}H^2 \boldsymbol{g}_{ij}$ (cf. (3.2.16)). ▮

Proposition 3.1.7. *If $g \in \boldsymbol{c}$ is an arbitrary metric extending g_Σ then*

$$p_{ij} = P_{(ij)_0} + H\mathring{I\!I}_{ij} + \frac{1}{2}K\boldsymbol{g}_{ij} \tag{3.1.17}$$

where P_{ij} denotes the projection $\Pi^a_i \Pi^b_j P_{ab}$ of the ambient Schouten tensor.

Proof. To verify this it suffices to check that the right hand side is unchanged under conformal rescalings of g to $\hat{g} = e^{2\Upsilon} g$ with $\Upsilon|_\Sigma \equiv 0$ (since using Lemma 3.1.3 we may take $\hat{g}$ to be minimal). Note that under such a rescaling g_Σ (and hence K) does not change. Note also that $\nabla_i \Upsilon = 0$ and $\Upsilon^a = \nabla^a \Upsilon$ is normal. Writing Υ_b as $(\Upsilon_c N^c) N_b$ (2.4.11) one then has

$$\begin{aligned}
\hat{P}_{(ij)_0} &= P_{(ij)_0} - \Pi^b_{(j} \nabla_{i)_0} \left[(\Upsilon_c N^c) N_b\right] \\
&= P_{(ij)_0} - (\Upsilon_c N^c) \Pi^b_{(j} \nabla_{i)_0} N_b && \text{(since } \Pi^b_j N_b = 0\text{)} \\
&= P_{(ij)_0} + (\Upsilon_c N^c) I\!I_{(ij)_0} && \text{(by (3.1.7))}
\end{aligned}$$

as required by the mean curvature transformation law (3.1.10).

(Alternatively, to directly compute (3.1.17) one may rescale g to a minimal scale $\hat{g} = e^{2\Upsilon} g$ with $\iota * \hat{g} = g_\Sigma$ by taking Υ as in the proof of Lemma 3.1.3 and then rewrite $\hat{P}_{(ij)_0} + \frac{1}{2}K\boldsymbol{g}_{ij}$ in terms of the metric g.) □

In the case where the conformal surface already carries an intrinsic Möbius structure, we may define the notion of a *Möbius conformal embedding* to be a conformal embedding for which the induced Möbius structure agrees with the intrinsic one. Just as we may regard the induced conformal structure on a nondegenerate submanifold of a conformal manifold as intrinsic, we may also regard the (normal) induced Möbius structure on a conformal surface as intrinsic. Given $g_\Sigma \in \boldsymbol{c}_\Sigma$ we may therefore refer to p_{ij} as the *intrinsic Rho-tensor.*

3.2 Conformal Hypersurfaces and Tractors

Here we present the basic conformally invariant calculus for hypersurfaces developed in [7, 21, 90, 128, 141]. As in §3.1 we consider a conformal embedding $\iota : \Sigma \hookrightarrow M$ between Riemannian signature conformal manifolds $(\Sigma^m, \boldsymbol{c}_\Sigma)$ and $(M^n, \boldsymbol{c})$, with $n - 1 = m \geq 2$.

We denote the standard tractor bundle of M by $\mathcal{E}^A$ as usual, using abstract indices from the beginning of the alphabet. The standard tractor bundle of Σ will be denoted by $\mathcal{E}^I$, and we will use abstract indices I, J, K, L, I', etcetera. When using index free notation we denote the standard tractor bundles of M and Σ respectively by $\mathcal{T}M$ and $\mathcal{T}\Sigma$.

3.2.1 The Normal Tractor

Following [7] we define the *(unit) normal tractor* of Σ to be the section N^A of $\mathcal{E}^A|_\Sigma$ given by

$$N^A \stackrel{g}{=} \begin{pmatrix} 0 \\ N^a \\ H \end{pmatrix} \tag{3.2.1}$$

for any $g \in \boldsymbol{c}$ where H is the mean curvature of Σ with respect to g. Comparing the transformation law (3.1.13) for the mean curvature with the tractor transformation law (2.7.8) we see that N^A is well-defined. Clearly $N^A N_A = 1$. Note that if g is a minimal scale then

$$N^A \stackrel{g}{=} \begin{pmatrix} 0 \\ N^a \\ 0 \end{pmatrix}. \tag{3.2.2}$$

Remark 3.2.1. The normal tractor is closely related to the *conformal Robin operator* δ_N, which supplies the conformal Laplacian with self adjoint elliptic boundary conditions (see, e.g., [21]). The conformal Robin operator can be defined on ambient densities of any weight (not just $1 - \frac{n}{2}$) and is given by

$$\delta_N f = N^a \nabla_a f + wHf \tag{3.2.3}$$

along Σ for any section f of $\mathcal{E}[w]$. ∎

The normal tractor allows us to split the standard tractor bundle of M along Σ as an orthogonal direct sum

$$\mathcal{T}M|_\Sigma = \mathcal{N}^\perp \oplus \mathcal{N} \tag{3.2.4}$$

where $\mathcal{N}$ is the real line bundle spanned by the normal tractor (the 'normal tractor bundle'). We write

$$\Pi_B^A = \delta_B^A - N^A N_B \tag{3.2.5}$$

for the orthogonal projection $\mathcal{T}M|_\Sigma \to \mathcal{N}^\perp$.

3.2.2 Relating Tractor Bundles

The following theorem was implicit in [21] and made explicit in [90] (for the case $m \geq 3$) cf. [78].

Theorem 3.2.2. *The hypersurface standard tractor bundle $\mathcal{T}\Sigma$ is canonically isomorphic to the orthogonal complement $\mathcal{N}^\perp$ of the normal tractor bundle by a metric and filtration preserving bundle isomorphism. The isomorphism $\mathcal{T}\Sigma \to \mathcal{N}^\perp$ is given explicitly by*

$$v^I \stackrel{g_\Sigma}{=} \begin{pmatrix} \sigma \\ \mu^i \\ \rho \end{pmatrix} \quad \stackrel{\Pi_I^A}{\longmapsto} \quad v^A \stackrel{g}{=} \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} \tag{3.2.6}$$

where $\mu^a = \Pi_i^a \mu^i$ and g is any minimal scale extending g_Σ.

Proof. Let us start by fixing $g_\Sigma \in \boldsymbol{c}_\Sigma$ and g a minimal scale for which $\iota^* g = g_\Sigma$. We need to show that the above map is unchanged if we replace g by $\hat{g} = \Omega^2 g$ and g_Σ by $\hat{g}_\Sigma = \Omega^2 g_\Sigma$, with $\hat{g}$ a minimal scale. If $\Upsilon_a = \Omega^{-1}\nabla_a\Omega$ and $\Upsilon_i = \Omega^{-1}D_i\Omega$ $(= \Pi_i^a \Upsilon_a)$ then by (3.1.14) along Σ we have

$$\Upsilon_a = \Pi_a^i \Upsilon_i.$$

From this we see that the following diagram

$$\begin{array}{ccccc} & [\mathcal{E}^I]_{g_\Sigma} & \longrightarrow & [\mathcal{E}^A]_g|_\Sigma & \\ \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^j & \delta_i^j & 0 \\ -\frac{1}{2}\Upsilon^k\Upsilon_k & -\Upsilon_i & 1 \end{pmatrix} & \Big\downarrow & & \Big\downarrow & \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^b & \delta_a^b & 0 \\ -\frac{1}{2}\Upsilon^c\Upsilon_c & -\Upsilon_a & 1 \end{pmatrix} \\ & [\mathcal{E}^J]_{\hat{g}_\Sigma} & \longrightarrow & [\mathcal{E}^B]_{\hat{g}}|_\Sigma & \end{array}$$

commutes, where the horizontal maps have respective matrix representations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi_i^a & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi_j^b & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore the map (3.2.6) is well-defined. □

If $g \in \boldsymbol{c}$ is an arbitrary metric (with corresponding scale σ) and $g_\Sigma = \iota^* g$ then the map (3.2.6) of Theorem 3.2.2 is given by (cf. [78, 90])

$$v^I \overset{g_\Sigma}{=} \begin{pmatrix} \sigma \\ \mu^i \\ \rho \end{pmatrix} \xmapsto{\Pi_I^A} v^A \overset{g}{=} \begin{pmatrix} \sigma \\ \mu^a - HN^a\sigma \\ \rho - \frac{1}{2}H^2\sigma \end{pmatrix}. \tag{3.2.7}$$

This can be seen by rescaling g to a minimal scale $\hat{g}$ with the same induced metric g_Σ as in the proof of Lemma 3.1.3 and then using the ambient tractor transformation law (2.7.8).

We refer to the map $\Pi_I^A : \mathcal{E}^I \to \mathcal{E}^A|_\Sigma$ as the *standard tractor map* of the conformal embedding. In index free notation we write this map as

$$\mathcal{T}\iota : \mathcal{T}\Sigma \to \mathcal{T}M. \tag{3.2.8}$$

Using the identifications of the respective standard tractor bundles with their duals given by the respective tractor metrics, one may define a map $\Pi_A^I : \mathcal{E}_I \to \mathcal{E}_A|_\Sigma$ (whose image annihilates N^A). In other words we define Π_A^I to be $g_{AB}\Pi_J^B g^{IJ}$, where g^{IJ} denotes the 'inverse' hypersurface tractor metric. We will often think of Π_I^A and Π_A^I as sections of $\mathcal{E}_I \otimes \mathcal{E}^A|_\Sigma$ and $\mathcal{E}^I \otimes \mathcal{E}_A|_\Sigma$ respectively, allowing for a flexibility of interpretation.

3.2.3 Relating Tractor Connections on $\mathcal{T}\Sigma$

The standard tractor map allows us to define a connection $\check{\nabla}$ on the submanifold tractor bundle $\mathcal{T}\Sigma$ induced by the ambient tractor connection. Given a standard tractor field v^J we define

$$\check{\nabla}_i v^J = \Pi_B^J \nabla_i (\Pi_K^B v^K) \tag{3.2.9}$$

where ∇ denenotes the ambient tractor connection (and ∇_i denotes the pullback connection $\iota^*\nabla$).

Using Theorem 3.2.2 and (3.1.4) it is easy to see that if $g \in \boldsymbol{c}$ is a minimal scale then, splitting $\mathcal{E}^J$ using $g_\Sigma = \iota^* g$, we have

$$\check{\nabla}_i \begin{pmatrix} \sigma \\ \mu^j \\ \rho \end{pmatrix} = \begin{pmatrix} D_i\sigma - \mu_i \\ D_i\mu^j + P_i{}^j\sigma + \delta_i^j\rho \\ D_i\rho - P_{ij}\mu^j \end{pmatrix} \tag{3.2.10}$$

where D is the Levi-Civita connection of g_Σ and P_{ij} is the projection $\Pi_i^a\Pi_j^b P_{ab}$ of the ambient Schouten tensor. (The fact that we are differentiating densities does not complicate the use of (3.1.4) in the computation; one may trivialise the density bundles using g and g_Σ for simplicity.)

3.2.3.1 The Difference Tractor

Compare (3.2.10) with the corresponding formula

$$D_i \begin{pmatrix} \sigma \\ \mu^j \\ \rho \end{pmatrix} = \begin{pmatrix} D_i\sigma - \mu_i \\ D_i\mu^j + p_i{}^j\sigma + \delta_i^j\rho \\ D_i\rho - p_{ij}\mu^j \end{pmatrix} \tag{3.2.11}$$

for the intrinsic tractor connection of Σ, where p_{ij} is the intrinsic Rho tensor (Schouten for $m \geq 3$). One can easily see that

$$\check{\nabla}_i v^J = D_i v^J + \mathsf{S}_i{}^J{}_K v^K \tag{3.2.12}$$

where

$$\mathsf{S}_i{}^J{}_K \overset{g_\Sigma}{=} \begin{pmatrix} 0 & 0 & 0 \\ P_i{}^j - p_i{}^j & 0 & 0 \\ 0 & p_{ij} - P_{ij} & 0 \end{pmatrix}. \tag{3.2.13}$$

The $\mathrm{End}(\mathcal{T}\Sigma)$ valued 1-form S is referred to as the *difference tractor*. Using the splitting tractors corresponding to g_Σ we have

$$\mathsf{S}_{iJK} = (P_{ij} - p_{ij})\left(Z_J^j X_K - Z_K^j X_J\right). \tag{3.2.14}$$

Noting that $2Z^j_{[J}X_{K]}$ is conformally invariant by the intrinsic version of (2.7.3) we see that

$$\mathcal{F}_{ij} = P_{ij} - p_{ij} \tag{3.2.15}$$

is independent of the choice of metrics $g_\Sigma \in \boldsymbol{c}_\Sigma$ and $g \in \boldsymbol{c}$ minimal with $\iota^* g = g_\Sigma$. We refer to the invariant $\mathcal{F}_{ij}$ as the *Fialkow tensor.*

3.2.3.2 Computing the Fialkow Tensor

If one repeats the above calculation (of §§§3.2.3.1) removing the assumption that g be a minimal scale then one arrives at the expression

$$\mathcal{F}_{ij} = P_{ij} - p_{ij} + H\mathring{I\!I}_{ij} + \frac{1}{2}H^2\boldsymbol{g}_{ij} \tag{3.2.16}$$

of [128] for the Fialkow tensor. Now by considering the 'Ricci decomposition' of the Gauss equation (3.1.11) when $m \geq 3$ one readily obtains that

$$\mathcal{F}_{ij} = \frac{1}{m-2}\left(W_{iNjN} + \mathring{I\!I}^2_{ij} - \frac{\mathring{I\!I}^{kl}\mathring{I\!I}_{kl}}{2(m-1)}\boldsymbol{g}_{ij}\right) \tag{3.2.17}$$

where W_{iNjN} denotes $\Pi^a_i\Pi^b_j W_{acbd}N^cN^d$ and $\mathring{I\!I}^2_{ij} = \mathring{I\!I}_{ik}\mathring{I\!I}^k{}_j$ (cf. Proposition 3.3.10).

Remark 3.2.3. The manifestly invariant formula (3.2.17) for the Fialkow tensor was first calculated by the author, and pointed out to Gover and Vyatkin (see [43, 141], cf. [98]). It was observed by Vyatkin that such an expression seems to have appeared first in the study [67] of Fialkow (whence the name 'Fialkow tensor'). ▮

In the case $m = 2$ by combining (3.1.17) and (3.2.16) we have

$$\mathcal{F}_{ij} = \frac{1}{2}(P_k{}^k + H^2 - K)\boldsymbol{g}_{ij}. \tag{3.2.18}$$

Note that if Σ is a closed surface in Euclidean space then $P_i{}^i \equiv 0$ and $\mathcal{F}_i{}^i = H^2 - K$, which is a form of the Willmore energy density [146] (the integral of K over Σ being the topological invariant $2\pi\chi(\Sigma)$ by the Gauss-Bonnet theorem). Substituting (2.4.5) for the ambient curvature and (2.4.7) for the submanifold curvature in the Gauss equation (3.1.11) and then contracting with $\boldsymbol{g}^{ik}$ we obtain

$$P_i{}^i\boldsymbol{g}_{jl} = K\boldsymbol{g}_{jl} + I\!I_{il}I\!I_j{}^i - 2HI\!I_{jl}. \tag{3.2.19}$$

Tracing (3.2.19) we obtain

$$\begin{aligned} 2P_i^{\ i} &= 2K + I\!I_{ij} I\!I^{ij} - 4H^2 \\ &= 2K + \mathring{I\!I}_{ij} \mathring{I\!I}^{ij} - 2H^2. \end{aligned} \tag{3.2.20}$$

Thus

$$\mathcal{F}_{ij} = \frac{1}{4} |\mathring{I\!I}|^2 \boldsymbol{g}_{ij} \tag{3.2.21}$$

where $|\mathring{I\!I}|^2 = \mathring{I\!I}^{kl} \mathring{I\!I}_{kl}$.

Remark 3.2.4. (1) The tensor Q^{V_Σ} appearing in [23] is the same as the Fialkow tensor defined here for any dimension $m \geq 2$. In the conformally flat ambient case the above formulae for the Fialkow tensor can be found in Section 11.4 of [23] by specialising the formulae there to the case $V = V_\Sigma$ (V_Σ being their notation for $\mathcal{N}^\perp \cong \mathcal{T}\Sigma$).

(2) In [141] the Fialkow tensor for the surface case is defined to be zero. This amounts to treating the (non normal) induced tractor connection $\check{\nabla}$ as intrinsic (cf. [21]). We take the point of view that $\check{\nabla}$ is extrinsic and the normal (or normalised) tractor connection D is intrinsic. ∎

3.2.4 The Tractor Gauss Formula

The standard tractor map also allows us to define a conformal tractor analogue of the (Riemannian) second fundamental form by the following Gauss-type formula

$$\nabla_i v^B = \Pi_J^B \check{\nabla}_i v^J + \mathbb{L}_{iK} N^B v^K \tag{3.2.22}$$

for all sections v^J of $\mathcal{E}^J$. The 1-form $\mathbb{L}$ takes values in the standard (co)tractor bundle and is referred to as the *tractor second fundamental form* (or 'tractor shape form'). Using (3.2.12) we obtain the *tractor Gauss formula*

$$\nabla_i v^B = \Pi_J^B \left(D_i v^J + \mathsf{S}_i{}^J{}_K v^K \right) + \mathbb{L}_{iK} N^B v^K. \tag{3.2.23}$$

Remark 3.2.5. Using $\mathcal{T}\iota$ to identify $\mathcal{T}\Sigma$ with $\mathcal{N}^\perp$ we may write the Gauss formula using index free notation as

$$\nabla_X v \;=\; \underbrace{D_X v + \mathsf{S}(X) v}_{\text{'tangential part'}} \; + \underbrace{\mathbb{L}(X, v) N}_{\text{'normal part'}} \tag{3.2.24}$$

for any $X \in \mathfrak{X}(\Sigma)$ and $v \in \Gamma(\mathcal{T}\Sigma)$, where N denotes the normal tractor. ∎

3.2.4.1 Computing the Tractor Second Fundamental Form

By contracting (3.2.22) on both sides with N_B (and using that $N_B v^B = 0$ implies $N_B \nabla_i v^B$ equals $-v^B \nabla_i N_B$) we see that

$$\mathbb{L}_{iJ} = -\Pi_J^B \nabla_i N_B. \tag{3.2.25}$$

Note that $N^B \nabla_i N_B = 0$ since $N^B N_B = 1$, so we also have

$$\mathbb{L}_{iJ} \Pi_B^J = -\nabla_i N_B. \tag{3.2.26}$$

From the formula (2.7.7) for the ambient tractor connection we have that

$$\nabla_i N^B \overset{g}{=} \begin{pmatrix} 0 \\ \nabla_i N^b + H \Pi_i^b \\ \nabla_i H - P_{iN} \end{pmatrix} \tag{3.2.27}$$

where $P_{iN} = \Pi_i^a P_{ab} N^b$, and from this (using (3.2.7)) we have

$$\mathbb{L}_{iJ} \overset{\iota^* g}{=} \begin{pmatrix} 0 \\ \mathring{I\!I}_{ij} \\ -\nabla_i H + P_{iN} \end{pmatrix}, \tag{3.2.28}$$

i.e. $\mathbb{L}_i{}^J \overset{\iota^* g}{=} (0, \mathring{I\!I}_i{}^j, -\nabla_i H + P_{iN})$. By substituting the ambient Ricci decomposition (2.4.5) into the Codazzi-Mainardi equation (3.1.12) and contracting one can show that

$$\nabla_i H - P_{iN} = \frac{1}{m-1} D^j \mathring{I\!I}_{ij} \tag{3.2.29}$$

in any ambient dimension $n \geq 3$ [7] (cf. Proposition 3.3.12). From this we see that $\mathring{I\!I} = 0 \Leftrightarrow \mathbb{L} = 0$, so the hypersurface Σ is totally umbilic if and only if the tractor second fundamental form is identically zero.

3.2.5 Tractor Gauss and Codazzi-Mainardi Equations

Having established the Gauss formula (3.2.23) relating the intrinsic and ambient tractor connections along the hypersurface one can easily compute tractor analogues of the Gauss and Codazzi-Mainardi equations for the components of the ambient tractor curvature along Σ.

3.2.5.1 The Tractor Gauss Equation

Let $\check{\kappa}$ denote the curvature of $\check{\nabla}$, defined in terms of the Ricci-type identity

$$2\check{\nabla}_{[i}\check{\nabla}_{j]}v^k = \check{\kappa}_{ij}{}^K{}_L v^L \tag{3.2.30}$$

for all $v^L \in \Gamma(\mathcal{E}^L)$ where $\check{\nabla}$ is coupled with any torsion free affine connection on Σ. Let κ_{ijJK} denote the full projection $\Pi_i^a \Pi_j^b \Pi_K^C \Pi_L^D \kappa_{abCD}$ of the ambient tractor curvature along Σ. We denote the intrinsic tractor curvature of Σ by κ^Σ.

Noting the formal similarity between (3.2.22) and the Riemannian Gauss formula (3.1.5) one obtains that

$$\kappa_{ijKL} = \check{\kappa}_{ijKL} + \mathbb{L}_{iL}\mathbb{L}_{jK} - \mathbb{L}_{iK}\mathbb{L}_{jL} \tag{3.2.31}$$

by arguing formally in the same manner as for the Gauss equation (3.1.11). From the relationship (3.2.12) between the two connections $\check{\nabla}$ and D on $\mathcal{T}\Sigma$ we have

$$\check{\kappa}_{ij}{}^K{}_L = \kappa^\Sigma{}_{ij}{}^K{}_L + 2D_{[i}\mathsf{S}_{j]}{}^K{}_L + \mathsf{S}_i{}^K{}_{L'}\mathsf{S}_j{}^{L'}{}_L - \mathsf{S}_j{}^K{}_{L'}\mathsf{S}_i{}^{L'}{}_L \tag{3.2.32}$$

where the intrinsic tractor connection D of Σ is coupled with any torsion free affine connection on Σ. Putting these together we obtain the *tractor Gauss formula*

$$\kappa_{ij}{}^K{}_L = \kappa^\Sigma{}_{ij}{}^K{}_L + 2D_{[i}\mathsf{S}_{j]}{}^K{}_L + 2\mathsf{S}_{[i|}{}^K{}_{L'}\mathsf{S}_{|j]}{}^{L'}{}_L - 2\mathbb{L}_{[i}{}^K\mathbb{L}_{j]L}. \tag{3.2.33}$$

Remark 3.2.6. In [141] equation (3.2.31) is termed the 'tractor Gauss equation'. Our tractor Gauss equation (3.2.33) is consistent with [22] in the flat ambient case (and with §§6.5.2 in the CR case). Similar comments apply to the discussion of the tractor Codazzi-Mainardi Equation below. ∎

3.2.5.2 The Tractor Codazzi-Mainardi Equation

Let κ_{ijKD} denote the projection $\Pi^a_i \Pi^b_j \Pi^C_K \kappa_{abCD}$ of the ambient tractor curvature along Σ. By arguing formally in the same manner as for the Riemannian Codazzi-Mainardi equation (3.1.12) one has that

$$\kappa_{ij}{}^K{}_D N^D = -2\check{\nabla}_{[i}\mathbb{L}_{j]}{}^K \tag{3.2.34}$$

where $\check{\nabla}$ is coupled with any torsion free affine connection on Σ. From (3.2.12) we then have the *tractor Codazzi-Mainardi equation*

$$\kappa_{ij}{}^K{}_D N^D = -2D_{[i}\mathbb{L}_{j]}{}^K - 2\mathsf{S}_{[i|}{}^K{}_L \mathbb{L}_{|j]}{}^L \tag{3.2.35}$$

where D is similarly coupled with any torsion free affine connection on Σ.

3.2.6 Invariants and Invariant Operators

The hypersurface tractor D-operator D_I can be extended to act on sections of any ambient tractor bundle along Σ; more generally, if $\mathcal{E}^{\tilde{\Phi}}$ is any ambient tractor bundle and $\mathcal{E}^{\Phi}$ is any submanifold tractor bundle then

$$\mathsf{D}_I : \mathcal{E}^{\Phi}[w] \otimes \mathcal{E}^{\tilde{\Phi}}|_\Sigma \to \mathcal{E}_I \otimes \mathcal{E}^{\Phi}[w] \otimes \mathcal{E}^{\tilde{\Phi}}|_\Sigma \tag{3.2.36}$$

is defined by the usual formula

$$\mathsf{D}_I f^{\Phi\tilde{\Phi}} \overset{g_\Sigma}{=} \begin{pmatrix} w(n+2w-2)f^{\Phi\tilde{\Phi}} \\ (n+2w-2)D_i f^{\Phi\tilde{\Phi}} \\ -(D^i D_i + wp)f^{\Phi\tilde{\Phi}} \end{pmatrix} \tag{3.2.37}$$

where now the Levi-Civita connection D of g_Σ is coupled not only with the hypersurface (intrinsic) tractor connection, but also with the (pulled back) ambient tractor connection. Note in particular that this allows us to define iterated 'derivatives'

$$\mathsf{D}_J \cdots \mathsf{D}_K N^A \tag{3.2.38}$$

of the normal tractor N^A. One may similarly 'twist' the double-D-operator $\mathbb{D}_{IJ}$ so as to act on sections of $\mathcal{E}^{\Phi}[w] \otimes \mathcal{E}^{\tilde{\Phi}}|_\Sigma$. As in the case of conformal manifolds (§§2.7.7) one can now proliferate weighted scalar and tensor invariants by mimicking Riemannian constructions using corresponding tractor objects (for further details see [141, 83]).

Besides constructing invariants the hypersurface tractor calculus can be used to produce (extrinsic) differential operators. In [21] the calculus is applied to produce self adjoint elliptic boundary conditions for the 'conformally invariant powers of the Laplacian' $\square_{2k}$ of [81] (these differ from the GJMS operators P_{2k} of [87] when $k \geq 3$) for all k if n is odd and $k < \frac{n}{2}$ if n is even. Invariant boundary conditions for the Paneitz operator in 4-dimensions (the $k = \frac{n}{2} = 2$ case) were found in [38], and in $n \geq 4$ dimensions using tractor calculus in [90]. Tractor methods have also been used in attempts to better understand the structure of the intrinsic GJMS operators (and related Q-curvatures) of conformal hypersurfaces [98]. In [84] it was shown that on a conformal hypersurface there exist (extrinsically defined) 'conformally invariant powers of the Laplacian' to all orders.

Remark 3.2.7. In [84] the authors use ideas from tractor calculus developed for the study of conformally compact Riemannian manifolds [76, 78, 85] which we have not presented here. (An exposition of these ideas can also be found in [44], with some related developments.) Using these tools the authors solve formally (along the hypersurface Σ) the Loewner-Nirenberg boundary problem for a constant mean curvature metric in the conformal class on $M \setminus \Sigma$ with Σ as conformal infinity. This supplies $\Sigma \subset M$ with a canonical 'defining density' $\sigma \in \Gamma(\mathcal{E}[1])$ (corresponding to a metric $g^o = \sigma^{-2}\boldsymbol{g}$ on $M \setminus \Sigma$) determined up to order $n = \mathrm{dim} M$ after which log terms appear in the formal solution. The authors are then able to give a 'holographic' construction of hypersurface invariants and invariant operators (meaning that they extract well-defined limits on the hypersurface Σ from quantities defined in terms of the metric g^o on $M \setminus \Sigma$). The coefficient of the first log term in the formal expansion for σ is termed the 'ASC obstruction density' and is a conformal invariant of the hypersurface. This log term was calculated already for surfaces in conformal 3-manifolds in [3], and turns out to agree with the well known Willmore invariant [84]. The 'ASC obstruction density' therefore generalises the Willmore invariant to hypersurfaces in higher dimensions (for other approaches to generalising the Willmore invariant cf. [89, 141]). The results of [84] have been applied to the variational theory of conformal hypersurfaces in [70]. ∎

3.3 Higher Codimension Embeddings

Here we show that the conformally invariant calculus for hypersurfaces of [7, 21, 90, 128, 141] extends straightforwardly to the higher codimension case. (This has in large part already been accomplished in [23].) The key additional input in our approach is Lemma 3.3.2 which generalises Lemma 3.1.3. We relate this approach to the work of Burstall and Calderbank [23] which is also based around the conformal standard tractor bundle and connection (though informally termed the 'Cartan bundle' and 'Cartan connection'). We also introduce the 'normal tractor form' of a conformal submanifold and show that a submanifold is umbilic if and only if the normal tractor form is parallel along the submanifold.

We now consider conformal embedding $\iota : \Sigma \hookrightarrow M$ between conformal manifolds $(\Sigma^m, \boldsymbol{c}_\Sigma)$ and $(M^n, \boldsymbol{c})$ with $n > m \geq 2$. Note that we now allow $(\Sigma^m, \boldsymbol{c}_\Sigma)$ and $(M^n, \boldsymbol{c})$ to have any signature. (Despite the heading we also allow Σ to be a hypersurface.) Our notation is as in §3.1 and §3.2 whenever this makes sense. Note that as before we may canonically identify the density bundles $\mathcal{E}[w]|_\Sigma$ and $\mathcal{E}_\Sigma[w]$.

3.3.1 Submanifolds and Scales

3.3.1.1 The Normal Bundle

We denote the normal bundle of Σ by $N\Sigma \subset TM|_\Sigma$. We write N for the orthogonal projection $TM|_\Sigma \to N\Sigma$ so that

$$\delta^a_b = \Pi^a_b + \mathrm{N}^a_b \tag{3.3.1}$$

along Σ. We let $N\Sigma[w]$ denote $N\Sigma \otimes \mathcal{E}[w]|_\Sigma$. Note that the ambient conformal metric induces a bundle metric on $N\Sigma[-1]$. Given $g \in \boldsymbol{c}$ we write $\nabla^\perp$ for the *normal Levi-Civita connection* on $N\Sigma$ (or $N\Sigma[w]$). On a normal field N^a this is defined by

$$\nabla^\perp_i N^b = \mathrm{N}^b_c \nabla_i N^c. \tag{3.3.2}$$

The curvature $R^\perp$ of $\nabla^\perp$ on the normal bundle $N\Sigma$ is defined by the Ricci-type identity

$$\nabla^\perp_{[i} \nabla^\perp_{j]} N^c = R^\perp_{ij}{}^c{}_d N^d \tag{3.3.3}$$

for any normal field N^c.

3.3.1.2 The Second Fundamental Form

Fix $g \in \boldsymbol{c}$ and let $g_\Sigma = \iota^* g$. It is well known that the ambient Levi-Civita connection ∇ induces the submanifold Levi-Civita connection D,

$$D_i V^j = \Pi^j_b \nabla_i V^b \tag{3.3.4}$$

where $V^b = \Pi^b_k V^k$; the *second fundamental form* $I\!I_{ij}{}^c$ is then defined by the *Gauss formula*

$$\nabla_i V^c = \Pi^c_j D_i V^j + I\!I_{ij}{}^c V^j. \tag{3.3.5}$$

If N is any section of $N\Sigma[w]$ for any weight w then contracting both sides of (3.3.5) with N_c (and using that $N_c V^c = 0$) yields

$$N_c I\!I_{ij}{}^c = -\Pi^c_j \nabla_i N_c. \tag{3.3.6}$$

Remark 3.3.1. If f is a defining function for a hypersurface containing Σ in M then one may take $N_c = \nabla_c f$ in (3.3.6) to obtain

$$N_c I\!I_{ij}{}^c = -\Pi^a_i \Pi^b_j \nabla_a \nabla_b f. \tag{3.3.7}$$

This is one way to see that $I\!I_{ij}{}^c = I\!I_{(ij)}{}^c$, since torsion freeness implies $\nabla_{[a}\nabla_{b]} f = 0$. ∎

Of course in the hypersurface case we may trivialise the (weighted) normal bundle using a unit normal field so that the second fundamental form becomes a (weighted) bilinear form (§§3.1.5).

From (3.3.6) it follows that under the conformal rescaling of g to $\hat{g} = \Omega^2 g$ the second fundamental form transforms according to

$$\hat{I\!I}_{ij}{}^c = I\!I_{ij}{}^c - \boldsymbol{g}_{ij} \Upsilon^d \mathrm{N}^c_d \tag{3.3.8}$$

where $\Upsilon_a = \nabla_a \log \Omega$. As before we see that $\mathring{I\!I}_{ij}{}^c = I\!I_{(ij)_0}{}^c$ is conformally invariant, and the *mean curvature vector* $H^c = \frac{1}{m} \boldsymbol{g}^{ij} I\!I_{ij}{}^c$ transforms according to

$$\hat{H}^a = H^a - \Upsilon^b \mathrm{N}^a_b. \tag{3.3.9}$$

3.3.1.3 The Gauss-Codazzi-Ricci Equations

Fixing a metric $g \in \boldsymbol{c}$ we have the Gauss, Codazzi, and Ricci equations along Σ (see, e.g., [115]). These may be written, respectively, as

$$R_{ijkl} = r_{ijkl} + 2\boldsymbol{g}_{cd} I\!I_{l[i}{}^{c} I\!I_{j]k}{}^{d}, \tag{3.3.10}$$

$$R_{ij}{}^{c}{}_{k} \mathrm{N}_{c}^{d} = 2D_{[i} I\!I_{j]k}{}^{d} \tag{3.3.11}$$

where the submanifold Levi-Civita connection D is coupled with the normal Levi-Civita connection $\nabla^{\perp}$, and

$$R_{ij}{}^{a}{}_{b} \mathrm{N}_{a}^{c} \mathrm{N}_{d}^{b} = R^{\perp}_{ij}{}^{c}{}_{d} + 2\boldsymbol{g}^{kl} I\!I_{l[i}{}^{c} I\!I_{j]kd} \tag{3.3.12}$$

where $R_{ij}{}^{a}{}_{b} = \Pi_{i}^{c}\Pi_{j}^{d} R_{cd}{}^{a}{}_{b}$.

3.3.1.4 Minimal Scales

Here we generalise Lemma 3.1.3 to higher codimension. Note that by our definition the mean curvature vector has conformal weight -2, whereas the mean curvature covector has weight 0.

Lemma 3.3.2. *Given any metric $g_\Sigma \in \boldsymbol{c}_\Sigma$ there exists a metric $g \in \boldsymbol{c}$ extending g_Σ (i.e. such that $g_\Sigma = \iota^* g$) for which the mean curvature vector of Σ vanishes.*

Proof. By a partition of unity argument it suffices to establish the lemma for a neighbourhood in M of any point in Σ. Let $(x^1, \ldots, x^n)$ be local slice coordinates centered at any given point in Σ. Then Σ is locally defined by the equations $x^{m+1} = \cdots = x^n = 0$. Given $g_\Sigma \in \boldsymbol{c}_\Sigma$ let $g \in \boldsymbol{c}$ be any metric with $\iota^* g = g_\Sigma$. Let $H_a = \boldsymbol{g}_{ab} H^b$ denote the mean curvature covector of Σ with respect to g. On the $x^{m+1} = \cdots = x^n = 0$ slice of the coordinate chart the mean curvature covector may be written as $H_{\mathrm{a}} dx^{\mathrm{a}}$, using Einstein summation convention with the index a running from 1 to n. Of course $H_{\mathrm{a}} = 0$ for $\mathrm{a} = 1 \ldots m$ (H_a being normal). The functions H_{a} depend only on the first m coordinates, and we may extend H_a to the whole coordinate patch by extending its components H_{a} to be independent of $x^{m+1}, \ldots, x^n$. Now let $\hat{g} = e^{2\Upsilon} g$ with $\Upsilon = H_{\mathrm{a}} x^{\mathrm{a}}$. Then $\mathrm{d}\Upsilon = H_{\mathrm{a}} dx^{\mathrm{a}}$ along Σ since $x^{\mathrm{a}}|_\Sigma \equiv 0$ for $\mathrm{a} = m+1, \ldots, n$ and $H_{\mathrm{a}} \equiv 0$ for $\mathrm{a} = 1, \ldots, m$. We rewrite this as $\nabla_a \Upsilon = H_a$ along Σ. From (3.3.9) with $\Upsilon^b = \boldsymbol{g}^{ab}\nabla_a \Upsilon = H^b$ we therefore have $\hat{H}^a = H^a - H^b \mathrm{N}_b^a = 0$. □

Definition 3.3.3. Extending Definition 3.1.7 we refer to an ambient scale σ (or the corresponding metric $g \in \boldsymbol{c}$) for which the mean curvature vector identically vanishes as a *minimal scale.*

Note that if g and $\hat{g} = \Omega^2 g$ are both minimal scales then by (3.3.9) $\Upsilon_a = \nabla_a \log \Omega$ satisfies

$$\Upsilon_a N^a_b = 0. \tag{3.3.13}$$

3.3.1.5 The Natural Möbius Structure on Conformally Embedded Surfaces

We define the *(normal) induced Möbius structure* precisely as in Definition 3.1.5 from the $n = 3$ case; the calculation to show that this is well-defined is formally the same as (3.1.16) in §§3.1.8. Generalising Proposition 3.1.7 we have:

Proposition 3.3.4. *If $g_\Sigma \in \boldsymbol{c}_\Sigma$ and $g \in \boldsymbol{c}$ is any metric extending g_Σ then the Rho-tensor of g_Σ is given by*

$$p_{ij} = P_{(ij)_0} + H_c \mathring{I\!I}_{ij}{}^c + \frac{1}{2} K \boldsymbol{g}_{ij} \tag{3.3.14}$$

where P_{ij} is the projection $\Pi^a_i \Pi^b_j P_{ab}$ of the ambient Schouten tensor.

Proof. By Lemma 3.3.2 to verify this one simply needs to show that the right hand side of (3.3.14) is unchanged under conformal rescalings of g to $e^{2\Upsilon} g$ with $\Upsilon|_\Sigma \equiv 0$. This is a straightforward calculation using (2.4.11), (3.3.9) and (3.3.6) with $N_c = H_c$. □

Remark 3.3.5. As before, our definition of the normal induced Möbius structure is consistent with the notion of 'induced conformal Möbius structure' in [23]. The 'induced Möbius structure' of [23] corresponds to the assignment of the 'Rho-tensor'

$$\check{p}_{ij} = \Pi^a_i \Pi^b_j P_{ab} + H_c \mathring{I\!I}_{ij}{}^c + \frac{1}{2} H_c H^c \boldsymbol{g}_{ij} \tag{3.3.15}$$

to each metric $g_\Sigma \in \boldsymbol{c}_\Sigma$, calculated with respect to any metric $g \in \boldsymbol{c}$ extending g_Σ; the 'Rho-tensor' $\check{p}_{ij}$ differs from p_{ij} only by trace (cf. Remarks 2.6.1 and 3.1.6). ▮

As before we may think of the (normal) induced Möbius structure as being intrinsic, so that the conformal embedding $\iota : \Sigma \hookrightarrow M$ becomes in a natural sense a *Möbius conformal embedding*. We may therefore refer to p_{ij} as the *intrinsic Rho-tensor.*

3.3.2 Submanifolds and Tractors

3.3.2.1 The Normal Tractor Bundle

Definition 3.3.6. We define the normal tractor bundle $\mathcal{N}$ of Σ to be the subbundle of $\mathcal{T}M|_\Sigma$ which is the image of $N\Sigma[-1]$ under the map $N\Sigma[-1] \to \mathcal{T}M|_\Sigma$ given by

$$N^a \mapsto N^A \stackrel{g}{=} \begin{pmatrix} 0 \\ N^a \\ H_a N^a \end{pmatrix} \tag{3.3.16}$$

where g is any metric in $\boldsymbol{c}$.

The map (3.3.16) is well-defined by (3.3.9). Note that if $g \in \boldsymbol{c}$ is a minimal scale then (3.3.16) becomes

$$N^a \mapsto N^A \stackrel{g}{=} \begin{pmatrix} 0 \\ N^a \\ 0 \end{pmatrix} \tag{3.3.17}$$

and by (3.3.13) this gives rise to a well-defined (i.e. g-independent) vector bundle isomorphism $N\Sigma[-1] \cong \mathcal{N}$. We alternatively denote $\mathcal{N}$ by $\mathcal{N}^A$. We denote the orthogonal projection $\mathcal{E}^B|_\Sigma \to \mathcal{N}^A$ by N^A_B and the complementary orthogonal projection by

$$\Pi^A_B = \delta^A_B - \mathrm{N}^A_B. \tag{3.3.18}$$

We write $\nabla^{\mathcal{N}}$ for the *normal tractor connection* on $N\Sigma$ (or $N\Sigma[w]$); on a normal tractor field N^A this is defined by

$$\nabla^{\mathcal{N}}_i N^A = \mathrm{N}^B_C \nabla_i N^C \tag{3.3.19}$$

where ∇_i denotes the pullback $\iota^*\nabla$ of the ambient tractor connection. The curvature $\kappa^{\mathcal{N}}$ of the normal tractor connection is defined by the Ricci-type identity

$$\nabla^{\mathcal{N}}_{[i} \nabla^{\mathcal{N}}_{j]} N^C = \kappa^{\mathcal{N}}_{ij}{}^C{}_D N^D \tag{3.3.20}$$

for any section N^C of $\mathcal{N}^C$ (where $\nabla^{\mathcal{N}}_i$ is coupled with any torsion free affine connection on Σ).

Remark 3.3.7. In the case where M is the conformal n-sphere $\mathbb{S}^n$ the normal tractor bundle $\mathcal{N}$ may be interpreted in terms of the classical notion of a *central sphere congruence*

[18, 138] (cf. [22]). Choosing a metric $g \in \boldsymbol{c}$ at any point $x \in \Sigma^m$ there is a unique totally umbilic m-sphere $\mathbb{S}_x$ in $\mathbb{S}^n$ which is tangent to Σ at x and whose mean curvature vector agrees with that of Σ at x. By (3.3.9) the sphere $\mathbb{S}_x$ does not depend on the choice of $g \in \boldsymbol{c}$. Realising the conformal n-sphere as the projectivisation $\mathbb{P}\mathcal{C}$ of the null cone $\mathcal{C}$ in $\mathbb{R}^{n+1,1}$ (§ 2.5) the m-sphere $\mathbb{S}_x \subset \mathbb{S}^n$ determines a subcone of $\mathcal{C}$ which is the intersection of $\mathcal{C}$ with a unique $(m+2)$-dimensional subspace V_x in $\mathbb{R}^{n+1,1}$. The orthogonal complement $\mathcal{N}_x^\perp$ of $\mathcal{N}_x$ is an $(m+2)$-dimensional subspace in $\mathcal{T}_x\mathbb{S}^n$. Identifying $\mathcal{T}_x\mathbb{S}^n$ with $\mathbb{R}^{n+1,1}$ one obtains from $\mathcal{N}_x^\perp \cap \mathcal{C}$ an m-sphere $\mathbb{S}'_x \subset \mathbb{S}^n$. The fiber of the normal tractor bundle of $\mathbb{S}'_x$ at x is clearly $\mathcal{N}_x$, in other words $\mathbb{S}'_x$ has matching tangent space and mean curvature vector with Σ at x (for any $g \in \boldsymbol{c}$). Thus $\mathbb{S}'_x = \mathbb{S}_x$ and $\mathcal{N}_x^\perp = V_x$. ▮

3.3.2.2 Relating Tractors

Using Lemma 3.3.2 we now generalise Theorem 3.2.2. (The following theorem alternatively follows from [23], Sections 9.3 and 11.3.)

Theorem 3.3.8. *The submanifold standard tractor bundle $\mathcal{T}\Sigma$ is canonically isomorphic to the orthogonal complement $\mathcal{N}^\perp$ of the normal tractor bundle by a metric and filtration preserving bundle isomorphism. The isomorphism $\mathcal{T}\iota : \mathcal{T}\Sigma \to \mathcal{N}^\perp$ is given explicitly by*

$$v^I \stackrel{g_\Sigma}{=} \begin{pmatrix} \sigma \\ \mu^i \\ \rho \end{pmatrix} \xmapsto{\Pi_I^A} v^A \stackrel{g}{=} \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} \tag{3.3.21}$$

where $\mu^a = \Pi_i^a \mu^i$ and g is any minimal scale extending g_Σ.

Proof. The proof is as in the hypersurface case, with (3.3.13) replacing (3.1.14). □

As in the hypersurface case we refer to $\mathcal{T}\iota$ as the *standard tractor map.*

Remark 3.3.9. Relaxing the condition that g be minimal the map (3.3.21) becomes

$$v^I \stackrel{g_\Sigma}{=} \begin{pmatrix} \sigma \\ \mu^i \\ \rho \end{pmatrix} \xmapsto{\Pi_I^A} v^A \stackrel{g}{=} \begin{pmatrix} \sigma \\ \mu^a - H^a\sigma \\ \rho - \frac{1}{2}H_a H^a \sigma \end{pmatrix}. \tag{3.3.22}$$

This can be easily checked using (3.3.9).

3.3.2.3 Relating Tractor Connections on $\mathcal{T}\Sigma$

As in §§3.2.3 we define the connection $\check{\nabla}$ on $\mathcal{T}\Sigma$ by

$$\check{\nabla}_i v^J = \Pi^J_B \nabla_i \left(\Pi^B_K v^K \right) \tag{3.3.23}$$

for any submanifold standard tractor field v^J. Computing with respect to a minimal scale $g \in \boldsymbol{c}$ (and $g_\Sigma = \iota^* g$) we obtain the explicit expression (3.2.10) for $\check{\nabla}$ exactly as in the hypersurface case. Thus if D is the submanifold intrinsic tractor connection then

$$\check{\nabla}_i v^J = D_i v^J + \mathsf{S}_i{}^J{}_K v^K \tag{3.3.24}$$

where

$$\mathsf{S}_{iJK} = 2(P_{ij} - p_{ij}) Z^j_{[J} X_{K]} \tag{3.3.25}$$

with respect to $g \in \boldsymbol{c}$ minimal and $g_\Sigma = \iota^* g$. (Here X_K denotes the intrinsic canonical tractor and Z^j_J the Z-splitting tractor of g_Σ). Since S_{iJK} and $2Z^j_{[J}X_{K]}$ are manifestly conformally invariant we see that

$$\mathcal{F}_{ij} = P_{ij} - p_{ij} \tag{3.3.26}$$

does not depend on the choice of pair $(g, \iota^* g)$ with $g \in \boldsymbol{c}$ minimal. As in the hypersurface case we term the invariant $\mathcal{F}_{ij}$ the *Fialkow tensor.*

3.3.2.4 Computing the Fialkow Tensor

Here we relate the Fialkow tensor to the ambient Weyl curvature and the trace free part of the second fundamental form, as in §§§3.2.3.2.

Proposition 3.3.10. *In the case $m \geq 3$ the Fialkow tensor is given by*

$$\mathcal{F}_{ij} = \frac{1}{m-2} \left(W_{icjd} \mathrm{N}^{cd} + \frac{W_{acbd} \mathrm{N}^{ab} \mathrm{N}^{cd}}{2(m-1)} \boldsymbol{g}_{ij} + \mathring{I\!I}_i{}^{kc} \mathring{I\!I}_{jkc} - \frac{\mathring{I\!I}^{klc} \mathring{I\!I}_{klc}}{2(m-1)} \boldsymbol{g}_{ij} \right) \tag{3.3.27}$$

where W_{icjd} denotes the projection $\Pi^a_i \Pi^b_j W_{acbd}$ of the ambient Weyl tensor and $\mathrm{N}^{ab} = \mathrm{N}^a_c \boldsymbol{g}^{cb}$. In the $m = 2$ case

$$\mathcal{F}_{ij} = \frac{1}{4} |\mathring{I\!I}|^2 \boldsymbol{g}_{ij} \tag{3.3.28}$$

where $|\mathring{I\!I}|^2 = \mathring{I\!I}_{ijc} \mathring{I\!I}^{ijc}$.

Proof. We give the proof for the $m \geq 3$ case. The $m = 2$ case is a straightforward generalisation of the argument detailed in §§§3.2.3.2.

For simplicity we work with a minimal scale $g \in \boldsymbol{c}$ and $g_\Sigma = \iota^* g$. Thus $I\!I = \mathring{I\!I}$. Applying the submanifold and ambient Ricci decompositions (2.4.5) in the Gauss equation we obtain

$$W_{ijkl} + P_{ik}\boldsymbol{g}_{jl} - P_{jk}\boldsymbol{g}_{il} - P_{il}\boldsymbol{g}_{jk} + P_{jl}\boldsymbol{g}_{ik} = w_{ijkl} + p_{ik}\boldsymbol{g}_{jl} - p_{jk}\boldsymbol{g}_{il} - p_{il}\boldsymbol{g}_{jk} + p_{jl}\boldsymbol{g}_{ik} + \boldsymbol{g}_{cd} I\!I_{li}{}^c I\!I_{jk}{}^d - \boldsymbol{g}_{cd} I\!I_{lj}{}^c I\!I_{ik}{}^d$$

where W_{ijkl} denotes the full projection $\Pi_i^a \Pi_j^b \Pi_k^c \Pi_l^d W_{abcd}$ of the ambient Weyl curvature, $P_{ij} = \Pi_i^a \Pi_j^b P_{ab}$, and w_{ijkl} denotes the submanifold intrinsic Weyl tensor. Applying the map $T_{ijkl} \mapsto \frac{1}{m-2}\left(T_{ikj}{}^k - \frac{T_{kl}{}^{kl}}{2(m-1)}\boldsymbol{g}_{ij}\right)$ on both sides of the above display we get

$$-\tfrac{1}{m-2}\left(W_{icjd}\mathrm{N}^{cd} + \tfrac{W_{acbd}\mathrm{N}^{ab}\mathrm{N}^{cd}}{2(m-1)}\boldsymbol{g}_{ij}\right) + P_{ij} = p_{ij} + \tfrac{1}{m-2}\left(\mathring{I\!I}_i{}^{kc}\mathring{I\!I}_{jkc} - \tfrac{\mathring{I\!I}^{klc}\mathring{I\!I}_{klc}}{2(m-1)}\boldsymbol{g}_{ij}\right)$$

noting that $\boldsymbol{g}^{kl} W_{ikjl} = \boldsymbol{g}^{kl}\Pi_k^c \Pi_l^d W_{icjd} = -N^{cd} W_{icjd}$ since W_{abcd} is trace free, and similarly $W_{kl}{}^{kl} = W_{acbd}\mathrm{N}^{ab}\mathrm{N}^{cd}$. The result then follows from (3.3.26). □

Remark 3.3.11. As in the hypersurface case, if one repeats the calculation of (3.2.10) without the restriction that $g \in \boldsymbol{c}$ be a minimal scale, one obtains the more general formula

$$\mathcal{F}_{ij} = P_{ij} - p_{ij} + H_c \mathring{I\!I}_{ij}{}^c + \frac{1}{2} H_c H^c \boldsymbol{g}_{ij} \tag{3.3.29}$$

for the Fialkow tensor in terms of g and $g_\Sigma = \iota^* g$ (which holds for $m \geq 2$). ▮

3.3.2.5 The Tractor Gauss Formula

We define the *tractor second fundamental form* $\mathbb{L}$ by the Gauss-type formula

$$\nabla_i v^B = \Pi_J^B \check{\nabla}_i v^J + \mathbb{L}_{iK}{}^B v^K \tag{3.3.30}$$

for all sections v^J of $\mathcal{E}^J$. The 1-form $\mathbb{L}$ takes values in $\mathcal{T}^*\Sigma \otimes \mathcal{N}$ where $\mathcal{T}^*\Sigma = (\mathcal{T}\Sigma)^*$. Using (3.3.24) we obtain the *tractor Gauss formula*

$$\nabla_i v^B = \Pi_J^B \left(D_i v^J + \mathsf{S}_i{}^J{}_K v^K\right) + \mathbb{L}_{iK}{}^B v^K. \tag{3.3.31}$$

3.3.2.6 Computing the Tractor Second Fundamental Form

By contracting (3.3.30) on both sides with N_B where N^A is any section of the normal tractor bundle we see that

$$N_B \mathbb{L}_{iJ}{}^B = -\Pi_J^B \nabla_i N_B. \tag{3.3.32}$$

This formula can be used to give an explicit expression for $\mathbb{L}_{iJ}{}^C$ in terms of a choice of ambient metric. For simplicity we compute with respect to a minimal ambient scale.

Proposition 3.3.12. *Let $g \in \boldsymbol{c}$ be a minimal scale and $g_\Sigma = \iota^* g$. Computing with respect to g and g_Σ we have*

$$\mathbb{L}_{iJ}{}^C = \mathring{I\!I}_{ij}{}^c Z_J^j Z_c^C + \frac{1}{m-1}\left(D^j \mathring{I\!I}_{ij}{}^c - \Pi_j^b W_{abde} \mathrm{N}^{ae} \mathrm{N}^{cd}\right) X_J Z_c^C \tag{3.3.33}$$

where Z_c^C and Z_J^i are the respective Z-splitting tractors of g and g_Σ, X_J is the submanifold intrinsic canonical tractor, and $\mathrm{N}^{ab} = \boldsymbol{g}^{ac} \mathrm{N}_c^b$.

Proof. Let N^a be a weight -1 normal vector field and let $N^A = N^a Z_a^A$. Then by (2.7.7) (or by (2.7.11)) one has $\nabla_i N^B = (\nabla_i N^b) Z_b^B - \Pi_i^a P_{ac} N^c X^B$ so that

$$\begin{aligned}\Pi_J^C \nabla_i N_C &= \Pi_j^c (\nabla_i N_c) Z_J^j - \Pi_i^a P_{ac} N^c X_J \\ &= -N_c I\!I_{ij}{}^c Z_J^j - \Pi_i^a P_{ac} N^c X_J\end{aligned} \tag{3.3.34}$$

using (3.3.6). Note that $I\!I = \mathring{I\!I}$ by minimality. Since (3.3.34) and (3.3.32) hold for all such normal fields (noting that $\mathbb{L}_{iJ}{}^C \Pi_C^D = 0$ and $I\!I_{ij}{}^c \Pi_d^c = 0$) we have

$$\mathbb{L}_{iJ}{}^C = \mathring{I\!I}_{ij}{}^c Z_J^j Z_c^C + \Pi_i^a P_{ad} \mathrm{N}^{cd} X_J Z_c^C. \tag{3.3.35}$$

Now by substituting the ambient Ricci decomposition (2.4.5) into the Codazzi equation (3.3.11) we obtain

$$W_{ij}{}^c{}_k \mathrm{N}_c^d + 2P_{[i}{}^c \boldsymbol{g}_{j]k} \mathrm{N}_c^d = 2D_{[i} I\!I_{j]k}{}^d$$

where $W_{ij}{}^c{}_k = \Pi_i^a \Pi_j^b \Pi_k^d W_{ab}{}^c{}_d$, $P_{ic} = \Pi_i^a P_{ac}$, and the Levi-Civita connection D of g_Σ is coupled with the normal Levi-Civita connection. Contracting with g^{ik} we obtain

$$W_{ij}{}^{ci} \mathrm{N}_c^d - (m-1) P_j{}^c \mathrm{N}_c^d = D^i I\!I_{ij}{}^d = D^i \mathring{I\!I}_{ij}{}^d. \tag{3.3.36}$$

Noting that $W_{ij}{}^{ci} = -\Pi_j^b W_{ab}{}^c{}_e \mathrm{N}^{ae}$ since the W_{abcd} the result follows from (3.3.35) and (3.3.36). □

3.3.2.7 Decomposing the Ambient Tractor Connection

Along Σ we may decompose the ambient standard tractor bundle $\mathcal{T}M$ as

$$\mathcal{T}M|_{\Sigma} = \mathcal{T}\Sigma \oplus \mathcal{N}. \tag{3.3.37}$$

If $v \in \Gamma(\mathcal{T}M|_{\Sigma})$ is given by $(v^{\top}, v^{\perp})$ with respect to this decomposition then by (3.3.31), (3.3.32), and (3.3.19) we have

$$\nabla_X v = \begin{pmatrix} D_X + \mathsf{S}(X) & -\mathbb{L}(X)^T \\ \mathbb{L}(X) & \nabla^{\mathcal{N}}_X \end{pmatrix} \begin{pmatrix} v^{\top} \\ v^{\perp} \end{pmatrix} \tag{3.3.38}$$

for any $X \in \mathfrak{X}(\Sigma)$, where $\mathbb{L}(X)^T$ is the transpose of $\mathbb{L}(X)$ with respect to the ambient tractor metric. We therefore write

$$\iota^*\nabla = \begin{pmatrix} D + \mathsf{S} & -\mathbb{L}^T \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \quad \text{on} \quad \begin{matrix} \mathcal{T}\Sigma \\ \oplus \\ \mathcal{N} \end{matrix} . \tag{3.3.39}$$

Remark 3.3.13. In [23] Burstall and Calderbank define a 'Möbius reduction' to be a rank $(m+2)$ subbundle V of $\mathcal{T}M|_{\Sigma}$ containing the rank $m+1$ subbundle spanned by the canonical tractor X^A and its covariant derivatives in submanifold tangential directions (with respect to the tractor connection coupled with the Levi-Civita connection of some, equivalently any, metric $g \in \boldsymbol{c}$). One then decomposes the ambient tractor connection along Σ as (using notation similar to the above)

$$\iota^*\nabla = \begin{pmatrix} \nabla^V & -(\mathbb{L}^V)^T \\ \mathbb{L}^V & \nabla^{V^{\perp}} \end{pmatrix} \quad \text{on} \quad \begin{matrix} V \\ \oplus \\ V^{\perp} \end{matrix} .$$

The definition of 'Möbius reduction' implies that $\mathbb{L}^V_{iJ}CX^J = 0$ and $\mathbb{L}^V_{iJ}CX_C = 0$, so that there is a well-defined projection $II^{V\,c}_{ij} = \mathbb{L}^{V\,C}_{iJ} Z^J_j Z^C_c$ of $\mathbb{L}^{V\,C}_{iJ}$. Burstall and Calderbank then define the unique 'canonical Möbius reduction' V_{Σ} by imposing an algebraic normalisation condition on

$$\begin{pmatrix} 0 & -(\mathbb{L}^V)^T \\ \mathbb{L}^V & 0 \end{pmatrix}$$

which they denote $\mathcal{N}^V$ (see Section 9.3 of [23]) similar to the algebraic normalisation condition imposed on the curvature of the normal Cartan/tractor connection [33, 27].

This algebraic normalisation condition amounts to the requirement that $\boldsymbol{g}^{ij} I\!I^{V\,c}_{ij} = 0$. Since by Proposition 3.3.12 the tractor second fundamental form has invariant projection $\mathring{I\!I}_{ij}{}^{c} = \mathbb{L}_{iJ}{}^{C} Z^{J}_{j} Z^{C}_{c}$ the ‘canonical Möbius reduction’ V_Σ is the same as the orthogonal complement $\mathcal{N}^\perp$ of the normal tractor bundle. ∎

3.3.2.8 Tractor Gauss-Codazzi-Ricci Equations

By writing the curvature $\iota^*\kappa$ of $\iota^*\nabla$ as

$$\iota^*\kappa = \begin{pmatrix} D + \mathsf{S} & -\mathbb{L}^T \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \wedge \begin{pmatrix} D + \mathsf{S} & -\mathbb{L}^T \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \tag{3.3.40}$$

one may easily obtain conformal tractor analogues of the Riemannian Gauss, Codazzi, and Ricci equations (cf. [23], Equations (9.13a) and (9.13b)). By evaluating (3.3.40) on $(v^\top, 0)$ one obtains (from the $\mathcal{T}\Sigma$ component) the *tractor Gauss equation*

$$\kappa_{ij}{}^{K}{}_{L} = \kappa^{\Sigma}{}_{ij}{}^{K}{}_{L} + 2D_{[i}\mathsf{S}_{j]}{}^{K}{}_{L} + 2\mathsf{S}_{[i|}{}^{K}{}_{L'}\mathsf{S}_{|j]}{}^{L'}{}_{L} - 2\mathbb{L}_{[i}{}^{KC}\mathbb{L}_{j]LC} \tag{3.3.41}$$

where $\kappa_{ij}{}^{K}{}_{L} = \Pi^a_i \Pi^b_j \kappa_{ab}{}^{C}{}_{D} \Pi^K_C \Pi^D_L$. From the $\mathcal{N}$ component one obtains the *tractor Codazzi equation*

$$\kappa_{ij}{}^{K}{}_{D} \mathrm{N}^D_E = -2D_{[i}\mathbb{L}_{j]}{}^{K}{}_{E} - 2\mathsf{S}_{[i|}{}^{K}{}_{L}\mathbb{L}_{|j]}{}^{L}{}_{E} \tag{3.3.42}$$

(from the $\mathcal{N}$ component) where $\kappa_{ijKD} = \Pi^a_i \Pi^b_j \Pi^C_K \kappa_{abCD}$ and D is coupled with the normal tractor connection (and also with any torsion free affine connection on Σ). Applying (3.3.40) to $(0, v^\perp)$ and evaluating the $\mathcal{N}$ component gives the *tractor Ricci equation*

$$\kappa_{ij}{}^{A}{}_{B} \mathrm{N}^C_A \mathrm{N}^B_D = \kappa^{\mathcal{N}}{}_{ij}{}^{C}{}_{D} + 2\mathbb{L}_{[i}{}^{KC}\mathbb{L}_{j]KD} \tag{3.3.43}$$

where $\kappa_{ij}{}^{C}{}_{D} = \Pi^a_i \Pi^b_j \kappa_{ab}{}^{C}{}_{D}$. Evaluating the $\mathcal{T}\Sigma$ component gives the tractor Codazzi equation again.

3.3.2.9 Invariants and Invariant Operators

One may extend the submanifold tractor-D operator D_I to act on sections of $\mathcal{E}^\Phi[w] \otimes \mathcal{E}^{\tilde{\Phi}}$ where $\mathcal{E}^\Phi[w]$ is any weighted submanifold tractor bundle and $\mathcal{E}^{\tilde{\Phi}}$ is any ambient tractor bundle by using the ambient tractor connection (precisely as in the hypersurface case §§3.2.6). Working locally if necessary we assume the normal bundle of Σ (equivalently the normal tractor bundle) to be oriented. We define the *normal tractor form* $N_{A_1 \cdots A_d}$ where $d = n - m$ to be the unique section of $\mathcal{E}_{A_1 \cdots A_d}|_\Sigma$ satisfying

$$N_{A_1 \cdots A_d} v^{A_1} = 0 \text{ for all } v \in \mathcal{N}^\perp \quad \text{and} \quad N_{A_1 \cdots A_d} N^{A_1 \cdots A_d} = 1 \tag{3.3.44}$$

which is compatible with the orientation of the normal bundle. The normal tractor form may be expressed in terms of a local orthonormal frame $N_1^A, \ldots, N_d^A$ for the normal tractor bundle as $N^{A_1 \cdots A_d} = N_1^{[A_1} \cdots N_d^{A_d]}$. The normal tractor form therefore encodes the normal tractor bundle in terms of a section of $\mathcal{E}_{A_1 \cdots A_d}|_\Sigma$.

It is clear that proceeding along the lines of the hypersurface case (see §§3.2.6 and the references therein) replacing N^A with $N^{A_1 \cdots A_d}$ one obtains a straightforward and very general construction of submanifold conformal invariants. Specific applications of the submanifold tractor calculus are being developed in current joint work with A. R. Gover. In particular we are seeking to develop a 'holographic' approach to conformal submanifold theory extending recent developments in the hypersurface case [84, 70]. This is partly motivated by [89, 86] which gives an alternative approach to the study of (low order) conformal invariants of submanifolds using 'Poincaré-Einstein holography'.

4 Introduction to CR Embedded Submanifolds of CR Manifolds

Hypersurface type CR geometry is motivated by the biholomorphic equivalence problem for complex domains, and is rooted in the result of Poincaré that the analogue of the Riemann mapping theorem fails for domains of complex dimension greater than one [120]. On the side of geometry key pioneering work was developed by Cartan, Tanaka, and Chern-Moser, in which it was seen that the structure is invariantly captured by a prolonged system now known as a Cartan connection [37, 41, 129]. The fundamental role of CR geometry in analysis was significantly strengthened by the result of Fefferman that any biholomorphic map between smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^{n+1}$ extends smoothly to the boundary, and so induces a CR diffeomorphism between the boundaries [61]; so Poincaré's result may be recovered by a simple counting of invariants argument (that was in fact proposed in [120]). This brought to the fore the role of CR invariants as tools for distinguishing domains. Hypersurface type CR geometry is an important example in a class of structures known as parabolic geometries that also includes conformal geometry, projective differential geometry, and many other structures. Seeking to determine the asymptotic expansion of the Bergman kernel, Fefferman initiated a programme for the explicit construction of CR, and more widely parabolic, invariants [63]. There has subsequently been much interest and progress on this [8, 92, 75].

The study of CR embeddings and immersions (in CR manifolds) is also closely connected with the study of holomorphic mappings between domains. Although open questions remain about when proper holomorphic mappings between domains in $\mathbb{C}^{m+1}$ and $\mathbb{C}^{n+1}$ extend smoothly [14, 143], if a holomorphic map between smoothly bounded domains does extend in this way then it induces a CR map between the boundaries. So again CR invariants of the boundaries play a fundamental role. The Chern-Moser moving frames approach to the CR Cartan connection has been effectively applied to the study of CR embeddings and immersions in the important work of Webster [144] on CR rigidity for real codimension two embeddings. This theme is significantly extended in the article [54] of Ebenfelt, Huang and Zaitsev where rigidity is established when the codimension is not too large. These works have strong applications to the study of proper holomorphic maps between balls and to the study of Milnor links of isolated singularities of analytic varieties [143, 54]. The Chern-Moser approach has also been applied in related work generalising the Schwarz reflection principle to several complex variables, where invariant nondegeneracy conditions on CR maps play a key role [57, 102].

Despite the strong specific results mentioned, and geometric studies by several authors [47, 48, 49, 111, 113, 133], a significant gap has remained in the general theory for CR embeddings and immersions. A basic general theory should enable the straightforward construction of local CR invariants, but in fact to this point very few invariants are known. In particular using existing approaches there has been no scope for a general theory of invariant construction, as the first step in a Fefferman-type invariants programme cf. [63]. Closely related is the need to construct CR invariant differential operators required for geometric analysis. Again no general theory for their construction has been previously advanced. The aim of this article is to close this gap. We develop a general CR invariant treatment that on the one hand is conceptual and on the other provides a practical and constructive approach to treating the problems mentioned. The final package may be viewed as, in some sense, an analogue of the usual Ricci calculus approach to Riemannian submanifold theory, which is in part based around the Gauss formula. Our hope is that this may be easily used by analysts or geometers not already strongly familiar with CR geometry; for this reason we have attempted to make the treatment largely self contained. The theory and tools developed here may also be viewed as providing a template for the general problem of treating parabolic submanifolds in parabolic geometries. This is reasonably well understood in the conformal setting [7, 23, 78, 90, 128, 141] but little is known in the general case. The CR case treated here is considerably more subtle than the conformal analogue as it involves dealing with a non-maximal parabolic.

4.1 CR Embeddings

Abstractly, a nondegenerate hypersurface-type CR manifold is a smooth manifold M^{2n+1} equipped with a contact distribution H on which there is a formally integrable complex structure $J : H \to H$. We refer to such manifolds simply as *CR manifolds*. A *CR mapping* between two CR manifolds is a smooth mapping whose tangent map restricts to a complex linear bundle map between the respective contact distributions. A *CR embedding* is a CR mapping which is also an embedding.

Typically in studying CR embeddings one works with an arbitrary choice of contact form for the contact distribution in the ambient manifold (ambient *pseudohermitian structure*). The ambient contact form then pulls back to a pseudohermitian contact form on the submanifold (assuming transversality when the ambient manifold has large signature). Associated with these contact forms are their respective Tanaka-Webster connections, and these can be used to construct pseudohermitian invariants of the embedding. The task of finding some, let alone all, pseudohermitian invariants which are in fact CR invariants (not depending on the additional choice of ambient contact form) is very difficult, unless one can find a manifestly invariant approach. We give such an approach. Our approach uses the natural invariant calculus on CR manifolds, the *CR tractor calculus*. In the CR tractor calculus the *standard tractor bundle* and *normal tractor (or Cartan) connection* play the role analogous to the (holomorphic) tangent bundle and Tanaka-Webster connection in pseudohermitian geometry.

4.2 Invariant Calculus on CR Manifolds

Due to the work of Cartan, Tanaka, and Chern-Moser we may view a CR manifold (M, H, J) as a Cartan geometry of type (G, P) with G a pseudo-special unitary group and P a parabolic subgroup of G. The *tractor bundles* are the associated vector bundles on M corresponding to representations of G, the standard tractor bundle corresponding to the standard representation. The normal Cartan connection then induces a linear connection on each tractor bundle [27]. In order to relate the CR tractor calculus to the Tanaka-Webster calculus of a choice of pseudohermitian contact form we work with the direct construction of the CR standard tractor bundle and connection given in [79]. This avoids the need to first construct the Cartan bundle.

To fully treat CR submanifolds one needs to work with *CR density line bundles*, and their Tanaka-Webster calculus. From the Cartan geometric point of view the CR density bundles $\mathcal{E}(w, w')$ on M^{2n+1} are the complex line bundles associated to one dimensional complex representations of P, and include the *canonical bundle* $\mathscr{K}$ as $\mathcal{E}(-n-2, 0)$. The bundle $\mathcal{E}(1, 0)$ is the dual of an $(n+2)^{th}$ root of $\mathscr{K}$ and

$$\mathcal{E}(w, w') = \mathcal{E}(1, 0)^{w} \otimes \overline{\mathcal{E}(1, 0)}^{w'}$$

where $w - w' \in \mathbb{Z}$ (and w, w' may be complex). Since the Tanaka-Webster connection acts on the canonical bundle it acts on all the density bundles.

4.3 Invariant Calculus on Submanifolds and Main Results

We seek to extend the CR tractor calculus to the setting of transversally CR embedded submanifolds of CR manifolds in order to deal with the problem of invariants. Our approach parallels the usual approach to Riemannian submanifold geometry; of central importance in the Riemannian theory of submanifolds is the second fundamental form.

4.3.1 Normal tractors and the tractor second fundamental form

One way to understand the Riemannian second fundamental form is in terms of the turning of normal fields (i.e. as the shape operator). To define a tractor analogue of the shape operator one needs a tractor analogue of the normal bundle for a CR embedding $\iota : \Sigma^{2m+1} \to M^{2n+1}$.

In §§6.2.1 we use a CR invariant differential splitting operator to give a CR analogue of the normal tractor of [7] associated to a weighted unit normal field in conformal submanifold geometry. It turns out that this *a priori* differential splitting gives a canonical bundle isomorphism between the Levi-orthogonal complement of $T^{1,0}\Sigma$ in $T^{1,0}M|_\Sigma$, tensored with the appropriate ambient density bundle, and a subbundle $\mathcal{N}$ of the ambient standard tractor bundle along Σ (Proposition 6.2.3). The ambient standard tractor bundle carries a parallel Hermitian metric and the *normal tractor bundle* $\mathcal{N}$ is nondegenerate since Σ and M are required to be nondegenerate, so the ambient tractor connection induces connections $\nabla^{\mathcal{N}}$ and $\nabla^{\mathcal{N}^\perp}$ on $\mathcal{N}$ and $\mathcal{N}^\perp$ respectively. We therefore obtain (§§6.5.2, see also §§6.2.1, §§6.2.5, and §§6.3.3):

Proposition 4.3.1. *The ambient standard tractor bundle $\mathcal{T}M$ splits along Σ as $\mathcal{N}^\perp \oplus \mathcal{N}$, and the ambient tractor connection ∇ splits as*

$$\iota^*\nabla = \begin{pmatrix} \nabla^{\mathcal{N}^\perp} & -\mathbb{L}^\dagger \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \qquad \textit{on} \qquad \mathcal{T}M|_\Sigma = \begin{matrix} \mathcal{N}^\perp \\ \oplus \\ \mathcal{N} \end{matrix}$$

where $\mathbb{L}^{\dagger}(X)$ *is the Hermitian adjoint of* $\mathbb{L}(X)$ *for any* $X \in \mathfrak{X}(\Sigma)$*.*

The $\mathrm{Hom}(\mathcal{N}, \mathcal{N}^{\perp})$ valued 1-form $\mathbb{L}^{\dagger}$ on Σ is the CR tractor analogue of the shape operator, and we term $\mathbb{L}$ the *CR tractor second fundamental form*. The ambient standard tractor bundle can be decomposed with respect to a choice of contact form. Here it is sensible to choose an ambient contact form whose Reeb vector field is tangent to the submanifold (called *admissible* [54]). We give the components of $\mathbb{L}$ with respect to an admissible ambient contact form in Proposition 6.3.6 (see also Proposition 6.2.15). The principal component of $\mathbb{L}$ is the *CR second fundamental form* $II_{\mu\nu}{}^{\gamma}$ of Σ in M, which appears, for example, in [54].

4.3.2 Relating submanifold and ambient densities and tractors

Another way to understand the Riemannian second fundamental form is in terms of the normal part of the ambient covariant derivative of a submanifold vector field in tangential directions. This is achieved via the Gauss formula. In the Riemannian Gauss formula a submanifold vector field is regarded as an ambient vector field along the submanifold using the pushforward of the embedding, which relies on the tangent map. In order to give a CR tractor analogue of the Gauss formula one needs to be able to pushforward submanifold tractors to give ambient tractors along the submanifold – one looks for a CR 'standard tractor map'. One might hope for a canonical isomorphism

$$\mathcal{T}\Sigma \to \mathcal{N}^{\perp}$$

between the submanifold standard tractor bundle and the orthogonal complement of the normal tractor bundle (these having the same rank). In the conformal case there is such a canonical isomorphism [22, 78, 90], however in the CR case it turns out that there is no natural 'standard tractor map' $\mathcal{T}\Sigma \to \mathcal{T}M$ in general.

The problem has to do with the necessity of relating corresponding submanifold and ambient CR density bundles. It turns out that these are not isomorphic along the submanifold, but are related by the top exterior power of the normal tractor bundle $\mathcal{N}$. Rather than seeking to identify these bundles we therefore define the ratio bundles of densities

$$\mathcal{R}(w, w') = \mathcal{E}(w, w')|_{\Sigma} \otimes \mathcal{E}_{\Sigma}(w, w')^{*}$$

where $\mathcal{E}(w, w')|_{\Sigma}$ is a bundle of ambient CR densities along Σ and $\mathcal{E}_{\Sigma}(w, w')^{*}$ is dual to the corresponding submanifold intrinsic density bundle. We obtain (in §§6.3.2, see also §§6.1.11):

Proposition 4.3.2. *Given a transversal CR embedding* $\iota : \Sigma^{2m+1} \to M^{2n+1}$ *we have a canonical isomorphism of complex line bundles*

$$\mathcal{R}(m+2,0) \cong \Lambda^d \mathcal{N}$$

where $d = n - m$. *The complex line bundles* $\mathcal{R}(w,w')$ *therefore carry a canonical CR invariant connection* $\nabla^{\mathcal{R}}$ *induced by* $\nabla^{\mathcal{N}}$.

The bundles $\mathcal{R}(w,w)$ are canonically trivial and the connection $\nabla^{\mathcal{R}}$ on these is flat. The bundles $\mathcal{R}(w,w')$ are therefore normed, and $\nabla^{\mathcal{R}}$ is a $\mathrm{U}(1)$-connection. Using the pseudohermitian Gauss and Ricci equations (§§§6.1.7.1 and §§§6.1.7.3) we calculate the curvature of $\nabla^{\mathcal{R}}$ (§§6.3.2, see also §§§6.1.11.1, and in particular Lemma 6.1.42) and see that this connection is not flat in general when $w \neq w'$. Thus rather than identifying corresponding density bundles we should keep the ratio bundles $\mathcal{R}(w,w')$ in the picture.

We are then able to show (from Theorem 6.2.6 combined with Definitions 6.2.8, 6.2.11, 6.2.13 and §§6.3.3):

Theorem 4.3.3. *Let* $\iota : \Sigma^{2m+1} \to M^{2n+1}$ *be a transversal CR embedding. Then there is a canonical, metric and filtration preserving, bundle map*

$$\mathcal{T}^{\mathcal{R}}\iota : \mathcal{T}\Sigma \to \mathcal{T}M|_{\Sigma} \otimes \mathcal{R}(1,0)$$

over ι, *which gives an isomorphism of* $\mathcal{T}\Sigma$ *with* $\mathcal{N}^{\perp} \otimes \mathcal{R}(1,0)$. *Moreover, the submanifold and ambient tractor connections are related by the tractor Gauss formula*

$$\nabla_X \iota_* u \;=\; \iota_*(D_X u + \mathsf{S}(X)u) + \mathbb{L}(X)\iota_* u$$

for all $u \in \Gamma(\mathcal{T}\Sigma)$ *and* $X \in \mathfrak{X}(\Sigma)$, *where* S *is an* $\mathrm{End}(\mathcal{T}\Sigma)$ *valued 1-form on* Σ, D *is the submanifold tractor connection,* ∇ *is the (pulled back) ambient tractor connection coupled with* $\nabla^{\mathcal{R}}$, *and the pushforward map* ι_* *is defined using using* $\mathcal{T}^{\mathcal{R}}\iota$.

By Proposition 4.3.1 the tractor Gauss formula implies

$$\nabla^{\mathcal{N}^{\perp}}_X \iota_* u \;=\; \iota_*(D_X u + \mathsf{S}(X)u)$$

for all $u \in \Gamma(\mathcal{T}\Sigma)$ and $X \in \mathfrak{X}(\Sigma)$, where $\nabla^{\mathcal{N}^\perp}$ is coupled with $\nabla^{\mathcal{R}}$. The *difference tractor* S measures the failure of the ambient tractor (or normal Cartan) connection to induce the submanifold one. The components of S with respect to an admissible ambient contact form are given in (6.2.19), (6.2.20), and (6.2.21) (in §§6.3.3 it is noted that these formulae hold in arbitrary codimension and signature). The principal component of S is the difference between ambient and submanifold pseudohermitian Schouten tensors $P_{\mu\bar{\nu}} - p_{\mu\bar{\nu}}$ for a pair of *compatible* contact forms (Definition 6.1.2); using the pseudohermitian Gauss equation (§§§6.1.7.1) one can give a manifestly invariant expression for this tensor involving the ambient Chern-Moser tensor and the CR second fundamental form (see Lemma 6.1.42 for the case $m = n - 1$).

4.3.3 Constructing invariants

In §6.4 we develop both the theoretical and practical aspects of constructing invariants of CR embeddings. We deal with the geometric part of the invariant theory problem, using the results stated above. In particular, in §§6.4.1 we demonstrate that the tractor second fundamental form $\mathbb{L}$, the difference tractor S, and the submanifold and ambient tractor (or Cartan) curvatures are the basic invariants of the CR embedding, in that they determine the higher jets of the structure (Proposition 6.4.2). By applying natural differential operators to these objects and making suitable contractions, one can start to proliferate local invariants of a CR embedding. In practice a more refined construction is useful. The algebraic problem of showing that one can make all invariants of a CR embedding, suitably polynomial in the jets of the structure, is beyond the scope of this article; despite much progress on the analogous problems for CR or conformal manifolds, these are still far from being completely solved (see, e.g., [8, 92]). We therefore turn in §§6.4.4 to considering practical constructions of invariants. In §§§6.4.4.1 and §§§6.4.4.2 we develop a richer calculus of invariants than that presented for theoretical purposes in §§6.4.1 and §§6.4.2. In §§§6.4.4.3 we illustrate this calculus with an example of an invariant section $\mathcal{I}$ of $\mathcal{E}_\Sigma(-2,-2)$ given by a manifestly invariant tractor expression (6.4.10) involving $\mathbb{L} \otimes \overline{\mathbb{L}}$; we show how to calculate $\mathcal{I}$ in terms of the pseudohermitian calculus of a pair of compatible contact forms, yielding the expression (6.4.16).

4.3.4 A CR Bonnet theorem

With the setup of Proposition 4.3.1 and Theorem 4.3.3 established it is straightforward to give CR tractor analogues of the Gauss, Codazzi and Ricci equations from Riemannian submanifold theory. These are given in §§6.5.2. Just as in the Riemannian theory, if we specialise to the ambient flat case the (tractor) Gauss, Codazzi and Ricci equations give the integrability conditions for a Bonnet theorem or fundamental theorem of embeddings. We have (Theorem 6.5.5):

Theorem 4.3.4. *Let (Σ^{2m+1}, H, J) be a signature (p, q) CR manifold and suppose we have a complex rank d vector bundle $\mathcal{N}$ on Σ equipped with a signature (p', q') Hermitian bundle metric $h^{\mathcal{N}}$ and metric connection $\nabla^{\mathcal{N}}$. Fix an $(m+2)^{th}$ root $\mathcal{R}$ of $\Lambda^d\mathcal{N}$, and let $\nabla^{\mathcal{R}}$ denote the connection induced by $\nabla^{\mathcal{N}}$. Suppose we have a $\mathcal{N}\otimes\mathcal{T}^*\Sigma\otimes\mathcal{R}$ valued 1-form $\mathbb{L}$ which annihilates the canonical tractor of Σ and an $\mathcal{A}^0\Sigma$ valued 1-form S on Σ such that the connection*

$$\nabla := \begin{pmatrix} D\otimes\nabla^{\mathcal{R}} + \mathsf{S} & -\mathbb{L}^{\dagger} \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \quad \text{on} \quad \begin{matrix} \mathcal{T}\Sigma\otimes\mathcal{R}^* \\ \oplus \\ \mathcal{N} \end{matrix}$$

is flat (where D is the submanifold tractor connection), then (locally) there exists a transversal CR embedding of Σ into the model $(p+p', q+q')$ hyperquadric $\mathcal{H}$, unique up to automorphisms of the target, realising the specified extrinsic data as the induced data.

4.4 Geometric Intuition

In the case where $M = \mathbb{S}^{2n+1}$ we can give a clear geometric interpretation of the normal tractor bundle $\mathcal{N}$ of a CR embedded submanifold, or rather of its orthogonal complement $\mathcal{N}^{\perp}$. In the conformal case a similar characterisation of the normal tractor bundle may be given via the notion of a central sphere congruence (see [22]).

One may explicitly realise the standard tractor bundle of $\mathbb{S}^{2n+1}$ by considering the sphere as the space of isotropic lines in the projectivisation of $\mathbb{C}^{n+1,1}$; if ℓ is a complex null line then a standard tractor at the point $\ell\in\mathbb{S}^{2n+1}$ is a constant vector field along ℓ in the ambient space $\mathbb{C}^{n+1,1}$. The tractor parallel transport on $\mathbb{S}^{2n+1}$ then comes from the affine structure of $\mathbb{C}^{n+1,1}$ and the standard tractor bundle is flat. Given a point x in our CR embedded submanifold $\Sigma^{2m+1}\subset\mathbb{S}^{2n+1}$ there is a unique totally chain CR subsphere $\mathbb{S}_x$ of dimension $2m+1$ which osculates Σ to first order at x. If we view $\mathbb{S}^{2n+1}$ as the

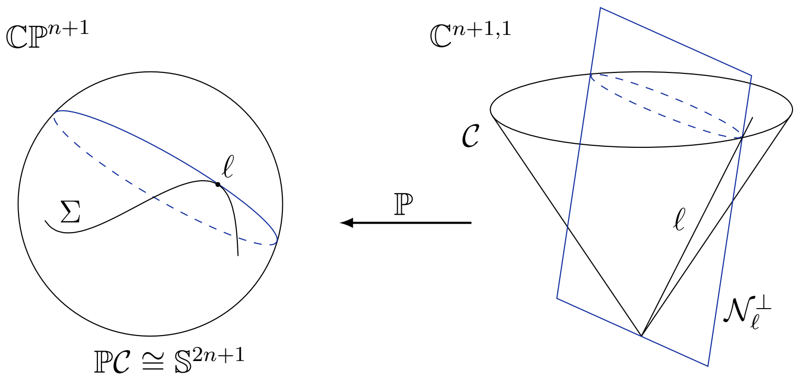

Figure 4.4.1: The orthogonal complement $\mathcal{N}^\perp$ of the normal tractor bundle when $M = \mathbb{S}^{2n+1}$.

unit sphere in $\mathbb{C}^{n+1}$ then $\mathbb{S}_x$ is the intersection of $\mathbb{S}^{2n+1}$ with the $(m+1)$-dimensional complex affine subspace of $\mathbb{C}^{n+1}$ generated by the tangent space to Σ at x. Viewing $\mathbb{S}^{2n+1}$ instead as a projective hyperquadric the sphere $\mathbb{S}_\ell$ with $x = \ell$ is the image under the projectivisation map of the intersection of the cone $\mathcal{C}$ of isotropic lines in $\mathbb{C}^{n+1,1}$ with a non-null complex $(m+2)$-dimensional subspace $\mathcal{N}_\ell^\perp$.

In this case the rank $d = n - m$ normal tractor bundle $\mathcal{N}$ may be viewed as giving a $\mathrm{Gr}(d, \mathbb{C}^{n+1,1})$ valued CR analogue of the Gauss map of an embedded Riemannian submanifold in Euclidean space.

4.5 Structure

We aim to produce a calculus of invariants for CR embeddings which is both simple and practical, and yields a machinery for constructing local CR invariants with formulae in terms the psuedohermitian (Tanaka-Webster) calculus. We thus emphasise heavily the connection between the CR tractor calculus and the pseudohermitian calculus of a fixed contact form. Although our final results have a simple interpretation in terms of tractor calculus, they are often established though explicit calculation using pseudohermitian calculus. For this reason we have devoted the first part of the article to giving a detailed exposition of the Tanaka-Webster calculus associated to a choice of pseudohermitian contact form (§5.2) and an explicit description of the CR tractor calculus in terms of this pseudohermitian calculus (§5.3). Although the results of §5.2 may largely be found elsewhere in the literature, proofs are often merely indicated; collecting these results, and establishing them by proof, provides the essential reference for verifying the CR invariance of our later constructions. These results are immediately applied in §5.3 where we present the CR tractor calculus, using the explicit description of the standard tractor bundle and normal connection given in [79]. For the purpose of invariant theory we introduce some CR analogues of parts of the conformal tractor calculus not yet developed in the CR case.

In §6.1 we discuss the pseudohermitian geometry of CR embeddings, working in particular with pairs of compatible ambient and submanifold contact forms (see Definition 6.1.2). We also discuss in this section the relationship between the submanifold and ambient CR density bundles. For simplicity we initially treat the minimal codimension strictly pseudoconvex case, generalising to nondegenerate transversal CR embeddings of arbitrary codimension between CR manifolds of any signature in §6.3.

In §6.2 we develop a manifestly CR invariant approach to studying CR submanifolds using tractor calculus. Again we restrict initially to the minimal codimension strictly pseudoconvex case, generalising in §6.3. In §6.4 we apply this calculus to the basic geometric problems of invariant theory for CR embeddings, addressing practical constructions of invariants in §§6.4.4. In §6.5 we prove a CR analogue of the Bonnet theorem (Theorem 6.5.5).

5 CR Geometry

5.1 Abstract CR Manifolds

A CR manifold of hypersurface type is a triple (M^{2n+1}, H, J) where M is a real $(2n+1)$-dimensional manifold, H is a corank one distribution in TM, and J is an almost complex structure on H satisfying the integrability condition

$$[X, Y] - [JX, JY] + J([JX, Y] + [X, JY]) = 0$$

for any two vector fields $X, Y \in \Gamma(H)$. The almost complex structure J extends by complex linearity to act on $H \otimes \mathbb{C}$, and since $J^2 = -\mathrm{id}$ the eigenvalues of J must be $\pm i$. It is easy to see that J acts by i on the bundle

$$T^{1,0}M := \{X - iJX \;:\; X \in H\} \subset H \otimes \mathbb{C}$$

and by $-i$ on the bundle $T^{0,1}M = \overline{T^{1,0}M}$. Moreover one has that $T^{1,0}M \cap T^{0,1}M = \emptyset$ and

$$H \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M.$$

From the integrability condition imposed on J it follows that $T^{1,0}M$ is *formally integrable*, that is

$$[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$$

where here we have used the same notation for the bundle $T^{1,0}M$ and its space of sections.

To simplify our discussion we assume that M is orientable. Since H carries an almost complex structure it must be an orientable vector bundle, thus the annihilator line bundle $H^{\perp} \subset T^*M$ must also be orientable (so there exists a global section of $H^{\perp}$ which is nowhere zero). We say that the CR manifold of hypersurface type (M^{2n+1}, H, J) is *nondegenerate* if H is a contact distribution for M, that is, for any global section θ of $H^{\perp}$

which is nowhere zero the $(2n+1)$-form $\theta \wedge d\theta^n$ is nowhere zero (this is equivalent to the antisymmetric bilinear form $d\theta$ being nondegenerate at each point when restricted to elements of H). If H is a contact distribution then a global section θ of $H^\perp$ which is nowhere zero is called a contact form. We assume that the line bundle $H^\perp$ has a fixed orientation so that we can talk about positive and negative elements and sections. We also assume that (M^{2n+1}, H, J) is nondegenerate.

Given a choice of contact form θ for (M, H, J) we refer to the quadruple (M, H, J, θ) as a *(nondegenerate) pseudohermitian structure*. Clearly for any two positive contact forms θ and $\hat{\theta}$ there is a smooth function $\Upsilon \in C^\infty(M)$ such that $\hat{\theta} = e^{\Upsilon}\theta$. One can therefore think of the CR manifold (M, H, J) as an equivalence class of pseudohermitian structures much as we may think of a conformal manifold as an equivalence class of Riemannian structures. In order to make calculations in CR geometry it is often convenient to fix a choice of contact form θ, calculate, and then observe how things change if we rescale θ. We will take this approach in the following, working primarily in terms of the pseudohermitian calculus associated with the Tanaka-Webster connection of the chosen contact form θ. In order to make real progress however we will need to make use of the CR invariant tractor calculus [79] as a tool to produce CR invariants and invariant operators which can be expressed in terms of the Tanaka-Webster calculus.

5.1.1 CR Densities

On a CR manifold (M^{2n+1}, H, J) we denote the annihilator subbundle of $T^{1,0}M$ by $\Lambda^{0,1}M \subset \mathbb{C}T^*M$ (where by $\mathbb{C}T^*M$ we mean the complexified cotangent bundle). Similarly we denote the annihilator subbundle of $T^{0,1}M$ by $\Lambda^{1,0}M \subset \mathbb{C}T^*M$. The bundle $\Lambda^{1,0}M$ has complex rank $n+1$ and hence $\mathscr{K} = \Lambda^{n+1}(\Lambda^{1,0}M)$ is a complex line bundle on M. The line bundle $\mathscr{K}$ is simply the bundle of $(n+1,0)$-forms on M, that is

$$\mathscr{K} = \Lambda^{n+1,0}M := \left\{\omega \in \Lambda^{n+1}M \;:\; V \lrcorner \omega = 0 \text{ for all } V \in T^{1,0}M\right\},$$

and is known as the *canonical bundle*. We assume that $\mathscr{K}$ admits an $(n+2)^{\text{th}}$ root $\mathcal{E}(-1,0)$ and we define $\mathcal{E}(1,0)$ to be $\mathcal{E}(-1,0)^*$. We then define the *CR density bundles* $\mathcal{E}(w,w')$ to be $\mathcal{E}(1,0)^w \otimes \overline{\mathcal{E}(1,0)}^{w'}$ where $w - w' \in \mathbb{Z}$.

Remark 5.1.1. The assumption that $\mathscr{K}$ admits an $(n+2)^{\text{th}}$ root is equivalent to saying that the Chern class $c_1(\mathscr{K})$ is divisible by $n+2$ in $H^2(M,\mathbb{Z})$. Note that if M is a real hypersurface in $\mathbb{C}^{n+1}$ then $\mathscr{K}$ is trivial and therefore admits such a root. ∎

Note that the bundles $\mathcal{E}(w,w')$ and $\mathcal{E}(w',w)$ are complex conjugates of one another. In particular, each diagonal density bundle $\mathcal{E}(w,w)$ is fixed under conjugation. We denote by $\mathcal{E}(w,w)_{\mathbb{R}}$ the real line subbundle of $\mathcal{E}(w,w)$ consisting of elements fixed by conjugation.

5.1.2 Abstract Index Notation

We freely use abstract index notation for the holomorphic tangent bundle $T^{1,0}M$, denoting it by $\mathcal{E}^{\alpha}$, allowing the use of lower case Greek abstract indices from the start of the alphabet: α, β, γ, δ, ϵ, α', β', and so on. Similarly we use the abstract index notation $\mathcal{E}^{\bar{\alpha}}$ for $T^{0,1}M$. We denote the dual bundle of $\mathcal{E}^{\alpha}$ by $\mathcal{E}_{\alpha}$ and the dual bundle of $\mathcal{E}^{\bar{\alpha}}$ by $\mathcal{E}_{\bar{\alpha}}$. Tensor powers of these bundles are denoted by attaching appropriate indices to the $\mathcal{E}$, so, for example, we denote $\mathcal{E}^{\alpha}\otimes\mathcal{E}_{\beta}$ by $\mathcal{E}^{\alpha}{}_{\beta}$ and $\mathcal{E}_{\alpha}\otimes\mathcal{E}_{\bar{\beta}}\otimes\mathcal{E}_{\gamma}$ by $\mathcal{E}_{\alpha\bar{\beta}\gamma}$. As usual we attach abstract indices to the elements or sections of our bundles to show which bundle they belong to, so a section V of $T^{1,0}M$ will be written as V^{α} and a section ϖ of $(T^{1,0}M)^{*}\otimes T^{0,1}M$ will be denoted by $\varpi_{\alpha}{}^{\bar{\beta}}$. The tensor product of V^{α} and $\varpi_{\gamma}{}^{\bar{\beta}}$ is written as $V^{\alpha}\varpi_{\gamma}{}^{\bar{\beta}}$, and repeated indices denote contraction, so $\varpi(V)$ is written as $V^{\alpha}\varpi_{\alpha}{}^{\bar{\beta}}$. We indicate a tensor product of some (unweighted) complex vector bundle $\mathcal{V}\to M$ with the density bundle $\mathcal{E}(w,w')$ by appending (w,w'), i.e. $\mathcal{V}(w,w')=\mathcal{V}\otimes\mathcal{E}(w,w')$.

We may conjugate elements (or sections) of $\mathcal{E}^{\alpha}$ to get elements (or sections) of $\mathcal{E}^{\bar{\alpha}}$: we write

$$V^{\bar{\alpha}} := \overline{V^{\alpha}}$$

to say that $V^{\bar{\alpha}}$ is the conjugate of V^{α}. This extends in the obvious way to (weighted) tensor product bundles; note that the complex conjugate bundle of $\mathcal{E}_{\alpha}{}^{\bar{\beta}}(w,w')$ is $\mathcal{E}_{\bar{\alpha}}{}^{\beta}(w',w)$.

We will occasionally use abstract index notation for the tangent bundle, denoting it by $\mathcal{E}^{a}$ and allowing lower case Latin abstract indices from the start of the alphabet.

5.1.3 The Reeb Vector Field

Given a choice of pseudohermitian contact form θ for (M,H,J) there is a unique vector field $T\in\mathfrak{X}(M)$ determined by the conditions that $\theta(T)=1$ and $T\lrcorner\mathrm{d}\theta=0$; this T is called the *Reeb vector field* of θ. The Reeb vector field gives us a direct sum decomposition of the tangent bundle

$$TM = H\oplus\mathbb{R}T$$

and of the complexified tangent bundle

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}T, \tag{5.1.1}$$

where $\mathbb{R}T$ (resp. $\mathbb{C}T$) denotes the real (resp. complex) line bundle spanned by T. Dually, given θ we have

$$\mathbb{C}T^*M \cong (T^{1,0}M)^* \oplus (T^{0,1}M)^* \oplus \mathbb{C}\theta. \tag{5.1.2}$$

5.1.4 Densities and Scales

Definition 5.1.2 ([103])**.** Given a contact form θ for H we say that a section ζ of $\mathscr{K}$ is *volume normalised* if it satisfies

$$\theta \wedge (\mathrm{d}\theta)^n = i^{n^2} n!(-1)^q \theta \wedge (T \lrcorner \zeta) \wedge (T \lrcorner \bar{\zeta}). \tag{5.1.3}$$

Given ζ volume normalised for θ clearly $\zeta' = e^{i\varphi}\zeta$ is also volume normalised for θ for any real valued smooth function φ on M, so that such a ζ is determined only up to phase at each point. Note however that $\zeta \otimes \bar{\zeta}$ does not depend on the choice of volume normalised ζ. Let us fix a real $(n+2)^{th}$ root ς of $\zeta \otimes \bar{\zeta}$ in $\mathcal{E}(-1,-1)$. If $\hat{\theta} = f\theta$ and $\hat{\zeta}$ is volume normalised for $\hat{\theta}$ then $f\varsigma$ is an $(n+2)^{th}$ root of $\hat{\zeta} \otimes \bar{\hat{\zeta}}$. The map taking θ to ς and $f\theta$ to $f\varsigma$ determines an isomorphism from $H^\perp$ to $\mathcal{E}(-1,-1)_\mathbb{R}$. Fixing this isomorphism simply corresponds to fixing an orientation of $\mathcal{E}(-1,-1)_\mathbb{R}$, and we henceforth assume this is fixed. The isomorphism

$$H^\perp \cong \mathcal{E}(-1,-1)_\mathbb{R} \tag{5.1.4}$$

defines a tautological $\mathcal{E}(1,1)_\mathbb{R}$ valued 1-form:

Definition 5.1.3. The *CR contact form* is the $\mathcal{E}(1,1)_\mathbb{R}$-valued 1-form $\boldsymbol{\theta}$ which is given by $\varsigma^{-1}\theta$ where θ is any pseudohermitian contact form and ς is the corresponding positive section of $\mathcal{E}(-1,-1)_\mathbb{R}$.

5.1.5 The Levi Form

The Levi form of a pseudohermitian contact form θ is the Hermitian form $h : T^{1,0}M \otimes T^{0,1}M \to \mathbb{C}$ defined by

$$(U, \overline{V}) \mapsto -2i\mathrm{d}\theta(U, \overline{V}) = 2i\theta([U, \overline{V}])$$

for $U, V \in \Gamma(T^{1,0}M)$. The Levi form of θ may be thought of as a section of $\mathcal{E}_{\alpha\bar{\beta}}$, which we write as $h_{\alpha\bar{\beta}}$; there is also an inverse of the Levi form $h^{\alpha\bar{\beta}}$ determined by the condition that $h^{\alpha\bar{\beta}}h_{\gamma\bar{\beta}} = \delta^{\alpha}{}_{\gamma}$ (where $\delta^{\alpha}{}_{\gamma}$ is the identity endomorphism of $\mathcal{E}^{\alpha}$). Note that if θ is replaced by $\hat{\theta} = e^{\Upsilon}\theta$ then $\hat{h}_{\alpha\bar{\beta}} = e^{\Upsilon}h_{\alpha\bar{\beta}}$ and consequently $\hat{h}^{\alpha\bar{\beta}} = e^{-\Upsilon}h^{\alpha\bar{\beta}}$, moreover it is clear that $\hat{\varsigma} = e^{\Upsilon}\varsigma$ (where $\theta = \varsigma\boldsymbol{\theta}$ and $\hat{\theta} = \hat{\varsigma}\boldsymbol{\theta}$). This allows us to define a canonical weighted Levi form:

Definition 5.1.4. The *CR Levi form* $\boldsymbol{h}_{\alpha\bar{\beta}} \in \Gamma\left(\mathcal{E}_{\alpha\bar{\beta}}(1,1)\right)$ is the $\mathcal{E}(1,1)$-valued Hermitian form given by $\varsigma^{-1}h_{\alpha\bar{\beta}}$ for any pseudohermitian contact form $\theta = \varsigma\boldsymbol{\theta}$.

In the following we use the CR Levi form $\boldsymbol{h}_{\alpha\bar{\beta}}$ and its inverse $\boldsymbol{h}^{\alpha\bar{\beta}}$ to raise and lower indices. Note that lowering indices with $\boldsymbol{h}_{\alpha\bar{\beta}}$ identifies $\mathcal{E}^{\alpha}$ with $\mathcal{E}_{\bar{\beta}}(1,1)$ so that weights generally change when indices are raised and lowered.

The CR Levi form could also have been defined by the map

$$(U, \overline{V}) \mapsto 2i\boldsymbol{\theta}([U, \overline{V}]).$$

By complexifying and dualising the isomorphism (5.1.4) we obtain an isomorphism of $\mathcal{E}(1,1)$ with $(\mathbb{C}H^{\perp})^{*} = \mathbb{C}TM/\mathbb{C}H$. This allows us to identify $\boldsymbol{h}$, up to a constant factor, with the usual $\mathbb{C}TM/\mathbb{C}H$-valued Levi form in CR geometry.

Remark 5.1.5. Given a contact form θ one may also define a pseudo-Riemannian metric g_{θ} on the tangent bundle of M by taking the direct sum of the bilinear form $\mathrm{d}\theta(\cdot, J\cdot)$ on H (which is precisely the real part of the Levi form h of θ) with $\theta \otimes \theta$ on $\mathbb{R}T$. This metric is called the *Webster metric.* ∎

5.1.6 Decomposing Tensors

Using the direct sum decomposition of $\mathbb{C}TM$ given by a choice of contact form θ a real tangent vector X may be represented by the triple

$$(X^\alpha, X^{\bar{\alpha}}, X^0)$$

where X^α is the holomorphic part of X, $X^{\bar{\alpha}}$ is the antiholomorphic part, and $X^0 = \boldsymbol{\theta}(X)$. Note that X^0 is a $(1,1)$ density and $X^{\bar{\alpha}} = \overline{X^\alpha}$. (We follow [79] in using $\boldsymbol{\theta}$ rather than θ in defining X^0, this simplifies later conformal transformation laws.) Similarly we may represent a real covector ω by the triple

$$(\omega_\alpha, \omega_{\bar{\alpha}}, \omega_0)$$

where ω_α is the restriction of ω to holomorphic directions, $\omega_{\bar{\alpha}} = \overline{\omega_\alpha}$ is the restriction of ω to antiholomorphic directions, and the $(-1,-1)$ density ω_0 is the $\boldsymbol{\theta}$-component of ω (i.e. $\varsigma\omega(T)$ where $\theta = \varsigma\boldsymbol{\theta}$). It is easy to see that the above decompositions extend to arbitrary tensors or tensor fields. For instance we can represent a real covariant 2-tensor T by the 9-tuple

$$(T_{\alpha\beta}, T_{\alpha\bar{\beta}}, T_{\bar{\alpha}\beta}, T_{\bar{\alpha}\bar{\beta}}, T_{\alpha 0}, T_{\bar{\alpha}0}, T_{0\beta}, T_{0\bar{\beta}}, T_{00});$$

moreover, by reality it is enough to specify the 5-tuple

$$(T_{\alpha\beta}, T_{\alpha\bar{\beta}}, T_{\alpha 0}, T_{0\beta}, T_{00})$$

since $T_{\bar{\alpha}\beta} = \overline{T_{\alpha\bar{\beta}}}$, $T_{\bar{\alpha}0} = \overline{T_{\alpha 0}}$, and $T_{0\bar{\beta}} = \overline{T_{0\beta}}$.

5.2 Psuedohermitian Calculus

5.2.1 The Tanaka-Webster Connection

Since a choice of contact form θ for (M, H, J) gives rise to a pseudo-Riemannian metric g_θ on M (Remark 5.1.5) one also obtains the Levi-Civita connection ∇^{g_θ} of g_θ. Calculating with this connection is highly inconvenient however, since it does not preserve the direct sum decomposition (5.1.1) of $\mathbb{C}TM$ induced by θ. We instead look for a connection ∇ on M which still satisfies

$$\nabla g_\theta = 0,$$

but whose parallel transport also preserves H and (as a connection on H) preserves J; such a connection cannot be torsion free, since by the contact condition there exist $X, Y \in \Gamma(H)$ with $[X, Y] \notin \Gamma(H)$ and hence

$$T^{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

cannot be zero since $\nabla_X Y - \nabla_Y X \in \Gamma(H)$. It turns out that these conditions do not determine a connection on M uniquely, but we can determine ∇ uniquely by imposing the following additional conditions on the torsion of ∇,

$$T^{\nabla}_{\alpha\bar{\beta}}{}^{\gamma} = 0,\ T^{\nabla}_{\alpha\bar{\beta}}{}^{\bar{\gamma}} = 0,\ T^{\nabla}_{\alpha\bar{\beta}}{}^{0} = i\boldsymbol{h}_{\alpha\bar{\beta}},$$

$$T^{\nabla}_{\alpha\beta}{}^{\gamma} = 0,\ T^{\nabla}_{\alpha\beta}{}^{\bar{\gamma}} = 0,\ T^{\nabla}_{\alpha\beta}{}^{0} = 0,$$

$$T^{\nabla}_{\alpha 0}{}^{\gamma} = 0,\ T^{\nabla}_{\alpha 0}{}^{\bar{\gamma}} = -A^{\bar{\gamma}}{}_{\alpha},\ \text{and}\ T^{\nabla}_{\alpha 0}{}^{0} = 0$$

for some $A^{\bar{\gamma}}{}_{\alpha} \in \Gamma(\mathcal{E}^{\bar{\gamma}}{}_{\alpha}(-1,-1))$ with $A_{\alpha\beta}$ symmetric (see [131], Proposition 3.1). The connection ∇ determined uniquely by these conditions is called the *Tanaka-Webster connection* of θ (it was discovered independently by Tanaka and Webster in [129, 142]), and $A_{\alpha\beta}$ is known as the *pseudohermitian torsion tensor*.

Since the Tanaka-Webster connection preserves H and g_θ it also preserves the g_θ-orthogonal complement of H, which is spanned by the Reeb vector field T. Since $g_\theta(T, T) = 1$ this implies that $\nabla T = 0$. Thus also

$$\nabla\theta = 0,$$

since $\theta(\,\cdot\,) = g_\theta(\,\cdot\,, T)$. By definition the Tanaka-Webster connection ∇ preserves the direct sum decomposition (5.1.1) of $\mathbb{C}TM$ induced by θ. So, by definition ∇ induces a linear connection on H and on $T^{1,0}M$. It therefore makes sense to take the Tanaka-Webster covariant derivative of the Levi form h of θ, and it is easily seen that $\nabla h = 0$.

5.2.1.1 Interpreting the torsion conditions

The conditions on the torsion tensor may be alternatively phrased by saying that for any function $f \in C^\infty(M)$ we have

$$\nabla_\alpha \nabla_{\bar{\beta}} f - \nabla_{\bar{\beta}} \nabla_\alpha f = -i\boldsymbol{h}_{\alpha\bar{\beta}} \nabla_0 f, \quad (5.2.1)$$

$$\nabla_\alpha \nabla_\beta f - \nabla_\beta \nabla_\alpha f = 0, \quad (5.2.2)$$

and

$$\nabla_\alpha \nabla_0 f - \nabla_0 \nabla_\alpha f = A^{\bar{\gamma}}{}_\alpha \nabla_{\bar{\gamma}} f \tag{5.2.3}$$

where $A_{\alpha\beta}$ is the symmetric pseudohermitian torsion tensor. We note that since the Tanaka-Webster connection preserves the direct sum decomposition (5.1.2) of $\mathbb{C}T^*M$ induced by θ there is no ambiguity in the notation used in the above displays; for instance one can equivalently think of $\nabla_\alpha \nabla_\beta f$ as the 'α-component' of ∇ acting on $\nabla_\beta f$ or as the '$\alpha\beta$-component' of $\nabla\nabla f$.

5.2.2 The Tanaka-Webster Connection on Densities

The Tanaka-Webster connection of a contact form acts on sections of any density bundle since it acts on sections of $\mathcal{E}(-1,0)^{n+2} = \mathscr{K}$. In equation (5.2.3) above we are already implicitly using the action of the connection on the density bundle $\mathcal{E}(-1,-1)$ in the expression $\nabla_\alpha \nabla_0 f$. It does not matter whether or not we think of $\nabla_0 f$ as density valued in such equations because of the following lemma.

Lemma 5.2.1. *The Tanaka-Webster connection ∇ of θ on $\mathcal{E}(-1,-1)$ is simply the flat connection corresponding to the trivialisation induced by the contact form θ, i.e. by the section ς satisfying $\theta = \varsigma\boldsymbol{\theta}$. In particular the isomorphism* (5.1.4) *is parallel for the Tanaka-Webster connection, i.e. it intertwines the actions of ∇ on $H^\perp$ and on $\mathcal{E}(-1,-1)_{\mathbb{R}}$.*

Proof. Suppose the section ζ of $\mathscr{K}$ is volume normalised for θ. Parallel transporting ζ along any curve must preserve ζ up to phase (since the result of parallel transport will still be volume normalised, θ, $\mathrm{d}\theta$, and T being parallel). This implies that $\zeta \otimes \bar{\zeta}$ is parallel, but by definition $\varsigma^{n+2} = \zeta \otimes \bar{\zeta}$ so that ς^{n+2} and hence ς is parallel. □

The lemma also tells us that for the Tanaka-Webster connection ∇ of any contact form θ we have

$$\nabla\boldsymbol{\theta} = 0 \qquad \text{and} \qquad \nabla\boldsymbol{h} = 0. \tag{5.2.4}$$

The advantage of raising and lowering indices with the CR Levi form $\boldsymbol{h}_{\alpha\bar{\beta}}$ is that these operations commute with *any* Tanaka-Webster covariant derivative.

5.2.3 Pseudohermitian Curvature

By equation (5.2.1) the operator

$$\nabla_\alpha \nabla_{\bar\beta} - \nabla_{\bar\beta}\nabla_\alpha + i\boldsymbol{h}_{\alpha\bar\beta}\nabla_0$$

annihilates smooth functions on M; moreover, this operator preserves $\mathcal{E}^{\bar\gamma}$. By the Leibniz rule the above displayed operator commutes with multiplication by smooth functions when acting on sections of $\mathcal{E}^{\bar\gamma}$. Thus there is a tensor $R_{\alpha\bar\beta}{}^{\bar\gamma}{}_{\bar\delta}$ such that

$$\nabla_\alpha \nabla_{\bar\beta} V^{\bar\gamma} - \nabla_{\bar\beta}\nabla_\alpha V^{\bar\gamma} + i\boldsymbol{h}_{\alpha\bar\beta}\nabla_0 V^{\bar\gamma} = -R_{\alpha\bar\beta}{}^{\bar\gamma}{}_{\bar\delta} V^{\bar\delta} \tag{5.2.5}$$

for all sections $V^{\bar\gamma}$ of $\mathcal{E}^{\bar\gamma}$. Equivalently $R_{\alpha\bar\beta}{}^{\bar\gamma}{}_{\bar\delta}$ is characterised by

$$\nabla_\alpha \nabla_{\bar\beta} V_{\bar\delta} - \nabla_{\bar\beta}\nabla_\alpha V_{\bar\delta} + i\boldsymbol{h}_{\alpha\bar\beta}\nabla_0 V_{\bar\delta} = R_{\alpha\bar\beta}{}^{\bar\gamma}{}_{\bar\delta} V_{\bar\gamma} \tag{5.2.6}$$

for all sections $V_{\bar\delta}$ of $\mathcal{E}_{\bar\delta}$. Our conventions agree with those of [79, 142]. We refer to this tensor, or to $R_{\alpha\bar\beta\gamma\bar\delta} = \boldsymbol{h}_{\gamma\bar\epsilon} R_{\alpha\bar\beta}{}^{\bar\epsilon}{}_{\bar\delta}$, as the *pseudohermitian curvature tensor*, and it has the following properties

$$R_{\alpha\bar\beta\gamma\bar\delta} = R_{\gamma\bar\beta\alpha\bar\delta} = \overline{R_{\beta\bar\alpha\delta\bar\gamma}} = R_{\alpha\bar\delta\gamma\bar\beta} = R_{\gamma\bar\delta\alpha\bar\beta} \tag{5.2.7}$$

which we derive in §§5.2.4 below. The trace

$$R_{\alpha\bar\beta} = R_{\alpha\bar\beta}{}^{\bar\gamma}{}_{\bar\gamma} \tag{5.2.8}$$

of the pseudohermitian curvature tensor is referred to as the *Webster-Ricci tensor* of θ and its trace

$$R = \boldsymbol{h}^{\alpha\bar\beta} R_{\alpha\bar\beta} \tag{5.2.9}$$

is called the *Webster scalar curvature* of θ. The pseudohermitian curvature tensor can be decomposed as

$$R_{\alpha\bar\beta\gamma\bar\delta} = S_{\alpha\bar\beta\gamma\bar\delta} + P_{\alpha\bar\beta}\boldsymbol{h}_{\gamma\bar\delta} + P_{\gamma\bar\delta}\boldsymbol{h}_{\alpha\bar\beta} + P_{\alpha\bar\delta}\boldsymbol{h}_{\gamma\bar\beta} + P_{\gamma\bar\beta}\boldsymbol{h}_{\alpha\bar\delta} \tag{5.2.10}$$

where $S_{\alpha\bar\beta\gamma\bar\delta}$ satisfies

$$S_{\alpha\bar\beta\gamma\bar\delta} = S_{\gamma\bar\beta\alpha\bar\delta} = \overline{S_{\beta\bar\alpha\delta\bar\gamma}} = S_{\alpha\bar\delta\gamma\bar\beta} = S_{\gamma\bar\delta\alpha\bar\beta},\ S_{\alpha\bar\beta}{}^{\bar\gamma}{}_{\bar\gamma} = 0 \tag{5.2.11}$$

and

$$P_{\alpha\bar{\beta}} = \frac{1}{n+2}\left(R_{\alpha\bar{\beta}} - \frac{1}{2(n+1)}R\boldsymbol{h}_{\alpha\bar{\beta}}\right). \tag{5.2.12}$$

The tensor $S_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is known as the *Chern-Moser tensor* and is a CR invariant, by which we mean that if $\hat{\theta}$ is another contact form for H then $\hat{S}_{\alpha\bar{\beta}\gamma\bar{\delta}} = S_{\alpha\bar{\beta}\gamma\bar{\delta}}$ (note that we are thinking of $S_{\alpha\bar{\beta}\gamma\bar{\delta}}$ as a weighted tensor field).

5.2.4 The Full Tanaka-Webster Curvature

The full curvature tensor of the Tanaka-Webster connection ∇ of a contact form θ is defined by

$$\nabla_a\nabla_b Y^c - \nabla_b\nabla_a Y^c + T_{ab}^{\nabla e}\nabla_e Y^c = -R_{ab}{}^c{}_d Y^d \tag{5.2.13}$$

for any tangent vector field Y^c, where T^∇ is the torsion of ∇ given in §§5.2.1. The pseudohermitian curvature tensor $R_{\alpha\bar{\beta}}{}^{\bar{\gamma}}{}_{\bar{\delta}}$ is just one component of the full curvature tensor, taken with respect to the direct sum decomposition

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}T \tag{5.2.14}$$

and its dual.

Lemma 5.2.2. *The full curvature tensor $R_{ab}{}^c{}_d$ of the Tanaka-Webster connection is completely determined by the components $R_{\alpha\bar{\beta}}{}^{\bar{\gamma}}{}_{\bar{\delta}}$, $R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}}$, and $R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}}$.*

Proof. Note that the tensor $R_{ab}{}^c{}_d$ is real, so that the component $R_{\bar{\alpha}\beta}{}^{\gamma}{}_{\delta}$ is simply the complex conjugate of $R_{\alpha\bar{\beta}}{}^{\bar{\gamma}}{}_{\bar{\delta}}$ and so on. Also, the symmetry $R_{ab}{}^c{}_d = -R_{ba}{}^c{}_d$ translates into $R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}} = -R_{\beta\alpha}{}^{\bar{\gamma}}{}_{\bar{\delta}}$, $R_{\bar{\alpha}0}{}^{\gamma}{}_{\delta} = -R_{0\bar{\alpha}}{}^{\gamma}{}_{\delta}$, etcetera. Now since the Tanaka-Webster connection preserves the splitting of sections of $\mathbb{C}TM$ according to (5.2.14) we must have

$$R_{ab}{}^{\gamma}{}_{\bar{\delta}} = 0,\ R_{ab}{}^{\gamma}{}_0 = 0,\ \text{and}\ R_{ab}{}^0{}_\delta = 0.$$

Since T is parallel we also have that

$$R_{ab}{}^0{}_0 = 0.$$

From this we see that, up to conjugation and swapping the first two indices, the only nonzero components of $R_{ab}{}^{c}{}_{d}$ are

$$R_{\alpha\bar{\beta}}{}^{\gamma}{}_{\delta}\ ,\ R_{\alpha\beta}{}^{\gamma}{}_{\delta}\ ,\ R_{\alpha 0}{}^{\gamma}{}_{\delta}\ ,\ R_{\alpha\bar{\beta}}{}^{\bar{\gamma}}{}_{\bar{\delta}}\ ,\ R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}}\ \text{ and }\ R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}}.$$

Our conclusion follows by observing that if we lower indices using the CR Levi form then we have that

$$R_{ab\gamma\bar{\delta}} = -R_{ab\bar{\delta}\gamma},$$

since $\boldsymbol{h}_{\gamma\bar{\delta}}$ is parallel. □

Remark 5.2.3. From the last display of the above proof we have that $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -R_{\alpha\bar{\beta}\bar{\delta}\gamma}$. It immediately follows that $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\bar{\beta}\alpha\bar{\delta}\gamma} = \overline{R_{\beta\bar{\alpha}\delta\bar{\gamma}}}$, establishing one of the claims from §§5.2.3. ▮

Using (5.2.13) the curvature component $R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}}$ may also be characterised by a Ricci-type identity

$$\nabla_\alpha\nabla_\beta V^{\bar{\gamma}} - \nabla_\beta\nabla_\alpha V^{\bar{\gamma}} = -R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}}V^{\bar{\delta}} \tag{5.2.15}$$

for any section $V^{\bar{\gamma}}$ of $\mathcal{E}^{\bar{\gamma}}$. Similarly for $R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}}$ we have

$$\nabla_\alpha\nabla_0 V^{\bar{\gamma}} - \nabla_0\nabla_\alpha V^{\bar{\gamma}} - A_\alpha^\epsilon\nabla_\epsilon V^{\bar{\gamma}} = -R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}}V^{\bar{\delta}}. \tag{5.2.16}$$

On a section $V_{\bar{\delta}}$ of $\mathcal{E}_{\bar{\delta}}$ we have

$$\nabla_\alpha\nabla_\beta V_{\bar{\delta}} - \nabla_\beta\nabla_\alpha V_{\bar{\delta}} = R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}}V_{\bar{\gamma}} \tag{5.2.17}$$

by duality, and likewise for $R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}}$.

5.2.4.1 The Bianchi symmetry

Recall that for a connection ∇ without torsion the Bianchi symmetry comes from observing that by torsion freeness one has

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\nabla_d f + (\nabla_b\nabla_d - \nabla_d\nabla_b)\nabla_a f + (\nabla_d\nabla_a - \nabla_a\nabla_d)\nabla_b f = 0$$

for any $f \in C^\infty(M)$, since the curvature tensor must then satisfy

$$(R_{ab}{}^{c}{}_{d} + R_{bd}{}^{c}{}_{a} + R_{da}{}^{c}{}_{b})\nabla_c f = 0.$$

This approach also works for a connection with torsion and we use it below to express the consequences of the Bianchi symmetry for the curvature of the Tanaka-Webster connection in terms of the components $R_{\alpha\bar{\beta}}{}^{\bar{\gamma}}{}_{\bar{\delta}}$, $R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}}$, $R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}}$, and the pseudohermitian torsion. Because so many components of $R_{ab}{}^{c}{}_{d}$ already vanish one obtains expressions for $R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}}$ and $R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}}$ in terms of the pseudohermitian torsion.

Proposition 5.2.4. *The Bianchi symmetry for the Tanaka-Webster curvature tensor* $R_{ab}{}^{c}{}_{d}$ *is equivalent to the following identities*

$$R_{\alpha\bar{\beta}}{}^{\bar{\gamma}}{}_{\bar{\delta}} = R_{\alpha\bar{\delta}}{}^{\bar{\gamma}}{}_{\bar{\beta}} \tag{5.2.18}$$

$$R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}} = i\boldsymbol{h}_{\alpha\bar{\delta}} A^{\bar{\gamma}}_{\beta} - i\boldsymbol{h}_{\beta\bar{\delta}} A^{\bar{\gamma}}_{\alpha} \tag{5.2.19}$$

$$R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}} = \nabla_{\bar{\delta}} A^{\bar{\gamma}}_{\alpha} \tag{5.2.20}$$

$$\nabla_{\alpha} A^{\bar{\gamma}}_{\delta} = \nabla_{\delta} A^{\bar{\gamma}}_{\alpha}. \tag{5.2.21}$$

Proof. By elementary considerations the cyclic sum of $R_{ab}{}^{c}{}_{d}$ with respect to the lower three indices is determined by the cyclic sums of $R_{\alpha\beta}{}^{c}{}_{\delta}$, $R_{\alpha\beta}{}^{c}{}_{\bar{\delta}}$, $R_{\alpha 0}{}^{c}{}_{\delta}$, and $R_{\alpha 0}{}^{c}{}_{\bar{\delta}}$ with respect to their lower three indices. In the case of $R_{\alpha\beta}{}^{c}{}_{\delta}$ we may instead cyclically permute the lower indices of $R_{\alpha\beta}{}^{\gamma}{}_{\delta}$ since for each permutation the only nonzero part of the tensor is obtained by replacing c with γ. By (5.2.2) $\nabla_{\alpha}\nabla_{\beta}$ is symmetric on smooth functions so that

$$(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})\nabla_{\delta} f + (\nabla_{\beta}\nabla_{\delta} - \nabla_{\delta}\nabla_{\beta})\nabla_{\alpha} f + (\nabla_{\delta}\nabla_{\alpha} - \nabla_{\alpha}\nabla_{\delta})\nabla_{\beta} f = 0$$

for all $f \in C^{\infty}(M)$, and hence

$$R_{\alpha\beta}{}^{\gamma}{}_{\delta} + R_{\delta\alpha}{}^{\gamma}{}_{\beta} + R_{\beta\delta}{}^{\gamma}{}_{\alpha} = 0. \tag{5.2.22}$$

This expression is not listed above because it is a consequence of (5.2.19), the latter being equivalent to

$$R_{\alpha\beta}{}^{\gamma}{}_{\delta} = -i\delta^{\gamma}_{\alpha} A_{\beta\delta} + i\delta^{\gamma}_{\beta} A_{\alpha\delta} \tag{5.2.23}$$

since $R_{\alpha\beta\gamma\bar{\delta}} = -R_{\alpha\beta\bar{\delta}\gamma}$.

Now let f be a smooth function on M. We similarly compute

$$\left(R_{\alpha\beta}{}^{c}{}_{\bar{\delta}} + R_{\bar{\delta}\alpha}{}^{c}{}_{\beta} + R_{\beta\bar{\delta}}{}^{c}{}_{\alpha}\right) \nabla_{c} f.$$

Noting that $R_{\alpha\beta}{}^{c}{}_{\bar{\delta}}\nabla_c f = R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}}\nabla_{\bar{\gamma}} f$, and so on, we get

$$\begin{aligned}
&R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}}\nabla_{\bar{\gamma}} f + R_{\bar{\delta}\alpha}{}^{\gamma}{}_{\beta}\nabla_{\gamma} f + R_{\beta\bar{\delta}}{}^{\gamma}{}_{\alpha}\nabla_{\gamma} f \\
&= (\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\nabla_{\bar{\delta}} f + (\nabla_{\bar{\delta}}\nabla_\alpha - \nabla_\alpha\nabla_{\bar{\delta}} - i\boldsymbol{h}_{\alpha\bar{\delta}}\nabla_0)\nabla_\beta f \\
&\quad + (\nabla_\beta\nabla_{\bar{\delta}} - \nabla_{\bar{\delta}}\nabla_\beta + i\boldsymbol{h}_{\beta\bar{\delta}}\nabla_0)\nabla_\alpha f \\
&= \nabla_\alpha(\nabla_\beta\nabla_{\bar{\delta}} f - \nabla_{\bar{\delta}}\nabla_\beta f) + \nabla_\beta(\nabla_{\bar{\delta}}\nabla_\alpha f - \nabla_\alpha\nabla_{\bar{\delta}} f) \\
&\quad + \nabla_{\bar{\delta}}(\nabla_\alpha\nabla_\beta f - \nabla_\beta\nabla_\alpha f) - i\boldsymbol{h}_{\alpha\bar{\delta}}\nabla_0\nabla_\beta f + i\boldsymbol{h}_{\beta\bar{\delta}}\nabla_0\nabla_\alpha f \\
&= \nabla_\alpha(-i\boldsymbol{h}_{\beta\bar{\delta}}\nabla_0 f) + \nabla_\beta(i\boldsymbol{h}_{\alpha\bar{\delta}}\nabla_0 f) \\
&\quad - i\boldsymbol{h}_{\alpha\bar{\delta}}\nabla_0\nabla_\beta f + i\boldsymbol{h}_{\beta\bar{\delta}}\nabla_0\nabla_\alpha f \\
&= i\boldsymbol{h}_{\alpha\bar{\delta}}(\nabla_\beta\nabla_0 f - \nabla_0\nabla_\beta f) + i\boldsymbol{h}_{\beta\bar{\delta}}(\nabla_0\nabla_\alpha f - \nabla_\alpha\nabla_0 f) \\
&= i\boldsymbol{h}_{\alpha\bar{\delta}}A^{\bar{\epsilon}}_{\beta}\nabla_{\bar{\epsilon}} f - i\boldsymbol{h}_{\beta\bar{\delta}}A^{\bar{\epsilon}}_{\alpha}\nabla_{\bar{\epsilon}} f
\end{aligned}$$

Since f was arbitrary the above display holds at any point for all functions f with $\nabla_\gamma f = 0$ (or with $\nabla_{\bar{\gamma}} f = 0$) at that point, and thus we conclude that

$$R_{\alpha\beta}{}^{\bar{\gamma}}{}_{\bar{\delta}} = i\boldsymbol{h}_{\alpha\bar{\delta}}A^{\bar{\gamma}}_{\beta} - i\boldsymbol{h}_{\beta\bar{\delta}}A^{\bar{\gamma}}_{\alpha}$$

and

$$R_{\bar{\delta}\alpha}{}^{\gamma}{}_{\beta} + R_{\beta\bar{\delta}}{}^{\gamma}{}_{\alpha} = 0.$$

By conjugation the last display is equivalent to (5.2.18), noting that $R_{\beta\bar{\delta}}{}^{\gamma}{}_{\alpha} = -R_{\bar{\delta}\beta}{}^{\gamma}{}_{\alpha}$.

Similarly computing

$$R_{\alpha 0}{}^{c}{}_{\bar{\delta}}\nabla_c f + R_{0\bar{\delta}}{}^{c}{}_{\alpha}\nabla_c f + R_{\bar{\delta}\alpha}{}^{c}{}_{0}\nabla_c f$$

we get (noting $R_{\bar{\delta}\alpha}{}^{c}{}_{0} = 0$)

$$R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}}\nabla_{\bar{\gamma}} f + R_{0\bar{\delta}}{}^{\gamma}{}_{\alpha}\nabla_\gamma f = -(\nabla_\alpha A^{\gamma}_{\bar{\delta}})\nabla_\gamma f + (\nabla_{\bar{\delta}} A^{\bar{\gamma}}_{\alpha})\nabla_{\bar{\gamma}} f$$

so that

$$R_{\alpha 0}{}^{\bar{\gamma}}{}_{\bar{\delta}} = \nabla_{\bar{\delta}} A^{\bar{\gamma}}_{\alpha}.$$

Finally, computing the cyclic sum for $R_{\alpha 0}{}^{c}{}_{\delta}\nabla_c f$ we obtain

$$R_{\alpha 0}{}^{\gamma}{}_{\delta}\nabla_\gamma f + R_{0\delta}{}^{\gamma}{}_{\alpha}\nabla_\gamma f = -(\nabla_\alpha A^{\bar{\gamma}}_{\delta})\nabla_{\bar{\gamma}} f + (\nabla_\delta A^{\bar{\gamma}}_{\alpha})\nabla_{\bar{\gamma}} f$$

so that

$$\nabla_\alpha A_\delta^{\bar{\gamma}} = \nabla_\delta A_\alpha^{\bar{\gamma}}$$

and

$$R_{\alpha 0}{}^{\gamma}{}_{\delta} = R_{\delta 0}{}^{\gamma}{}_{\alpha}. \tag{5.2.24}$$

The identity (5.2.24) follows already from (5.2.20) since by lowering indices we have $R_{\alpha 0 \gamma \bar{\delta}} = \nabla_{\bar{\delta}} A_{\alpha\gamma}$ and using that $R_{\alpha 0 \bar{\delta} \gamma} = -R_{\alpha 0 \gamma \bar{\delta}}$ we get that

$$R_{\alpha 0}{}^{\gamma}{}_{\delta} = -\nabla^{\gamma} A_{\alpha\delta}. \tag{5.2.25}$$

□

The expressions (5.2.20) and (5.2.23) agree with those given in section 1.4.2 of [49], after adjusting (5.2.23) by factor of two to account for their slightly different conventions (see (1.84) in [49]).

Note that (5.2.18) implies that the pseudohermitian curvature tensor satisfies

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}}$$

(as was previously claimed) from which we also deduce that

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\alpha\bar{\delta}\gamma\bar{\beta}} = R_{\gamma\bar{\delta}\alpha\bar{\beta}}.$$

5.2.5 Curvature of the Density Bundles

Although the Tanaka-Webster connection ∇ of a contact form θ is flat on the diagonal density bundles $\mathcal{E}(w,w)$ it is not flat on density bundles in general. The curvature of the density bundles was calculated in [79, Prop. 2.2]. We give this proposition with an alternate proof:

Proposition 5.2.5. *Let θ be a pseudohermitian contact form and ∇ its Tanaka-Webster* *connection. On a section f of $\mathcal{E}(w, w')$ we have*

$$\nabla_\alpha \nabla_\beta f - \nabla_\beta \nabla_\alpha f = 0, \tag{5.2.26}$$

$$\nabla_\alpha \nabla_{\bar\beta} f - \nabla_{\bar\beta} \nabla_\alpha f + i\boldsymbol{h}_{\alpha\bar\beta}\nabla_0 f = \frac{w - w'}{n+2} R_{\alpha\bar\beta} f, \tag{5.2.27}$$

$$\nabla_\alpha \nabla_0 f - \nabla_0 \nabla_\alpha f - A_{\alpha\gamma}\nabla^\gamma f = \frac{w - w'}{n+2} (\nabla^\gamma A_{\gamma\alpha}) f. \tag{5.2.28}$$

Proof. We first consider sections $f = \zeta$ of $\mathcal{E}(-n-2, 0) = \mathscr{K}$. The map

$$\zeta \mapsto (T \lrcorner \zeta)|_{T^{1,0}M}$$

induces an isomorphism between the complex line bundles $\mathscr{K}$ and $\Lambda^n \left(T^{1,0}M\right)^*$. This isomorphism between $\mathscr{K}$ and $\mathcal{E}_{[\alpha_1 \cdots \alpha_n]}$ intertwines the action of the Tanaka-Webster connection ∇ since the Reeb vector field T is parallel and ∇ preserves $T^{1,0}M$, so the two line bundles have the same curvature. The curvature of the top exterior power $\mathcal{E}_{[\alpha_1 \cdots \alpha_n]}$ of $\mathcal{E}_\alpha$ is simply obtained by tracing the curvature of $\mathcal{E}_\alpha$.

Now let f be a section of $\mathcal{E}(-n-2, 0) = \mathscr{K}$. By tracing the conjugate of (5.2.19) and using the appropriate Ricci-type identity for $R_{\bar\alpha\bar\beta}{}^\gamma{}_\delta$ we therefore obtain

$$\nabla_{\bar\alpha} \nabla_{\bar\beta} f - \nabla_{\bar\beta} \nabla_{\bar\alpha} f = 0$$

since $R_{\bar\alpha\bar\beta}{}^\gamma{}_\gamma = 0$. Similarly, using that $R_{\alpha\beta\gamma\bar\delta} = -R_{\alpha\beta\bar\delta\gamma}$, we obtain from (5.2.19) that $R_{\alpha\beta}{}^\gamma{}_\gamma = 0$ and hence

$$\nabla_\alpha \nabla_\beta f - \nabla_\beta \nabla_\alpha f = 0.$$

Using that $R_{\alpha\bar\beta}{}^\gamma{}_\gamma = R_{\alpha\bar\beta\bar\gamma}{}^{\bar\gamma} = -R_{\alpha\bar\beta}{}^{\bar\gamma}{}_{\bar\gamma} = -R_{\alpha\bar\beta}$ and the appropriate Ricci-type identity for $R_{\alpha\bar\beta}{}^\gamma{}_\delta$ we get

$$\nabla_\alpha \nabla_{\bar\beta} f - \nabla_{\bar\beta} \nabla_\alpha f + i\boldsymbol{h}_{\alpha\bar\beta}\nabla_0 f = -R_{\alpha\bar\beta} f.$$

Finally, by tracing the conjugate of (5.2.20) we obtain

$$\nabla_{\bar\alpha} \nabla_0 f - \nabla_0 \nabla_{\bar\alpha} f - A^\gamma_{\bar\alpha} \nabla_\gamma f = (\nabla_\gamma A^\gamma_{\bar\alpha}) f$$

and using $R_{\alpha 0\gamma\bar\delta} = -R_{\alpha 0\bar\delta\gamma}$ we obtain from (5.2.20) that $R_{\alpha 0}{}^\gamma{}_\delta = -\nabla^\gamma A_{\alpha\delta}$ so that

$$\nabla_\alpha \nabla_0 f - \nabla_0 \nabla_\alpha f - A^{\bar\gamma}_\alpha \nabla_{\bar\gamma} f = -(\nabla^\gamma A_{\alpha\gamma}) f.$$

This establishes the proposition for (w, w') equal to $(-n-2, 0)$, and for (w, w') equal to $(0, -n-2)$ by conjugating.

Considering the action of the curvature operator(s) on $(n+2)^{th}$ powers of sections of $\mathcal{E}(-1,0)$ and $\mathcal{E}(0,-1)$ we obtain the result for (w, w') equal to $(-1,0)$ or $(0,-1)$. Taking powers and tensor products then gives the full proposition. □

5.2.6 Changing Contact Form

Here we establish how the various pseudohermitian objects we have introduced transform under conformal rescaling of the contact form. The first thing to consider is the Reeb vector field:

Lemma 5.2.6. *Under the transformation $\hat{\theta} = e^{\Upsilon}\theta$ of pseudohermitian contact forms, $\Upsilon \in C^{\infty}(M)$, the Reeb vector field transforms according to*

$$\hat{T} = e^{-\Upsilon}\left[T + ((\mathrm{d}\Upsilon)|_H \circ J)^{\sharp}\right] \tag{5.2.29}$$

where $\sharp$ denotes the usual isomorphism $H^* \to H$ induced by the bundle metric $\mathrm{d}\theta(\cdot, J\cdot)|_H$ on the contact distribution.

Proof. Defining $\hat{T}$ by (5.2.29) one has $\hat{\theta}(\hat{T}) = 1$ and

$$\hat{T} \lrcorner \mathrm{d}\hat{\theta} = \hat{T} \lrcorner \mathrm{d}(e^{\Upsilon}\theta) = \mathrm{d}\Upsilon(\hat{T})\hat{\theta} - \mathrm{d}\Upsilon + e^{\Upsilon}\hat{T} \lrcorner \mathrm{d}\theta.$$

We observe that

$$\mathrm{d}\hat{\theta}(\hat{T}, JY) = -\mathrm{d}\Upsilon(JY) + \mathrm{d}\theta(e^{\Upsilon}\hat{T}, JY) = 0$$

for any $Y \in \Gamma(H)$, using that $\hat{\theta}(JY) = 0$ and

$$\mathrm{d}\theta(e^{\Upsilon}\hat{T}, JY) = \mathrm{d}\theta(((\mathrm{d}\Upsilon)|_H \circ J)^{\sharp}, JY) = \mathrm{d}\Upsilon(JY).$$

Now since $\mathrm{d}\hat{\theta}(\hat{T}, \hat{T}) = 0$ we have $\hat{T} \lrcorner \mathrm{d}\hat{\theta} = 0$. □

If η is a 1-form whose restrictions to $\mathcal{E}^{\alpha}$ and $\mathcal{E}^{\bar{\alpha}}$ are η_{α} and $\eta_{\bar{\alpha}}$ respectively then $(\eta|_H)^{\sharp}$ is a contact vector field whose antiholomorphic component is $h^{\beta\bar{\alpha}}\eta_{\beta}$ and whose holomorphic component is $h^{\alpha\bar{\beta}}\eta_{\bar{\beta}}$. It is easy to see that the restriction of $(\mathrm{d}\Upsilon)|_H \circ J$ to $\mathcal{E}^{\alpha}$ is $i\nabla_{\alpha}\Upsilon$ and the restriction to $\mathcal{E}^{\bar{\alpha}}$ is $-i\nabla_{\bar{\alpha}}\Upsilon$. These observations imply:

Lemma 5.2.7. *If a 1-form ω has components $(\omega_\alpha, \omega_{\bar{\alpha}}, \omega_0)$ with respect to some contact form θ, then the components of ω with respect to $e^\Upsilon\theta$ are*

$$(\omega_\alpha,\ \omega_{\bar{\alpha}},\ \omega_0 + i\Upsilon^{\bar{\alpha}}\omega_{\bar{\alpha}} - i\Upsilon^\alpha\omega_\alpha)$$

where $\Upsilon^{\bar{\alpha}} = \boldsymbol{h}^{\beta\bar{\alpha}}\nabla_\beta\Upsilon$ and $\Upsilon^\alpha = \boldsymbol{h}^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\Upsilon$.

5.2.6.1 The Tanaka-Webster transformation laws

We need to see how the Tanaka-Webster connection transforms under rescaling of the contact form.

Proposition 5.2.8. *Under the transformation $\hat{\theta} = e^\Upsilon\theta$ of pseudohermitian contact forms, $\Upsilon \in C^\infty(M)$, the Tanaka-Webster connection on sections τ_β of $\mathcal{E}_\beta$ transforms according to*

$$\hat{\nabla}_\alpha\tau_\beta = \nabla_\alpha\tau_\beta - \Upsilon_\beta\tau_\alpha - \Upsilon_\alpha\tau_\beta \tag{5.2.30}$$

$$\hat{\nabla}_{\bar{\alpha}}\tau_\beta = \nabla_{\bar{\alpha}}\tau_\beta + \boldsymbol{h}_{\beta\bar{\alpha}}\Upsilon^\gamma\tau_\gamma \tag{5.2.31}$$

$$\hat{\nabla}_0\tau_\beta = \nabla_0\tau_\beta + i\Upsilon^{\bar{\gamma}}\nabla_{\bar{\gamma}}\tau_\beta - i\Upsilon^\gamma\nabla_\gamma\tau_\beta - i(\Upsilon^\gamma{}_\beta - \Upsilon^\gamma\Upsilon_\beta)\tau_\gamma \tag{5.2.32}$$

where $\Upsilon_\alpha = \nabla_\alpha\Upsilon$, $\Upsilon_{\bar{\alpha}} = \nabla_{\bar{\alpha}}\Upsilon$, $\Upsilon_{\alpha\bar{\beta}} = \nabla_{\bar{\beta}}\nabla_\alpha\Upsilon$, and indices are raised using $\boldsymbol{h}^{\alpha\bar{\beta}}$.

Remark. Note that on the left hand side of the last equation in the above display the '0-component' is taken with respect to the splitting of the cotangent bundle induced by $\hat{\theta}$, whereas on the right hand side it is taken with respect to the splitting of the cotangent bundle induced by θ (recall §§5.1.4 and §§5.1.6). In other words the operator $\hat{\nabla}_0$ is taken to be $\hat{\varsigma}\hat{\nabla}_{\hat{T}}$ where $\hat{\theta} = \hat{\varsigma}\boldsymbol{\theta}$, whereas $\nabla_0 = \varsigma\nabla_T$ where $\theta = \varsigma\boldsymbol{\theta}$. No overall factor appears in the transformation law since $\boldsymbol{\theta}(\varsigma T) = \theta(T) = 1$ and similarly $\boldsymbol{\theta}(\hat{\varsigma}\hat{T}) = 1$.

Proof of Proposition 5.2.8. We define the connection $\hat{\nabla}$ on $\mathcal{E}_\beta$ by the formulae above, and extend $\hat{\nabla}$ to a connection on TM in the obvious way. Precisely, we define $\hat{\nabla}$ to act on $\mathcal{E}_{\bar{\beta}}$ by the conjugates of the above formulae, so that, e.g., $\hat{\nabla}_\alpha\tau_{\bar{\beta}}$ is the conjugate of $\hat{\nabla}_{\bar{\alpha}}\tau_\beta$ with $\tau_\beta = \overline{\tau_{\bar{\beta}}}$. This gives a connection on $\mathbb{C}H^*$ which preserves the real subbundle H^* and preserves J. Thus by requiring $\hat{\theta}$ to be parallel for $\hat{\nabla}$ we obtain a connection on T^*M and hence on TM. To show that this is the Tanaka-Webster connection of $\hat{\theta}$ it remains only to show that $\hat{\nabla}g_{\hat{\theta}} = 0$ and to verify the torsion conditions.

To show that $\hat{\nabla} g_{\hat{\theta}} = 0$ it is sufficient to show that $\hat{\nabla}$ preserves the Levi form $\hat{h}$ is of $\hat{\theta}$. This is a computation using the formulae in the proposition: By the Leibniz rule we have

$$\hat{\nabla}_\alpha(\tau_\beta \zeta_{\bar{\gamma}}) = \nabla_\alpha(\tau_\beta \zeta_{\bar{\gamma}}) - \Upsilon_\alpha(\tau_\beta \zeta_{\bar{\gamma}}) - \Upsilon_\beta(\tau_\alpha \zeta_{\bar{\gamma}}) + \boldsymbol{h}_{\alpha\bar{\gamma}} \Upsilon^{\bar{\delta}}(\tau_\beta \zeta_{\bar{\delta}})$$

for a simple section of $\mathcal{E}_{\beta\bar{\gamma}}$. By $\mathbb{C}$-linearity we obtain

$$\hat{\nabla}_\alpha \hat{h}_{\beta\bar{\gamma}} = \nabla_\alpha \hat{h}_{\beta\bar{\gamma}} - \Upsilon_\alpha \hat{h}_{\beta\bar{\gamma}} - \Upsilon_\beta \hat{h}_{\alpha\bar{\gamma}} + \boldsymbol{h}_{\alpha\bar{\gamma}} \Upsilon^{\bar{\delta}} \hat{h}_{\beta\bar{\delta}};$$

the terms on the right hand side of the above display cancel in pairs since $\hat{h}_{\beta\bar{\gamma}} = e^{\Upsilon} h_{\beta\bar{\gamma}}$. By conjugate symmetry we also get $\hat{\nabla}_{\bar{\alpha}} \hat{h}_{\beta\bar{\gamma}} = 0$. Similarly one computes that

$$\begin{aligned}\hat{\nabla}_0 \hat{h}_{\alpha\bar{\beta}} = {} & \nabla_0 \hat{h}_{\alpha\bar{\beta}} + i\Upsilon^{\bar{\gamma}} \nabla_{\bar{\gamma}} \hat{h}_{\alpha\bar{\beta}} - i\Upsilon^{\gamma} \nabla_{\gamma} \hat{h}_{\alpha\bar{\beta}} \\ & - i\left(\Upsilon^{\gamma}{}_{\alpha} - \Upsilon^{\gamma}\Upsilon_{\alpha}\right) \hat{h}_{\gamma\bar{\beta}} + i\left(\Upsilon^{\bar{\gamma}}{}_{\bar{\beta}} - \Upsilon^{\bar{\gamma}}\Upsilon_{\bar{\beta}}\right) \hat{h}_{\alpha\bar{\gamma}}.\end{aligned}$$

The terms on the right hand side of the above display all cancel since $\nabla_0 \hat{h}_{\alpha\bar{\beta}} = \Upsilon_0 \hat{h}_{\alpha\bar{\beta}}$ and $\Upsilon_{\alpha\bar{\beta}} - \Upsilon_{\bar{\beta}\alpha} = i\boldsymbol{h}_{\alpha\bar{\beta}} \Upsilon_0$, where $\Upsilon_0 = \nabla_0 \Upsilon$ and $\Upsilon_{\bar{\beta}\alpha} = \nabla_\alpha \nabla_{\bar{\beta}} \Upsilon$.

Substituting $\hat{\nabla}_\beta f$ $(= \nabla_\beta f)$ for τ_β in equation (5.2.30) we see that

$$\hat{\nabla}_\alpha \hat{\nabla}_\beta f - \hat{\nabla}_\beta \hat{\nabla}_\alpha f = 0,$$

since $-\Upsilon_\beta \tau_\alpha - \Upsilon_\alpha \tau_\beta$ is symmetric. Similarly from (5.2.31) we obtain

$$\hat{\nabla}_\alpha \hat{\nabla}_{\bar{\beta}} f - \hat{\nabla}_{\bar{\beta}} \hat{\nabla}_\alpha f = -i\boldsymbol{h}_{\alpha\bar{\beta}} \left(\nabla_0 f + i\Upsilon^{\bar{\gamma}} \nabla_{\bar{\gamma}} f - i\Upsilon^{\gamma} \nabla_{\gamma} f\right),$$

and from Lemma 5.2.7 we have that

$$\hat{\nabla}_0 f = \nabla_0 f + i\Upsilon^{\bar{\gamma}} \nabla_{\bar{\gamma}} f - i\Upsilon^{\gamma} \nabla_{\gamma} f. \tag{5.2.33}$$

From (5.2.32) and (5.2.33) one has that

$$\begin{aligned}\hat{\nabla}_\alpha \hat{\nabla}_0 f - \hat{\nabla}_0 \hat{\nabla}_\alpha f = {} & \hat{\nabla}_\alpha \left(\nabla_0 f - i\Upsilon^{\gamma} \nabla_{\gamma} f + i\Upsilon^{\bar{\gamma}} \nabla_{\bar{\gamma}} f\right) \\ & - \left(\nabla_0 \nabla_\alpha f + i\Upsilon^{\bar{\gamma}} \nabla_{\bar{\gamma}} \nabla_\alpha f - i\Upsilon^{\gamma} \nabla_{\gamma} \nabla_\alpha f\right. \\ & \left. -i(\Upsilon^{\gamma}{}_{\alpha} - \Upsilon^{\gamma}\Upsilon_{\alpha}) \nabla_{\gamma} f\right).\end{aligned} \tag{5.2.34}$$

One can easily compute directly that

$$\hat{\nabla}_\alpha \left(\nabla_0 f - i\Upsilon^\gamma \nabla_\gamma f + i\Upsilon^{\bar{\gamma}} \nabla_{\bar{\gamma}} f\right) = \nabla_\alpha \left(\nabla_0 f - i\Upsilon^\gamma \nabla_\gamma f + i\Upsilon^{\bar{\gamma}} \nabla_{\bar{\gamma}} f\right) + \Upsilon_\alpha \left(\nabla_0 f - i\Upsilon^\gamma \nabla_\gamma f + i\Upsilon^{\bar{\gamma}} \nabla_{\bar{\gamma}} f\right)$$

(cf. the proof of Proposition 5.2.9). Substituting this into (5.2.34), expanding using the Leibniz rule and simplifying one obtains

$$\hat{\nabla}_\alpha \hat{\nabla}_0 f - \hat{\nabla}_0 \hat{\nabla}_\alpha f = \left(A_{\alpha\gamma} + i\Upsilon_{\alpha\gamma} - i\Upsilon_\alpha \Upsilon_\gamma\right) \hat{\nabla}^\gamma f$$

where $\Upsilon_{\alpha\gamma} = \nabla_\alpha \nabla_\gamma \Upsilon$ is symmetric. □

Note that in the course of the proof we have established the transformation law

$$\hat{A}_{\alpha\beta} = A_{\alpha\beta} + i\Upsilon_{\alpha\beta} - i\Upsilon_\alpha \Upsilon_\beta \tag{5.2.35}$$

for the pseudohermitian torsion.

5.2.6.2 The transformation law for the pseudohermitian curvature tensor

From the transformation laws for the Tanaka-Webster connection one can directly compute that

$$\hat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\alpha\bar{\beta}\gamma\bar{\delta}} + \Lambda_{\alpha\bar{\beta}} \boldsymbol{h}_{\gamma\bar{\delta}} + \Lambda_{\gamma\bar{\delta}} \boldsymbol{h}_{\alpha\bar{\beta}} + \Lambda_{\alpha\bar{\delta}} \boldsymbol{h}_{\gamma\bar{\beta}} + \Lambda_{\gamma\bar{\beta}} \boldsymbol{h}_{\alpha\bar{\delta}} \tag{5.2.36}$$

where

$$\Lambda_{\alpha\bar{\beta}} = -\frac{1}{2}(\Upsilon_{\alpha\bar{\beta}} + \Upsilon_{\bar{\beta}\alpha}) - \frac{1}{2}\Upsilon^\gamma \Upsilon_\gamma \boldsymbol{h}_{\alpha\bar{\beta}}. \tag{5.2.37}$$

In particular this tells us that $\hat{S}_{\alpha\bar{\beta}\gamma\bar{\delta}} = S_{\alpha\bar{\beta}\gamma\bar{\delta}}$ and

$$\hat{P}_{\alpha\bar{\beta}} = P_{\alpha\bar{\beta}} + \Lambda_{\alpha\bar{\beta}}. \tag{5.2.38}$$

5.2.6.3 The transformation laws for the Tanaka-Webster connection on densities

We also need to know how the Tanaka-Webster connection transforms when acting on densities. These transformation laws follow from the above since it suffices to compare the action of ∇ and $\hat{\nabla}$ on sections of the canonical bundle $\mathscr{K}$.

Proposition 5.2.9. *[79, Prop. 2.3] Under the transformation $\hat{\theta} = e^{\Upsilon}\theta$ of pseudohermitian contact forms, $\Upsilon \in C^\infty(M)$, the Tanaka-Webster connection acting on sections f of $\mathcal{E}(w, w')$ transforms according to*

$$\begin{aligned}
\hat{\nabla}_\alpha f &= \nabla_\alpha f + w\Upsilon_\alpha f \\
\hat{\nabla}_{\bar{\alpha}} f &= \nabla_{\bar{\alpha}} f + w'\Upsilon_{\bar{\alpha}} f \\
\hat{\nabla}_0 f &= \nabla_0 f + i\Upsilon^{\bar{\gamma}}\nabla_{\bar{\gamma}} f - i\Upsilon^{\gamma}\nabla_{\gamma} f \\
&\qquad + \tfrac{1}{n+2}\left[(w+w')\Upsilon_0 + iw\Upsilon^{\gamma}{}_{\gamma} - iw'\Upsilon^{\bar{\gamma}}{}_{\bar{\gamma}} + i(w'-w)\Upsilon^{\gamma}\Upsilon_{\gamma}\right] f.
\end{aligned}$$

Proof. Since ∇ preserves T and $\Gamma(T^{1,0}M)$ the map

$$I_\theta : \zeta \mapsto (T \lrcorner \zeta)|_{T^{1,0}M}$$

taking sections of $\mathscr{K}$ to sections of $\mathcal{E}_{[\alpha_1 \cdots \alpha_n]}$ commutes with ∇_X for all $X \in \mathfrak{X}(M)$. On the other hand $I_{\hat{\theta}} : \zeta \mapsto (\hat{T} \lrcorner \zeta)|_{T^{1,0}M}$ intertwines the action of the connection $\hat{\nabla}$. Now $I_{\hat{\theta}} = e^{-\Upsilon} \circ I_\theta$ since if $Y = \hat{T} - e^{-\Upsilon}T$ then $(Y \lrcorner \zeta)|_{T^{1,0}M} = 0$ for any $(n+1,0)$-form ζ; to see this note that Y is contact (by Lemma 5.2.6) and the antiholomorphic part of Y hooks into ζ to give zero, but also $\zeta|_{T^{1,0}M} = 0$ since the rank of $T^{1,0}M$ is n. Thus we have

$$\begin{aligned}
I_\theta(\hat{\nabla}_X \zeta) &= e^{\Upsilon} I_{\hat{\theta}}(\hat{\nabla}_X \zeta) = e^{\Upsilon}\hat{\nabla}_X I_{\hat{\theta}}(\zeta) = e^{\Upsilon}\hat{\nabla}_X[e^{-\Upsilon} I_\theta(\zeta)] \\
&= \hat{\nabla}_X I_\theta(\zeta) - \mathrm{d}\Upsilon(X) I_\theta(\zeta)
\end{aligned}$$

for any $X \in \mathfrak{X}(M)$, $\zeta \in \Gamma(\mathscr{K})$. So the action of $\hat{\nabla}$ on $\mathscr{K}$ is conjugate under I_θ to the action of $\hat{\nabla} - \mathrm{d}\Upsilon$ on $\mathcal{E}_{[\alpha_1 \cdots \alpha_n]}$. One now easily translates using I_θ the transformation laws for the Tanaka-Webster connection on $\mathcal{E}_{[\alpha_1 \cdots \alpha_n]}$ (obtained from Proposition 5.2.8 by taking traces) to the transformation laws for the Tanaka-Webster connection on the canonical bundle $\mathscr{K}$. The transformation laws for $\mathcal{E}(w, w')$ then follow from those for $\mathscr{K} = \mathcal{E}(-n-2, 0)$ in the obvious way. □

5.3 The Tractor Calculus

It is well known that nondegenerate (hypersurface type) CR geometries admit an equivalent description as parabolic Cartan geometries. The Cartan geometric description of CR manifolds was introduced by Cartan [37] in the case of 3-dimensional CR manifolds, and by Tanaka [129, 130, 132] and Chern and Moser [41] in the general case. To a signature (p, q) CR manifold (M, H, J) there is associated a canonical Cartan geometry $(\mathcal{G}, \omega)$ of type $(\mathrm{SU}(p+1, q+1), P)$ where the subgroup P of $\mathrm{SU}(p+1, q+1)$ is the stabiliser of a complex null line in $\mathbb{C}^{p+1,q+1}$. Moreover, any local CR diffeomorphism of (M, H, J) with another CR manifold (M', H', J') lifts to a local equivalence of the canonical Cartan geometries $(\mathcal{G}, \omega)$ and $(\mathcal{G}', \omega')$. In the model case of the CR sphere $\mathcal{G}$ is simply the group $G = \mathrm{SU}(n+1, 1)$ as a principal bundle over $\mathbb{S}^{2n+1} = G/P$ and ω is the left Maurer-Cartan form of G. Strictly speaking, if we do not wish to impose any global assumptions in the general case we need to quotient $\mathrm{SU}(p+1, q+1)$ and P by their common finite cyclic center, but for the purpose of local calculus we can ignore this.

Given any representation $\mathbb{V}$ of $\mathrm{SU}(p+1, q+1)$ there is associated to the CR Cartan bundle $\mathcal{G}$ a vector bundle $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$ over M. The CR Cartan connection ω induces on $\mathcal{V}$ a linear connection $\nabla^{\mathcal{V}}$. Such bundles $\mathcal{V}$ are known as *tractor bundles*, and the connection $\nabla^{\mathcal{V}}$ is the (canonical) *tractor connection* [27]. If $\mathbb{T}$ is the standard representation $\mathbb{C}^{p+1,q+1}$ of $\mathrm{SU}(p+1, q+1)$ then $\mathcal{T} = \mathcal{G} \times_P \mathbb{T}$ is known as the *standard tractor bundle.* Since $\mathbb{T}$ is a faithful representation of P the CR Cartan bundle $\mathcal{G}$ may be recovered from $\mathcal{T}$ as an adapted frame bundle. The Cartan connection ω is easily recovered from $\nabla^{\mathcal{T}}$. Elementary representation theory tells us that all other irreducible representations of $\mathrm{SU}(p+1, q+1)$ are subbundles of tensor representations constructed from $\mathbb{T}$ (and $\mathbb{T}^*$) given by imposing certain tensor symmetries, so knowing the standard tractor bundle $\mathcal{T}$ and its tractor connection one can easily explicitly obtain all tractor bundles and connections.

The tractor bundles and their tractor connections, along with certain invariant differential (splitting) operators from irreducible tensor bundles on the CR manifold into tractor bundles, form the basis of a calculus of local invariants and invariant operators for CR manifolds known as the *CR tractor calculus.*

5.3.1 The Standard Tractor Bundle

There are various ways to construct the CR Cartan bundle and hence the standard tractor bundle. However for our purposes it is much better to use the direct construction of the CR standard tractor bundle and connection found in [79]. This allows a very concrete description of the standard tractor bundle and connection in terms of the weighted tensor bundles and Tanaka-Webster calculus of §5.2.

Since the subgroup P of $\mathrm{SU}(p+1, q+1)$ stabilises a null complex line in $\mathcal{T}$ it also stabilises the orthogonal complement of this null line and so there is a filtration

$$\mathcal{T}^1 \subset \mathcal{T}^0 \subset \mathcal{T}^{-1} = \mathcal{T} \tag{5.3.1}$$

of $\mathcal{T}$ by subbundles where $\mathcal{T}^1$ has complex rank 1 and $\mathcal{T}^0$ has complex rank $n+1$ (and corank 1). The starting point for the explicit construction of $\mathcal{T}$ in [79] is the observation that

$$\mathcal{T}^1 = \mathcal{E}(-1,0), \;\; \mathcal{T}^0/\mathcal{T}^1 = \mathcal{E}^\alpha(-1,0), \;\text{ and }\; \mathcal{T}^{-1}/\mathcal{T}^0 = \mathcal{E}(0,1). \tag{5.3.2}$$

Let us introduce abstract index notation $\mathcal{E}^A$ for $\mathcal{T}$, allowing the use of capitalised Latin indices from the start of the alphabet. The dual of $\mathcal{T} = \mathcal{E}^A$ is denoted by $\mathcal{E}_A$ and the conjugate by $\mathcal{E}^{\bar{A}}$. Following [79] we present the standard cotractor bundle $\mathcal{E}_A$ rather than $\mathcal{E}^A$ (it makes little difference since there will be a parallel Hermitian metric around). The bundle $\mathcal{E}_A$ comes with a naturally defined filtration, dual to (5.3.1). Given a choice of contact form θ for (M, H, J) we identify the standard cotractor bundle $\mathcal{E}_A$ with

$$[\mathcal{E}_A]_\theta = \mathcal{E}(1,0) \oplus \mathcal{E}_\alpha(1,0) \oplus \mathcal{E}(0,-1);$$

we write $v_A \overset{\theta}{=} (\sigma, \tau_\alpha, \rho)$,

$$v_A \overset{\theta}{=} \begin{pmatrix} \sigma \\ \tau_\alpha \\ \rho \end{pmatrix}, \quad \text{or} \quad [v_A]_\theta = \begin{pmatrix} \sigma \\ \tau_\alpha \\ \rho \end{pmatrix}$$

if an element or section of $\mathcal{E}_A$ is represented by $(\sigma, \tau_\alpha, \rho)$ with respect to this identification; the identifications given by two contact forms θ and $\hat{\theta} = e^\Upsilon \theta$ are related by the transformation law

$$[\mathcal{E}_A]_\theta \ni \begin{pmatrix} \sigma \\ \tau_\alpha \\ \rho \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_\alpha & \delta_\alpha^\beta & 0 \\ -\frac{1}{2}(\Upsilon^\beta \Upsilon_\beta + i\Upsilon_0) & -\Upsilon^\beta & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \tau_\beta \\ \rho \end{pmatrix} \in [\mathcal{E}_A]_{\hat{\theta}} \tag{5.3.3}$$

where $\Upsilon_\alpha = \nabla_\alpha \Upsilon$ and $\Upsilon_0 = \nabla_0 \Upsilon$. This transformation law comes from the action of the nilpotent part P_+ of P on $\mathbb{T}^*$ (see [33] for the general theory) so that $\sim$ is indeed an equivalence relation on the disjoint union of the spaces $[\mathcal{E}_A]_\theta$. We can thus take the standard cotractor bundle $\mathcal{E}_A$ to be the quotient of the disjoint union of the $[\mathcal{E}_A]_\theta$ over all pseudohermitian contact forms θ by the equivalence relation (5.3.3).

5.3.2 Splitting Tractors

From (5.3.3) it is clear that there is an invariant inclusion of $\mathcal{E}(0,-1)$ into $\mathcal{E}_A$ given with respect to any contact form θ by the map

$$\rho \mapsto \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}.$$

Correspondingly there is an invariant section Z_A of $\mathcal{E}_A(0,1)$ such that the above displayed map is given by $\rho \mapsto \rho Z_A$. The weight $(0,1)$ *canonical tractor* Z_A can be written as

$$Z_A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

with respect to any choice of contact form θ.

Given a fixed choice of θ, we also get the corresponding *splitting tractors*

$$W_A^\beta \overset{\theta}{=} \begin{pmatrix} 0 \\ \delta_\alpha^\beta \\ 0 \end{pmatrix} \quad \text{and} \quad Y_A \overset{\theta}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which both have weight $(-1,0)$. A standard cotractor $v_A \overset{\theta}{=} (\sigma, \tau_\alpha, \rho)$ may instead be written as $v_A = \sigma Y_A + W_A^\beta \tau_\beta + \rho Z_A$ were we understand that Y_A and W_A^β are defined in terms of the splitting induced by θ. If $\hat{\theta} = e^\Upsilon \theta$ then by (5.3.3) we have

$$\hat{W}_A^\beta = W_A^\beta + \Upsilon^\beta Z_A, \tag{5.3.4}$$

$$\hat{Y}_A = Y_A - \Upsilon_\beta W_A^\beta - \frac{1}{2}(\Upsilon_\beta \Upsilon^\beta - i\Upsilon_0) Z_A. \tag{5.3.5}$$

5.3.3 The Tractor Metric

Since the group P preserves the inner product on $\mathbb{T} = \mathbb{C}^{p+1,q+1}$ the standard tractor bundle $\mathcal{T} = \mathcal{G} \times_P \mathbb{T}$ carries a natural signature $(p+1, q+1)$ Hermitian bundle metric.

We denote this bundle metric by $h_{A\bar{B}}$, and its inverse by $h^{A\bar{B}}$. Explicitly the tractor metric $h_{A\bar{B}}$ is given with respect to any contact form θ by

$$h_{A\bar{B}} = Z_A Y_{\bar{B}} + \boldsymbol{h}_{\alpha\bar{\beta}} W^{\alpha}_{A} W^{\bar{\beta}}_{\bar{B}} + Y_A Z_{\bar{B}} \tag{5.3.6}$$

where $Z_{\bar{B}}$, $W^{\bar{\beta}}_{\bar{B}}$, and $Y_{\bar{B}}$ are the respective conjugates of Z_B, W^{β}_{B}, and Y_B. One can easily check directly using (5.3.4) and (5.3.5) that the above expression does not depend on the choice of θ. Dually, the inverse tractor metric is given by

$$h^{A\bar{B}} v'_A \overline{v_B} = \sigma\overline{\rho'} + \boldsymbol{h}^{\alpha\bar{\beta}} \tau_\alpha \overline{\tau'_\beta} + \rho\overline{\sigma'} \tag{5.3.7}$$

for any two sections $v_A \overset{\theta}{=} (\sigma, \tau_\alpha, \rho)$ and $v'_A \overset{\theta}{=} (\sigma', \tau'_\alpha, \rho')$ of $\mathcal{E}_A$.

We use the tractor metric to identify $\mathcal{E}^A$ with $\mathcal{E}_{\bar{A}}$, the latter of which can be described explicitly as the disjoint union of the spaces

$$[\mathcal{E}_{\bar{A}}]_\theta = \mathcal{E}(0,1) \oplus \mathcal{E}_{\bar{\alpha}}(0,1) \oplus \mathcal{E}(-1,0) \tag{5.3.8}$$

(over all pseudohermitian contact forms θ) modulo the equivalence relation obtained by conjugating (5.3.3). Identifying $\mathcal{E}_{\bar{\alpha}}(0,1)$ with $\mathcal{E}^{\alpha}(-1,0)$ via the CR Levi form we write a standard tractor as $v^A \overset{\theta}{=} (\sigma, \tau^\alpha, \rho)$ or

$$v^A \overset{\theta}{=} \begin{pmatrix} \sigma \\ \\ \tau^\alpha \\ \\ \rho \end{pmatrix} \in \begin{matrix} \mathcal{E}(0,1) \\ \oplus \\ \mathcal{E}^\alpha(-1,0) \\ \oplus \\ \mathcal{E}(-1,0) \end{matrix} .$$

We may also raise and lower indices on the splitting tractors in order to write $v^A = \sigma Y^A + \tau^\beta W^A_\beta + \rho Z^A$ with respect to θ.

With these conventions the pairing between $v^A \overset{\theta}{=} (\sigma, \tau^\alpha, \rho)$ and $v'_A \overset{\theta}{=} (\sigma', \tau'_\alpha, \rho')$ is given by

$$v^A v'_A = \sigma\rho' + \tau^\alpha \tau'_\alpha + \rho\sigma'. \tag{5.3.9}$$

The various contractions of the splitting tractors (for a given θ) are described by the following table

$$\begin{array}{c|ccc} & Y_A & W_{A\bar{\beta}} & Z_A \\ \hline Y^A & 0 & 0 & 1 \\ W^A_\alpha & 0 & \boldsymbol{h}_{\alpha\bar{\beta}} & 0 \\ Z^A & 1 & 0 & 0 \end{array} \tag{5.3.10}$$

which reflects the form of the tractor metric $h_{A\bar{B}}$.

5.3.4 The Tractor Connection

In order to define the canonical (normal) tractor connection we need two further curvature objects. These are

$$T_\alpha = \frac{1}{n+2}(\nabla_\alpha P_\beta{}^\beta - i\nabla^\beta A_{\alpha\beta}) \tag{5.3.11}$$

and

$$S = -\frac{1}{n}(\nabla^\alpha T_\alpha + \nabla^{\bar{\alpha}} T_{\bar{\alpha}} + P_{\alpha\bar{\beta}} P^{\alpha\bar{\beta}} - A_{\alpha\beta} A^{\alpha\beta}). \tag{5.3.12}$$

These expressions appear in [79] and can be determined from the following formulae for the tractor connection by the normalisation condition on the tractor curvature (which amounts to certain traces of curvature tensors vanishing, see [29]). Of course the S and T_α terms are also needed to make sure that the formulae for the tractor connection given below transform correctly so as to give a well-defined connection on $\mathcal{E}_A$.

On any section $v_A \overset{\theta}{=} (\sigma, \tau_\alpha, \rho)$ of $\mathcal{E}_A$ the *standard tractor connection* $\nabla^\mathcal{T}$ (or simply ∇) is defined by the following formulae

$$\nabla_\beta v_A \overset{\theta}{=} \begin{pmatrix} \nabla_\beta \sigma - \tau_\beta \\ \nabla_\beta \tau_\alpha + i A_{\alpha\beta}\sigma \\ \nabla_\beta \rho - P_\beta{}^\alpha \tau_\alpha + T_\beta \sigma \end{pmatrix}, \tag{5.3.13}$$

$$\nabla_{\bar{\beta}} v_A \overset{\theta}{=} \begin{pmatrix} \nabla_{\bar{\beta}} \sigma \\ \nabla_{\bar{\beta}} \tau_\alpha + \boldsymbol{h}_{\alpha\bar{\beta}}\rho + P_{\alpha\bar{\beta}}\sigma \\ \nabla_{\bar{\beta}} \rho - i A_{\bar{\beta}}{}^\alpha \tau_\alpha - T_{\bar{\beta}} \sigma \end{pmatrix}, \tag{5.3.14}$$

and

$$\nabla_0 v_A \overset{\theta}{=} \begin{pmatrix} \nabla_0 \sigma + \frac{i}{n+2} P\sigma - i\rho \\ \nabla_0 \tau_\alpha + \frac{i}{n+2} P\tau_\alpha - i P_\alpha{}^\beta \tau_\beta + 2i T_\alpha \sigma \\ \nabla_0 \rho + \frac{i}{n+2} P\rho + 2i T^\alpha \tau_\alpha + i S\sigma \end{pmatrix} \tag{5.3.15}$$

where $P = P_\beta{}^{\bar{\beta}}$. Using (5.2.35), (5.2.38), and Proposition 5.2.8 combined with Proposition 5.2.9 one may check directly that the above formulae transform appropriately under rescaling of the contact form θ (i.e. are compatible with (5.3.3), and with Lemma 5.2.6 in the case of (5.3.15)) so that they give a well-defined connection on $\mathcal{E}_A$.

Coupling the tractor connection with the Tanaka-Webster connection of some contact form θ, the tractor connection is given on the splitting operators by (cf. [29], and also [54, Proposition 3.1])

$$\begin{aligned}
\nabla_\beta Y_A &= iA_{\alpha\beta}W_A^\alpha + T_\beta Z_A && (5.3.16)\\
\nabla_\beta W_A^\alpha &= -\delta_\beta^\alpha Y_A - P_\beta{}^\alpha Z_A && (5.3.17)\\
\nabla_\beta Z_A &= 0 && (5.3.18)\\
\nabla_{\bar\beta} Y_A &= P_{\alpha\bar\beta}W_A^\alpha - T_{\bar\beta} Z_A && (5.3.19)\\
\nabla_{\bar\beta} W_A^\alpha &= iA_{\bar\beta}^\alpha Z_A && (5.3.20)\\
\nabla_{\bar\beta} Z_A &= \boldsymbol{h}_{\alpha\bar\beta}W_A^\alpha, && (5.3.21)
\end{aligned}$$

and

$$\begin{aligned}
\nabla_0 Y_A &= \tfrac{i}{n+2}PY_A + 2iT_\alpha W_A^\alpha + iSZ_A && (5.3.22)\\
\nabla_0 W_A^\alpha &= -iP_\beta{}^\alpha W_A^\beta + \tfrac{i}{n+2}PW_A^\alpha + 2iT^\alpha Z_A && (5.3.23)\\
\nabla_0 Z_A &= -iY_A + \tfrac{i}{n+2}PZ_A. && (5.3.24)
\end{aligned}$$

Using either set of formulae for the tractor connection one can easily show by direct calculation that ∇ preserves $h_{A\bar B}$.

5.3.4.1 Weyl connections on the tangent bundle

The expression (5.3.15) for $\nabla_0 v_A$ may be simplified if one absorbs the terms involving $P_\alpha{}^\beta$ and its trace P into the definition of the connection on the tangent bundle we are using. This amounts to working with, in the terminology of [33], the *Weyl connection* determined by θ rather than the Tanaka-Webster connection of θ. The Weyl connection ∇^W determined by θ agrees with the Tanaka-Webster connection when differentiating in contact directions, but when differentiating in the Reeb direction one has

$$\nabla_0^W \tau_\alpha = \nabla_0 \tau_\alpha - iP_\alpha{}^\beta \tau_\beta \qquad (5.3.25)$$

for a section τ_α of $\mathcal{E}_\alpha$ (the action on $\mathcal{E}_{\bar{\alpha}}$ is given by conjugating (5.3.25) and $\nabla_0^W T = 0$). Using the isomorphism I_θ of $\mathcal{E}(-n-2, 0)$ with $\mathcal{E}_{[\alpha_1 \cdots \alpha_n]}$ from the proof of Proposition 5.2.9 one obtains from (5.3.25) that

$$\nabla_0^W \sigma = \nabla_0 \sigma + \frac{i}{n+2} P\sigma \tag{5.3.26}$$

for a section σ of $\mathcal{E}(1, 0)$. Using the Weyl connection rather than the Tanaka-Webster connection in the expression (5.3.15) for $\nabla_0 v_A$ one has the simpler expression

$$\nabla_0 v_A \stackrel{\theta}{=} \begin{pmatrix} \nabla_0^W \sigma - i\rho \\ \nabla_0^W \tau_\alpha + 2iT_\alpha \sigma \\ \nabla_0^W \rho + 2iT^\alpha \tau_\alpha + iS\sigma \end{pmatrix}. \tag{5.3.27}$$

5.3.5 The Adjoint Tractor Bundle

Another important bundle on CR manifolds is the *adjoint tractor bundle* $\mathcal{A} = \mathcal{G} \times_P \mathfrak{g}$. Since $\mathfrak{g}$ is the space of skew-Hermitian endomorphisms of $\mathbb{C}^{p+1,q+1}$ with respect to the signature $(p+1, q+1)$ inner product we may identify $\mathcal{A}$ with the bundle of $h_{A\bar{B}}$-skew-Hermitian endomorphisms of the standard tractor bundle. Thus we think of $\mathcal{A}$ as the subbundle of $\mathcal{E}_A{}^B = \mathcal{E}_A \otimes \mathcal{E}^B$ whose sections $t_A{}^B$ satisfy

$$t_{A\bar{B}} = -\overline{t_{B\bar{A}}}.$$

Since the standard tractor connection $\nabla^{\mathcal{T}}$ is Hermitian, it induces a connection on $\mathcal{A} \subset \mathrm{End}(\mathcal{T})$ and this is the usual (normal) tractor connection on $\mathcal{A}$.

The adjoint tractor bundle carries a natural filtration

$$\mathcal{A}^2 \subset \mathcal{A}^1 \subset \mathcal{A}^0 \subset \mathcal{A}^{-1} \subset \mathcal{A}^{-2} = \mathcal{A} \tag{5.3.28}$$

corresponding to a P-invariant filtration of $\mathfrak{su}(p+1, q+1)$. In particular, $\mathcal{A}^0 = \mathcal{G} \times_P \mathfrak{p}$ where $\mathfrak{p} = \mathrm{Lie}(P)$. Sections t of $\mathcal{A}^{-1}$ are those skew-Hermitian endomorphisms which satisfy $t_A{}^B Z^A Z_B = 0$, and sections $\mathcal{A}^0$ are those which additionally satisfy $t_A{}^B Z^A W_B^\beta = 0$. In any parabolic geometry the subbundle $\mathcal{A}^1 = \mathcal{G} \times \mathfrak{p}_+$, where $\mathfrak{p}_+$ is the nilpotent part of $\mathfrak{p}$ (in this case $\mathfrak{p}_+$ is a Heisenberg algebra), is canonically isomorphic to T^*M. Here the isomorphism is given explicitly by the map

$$(v_\alpha, v_{\bar{\alpha}}, v_0) \mapsto v_\alpha W_A^\alpha Z_{\bar{B}} - v_{\bar{\beta}} Z_A W_{\bar{B}}^{\bar{\beta}} - i v_0 Z_A Z_{\bar{B}} \tag{5.3.29}$$

with respect to any contact form θ. Dual to (5.3.29) there is a bundle projection from $\mathcal{A}^* \cong \mathcal{A}$ to TM. Explicitly, the resulting isomorphism of $\mathcal{A}/\mathcal{A}^0$ with TM is given with respect to θ by

$$X^\alpha W^A_\alpha Y^{\bar{B}} - X^{\bar{\beta}} Y^A W^{\bar{B}}_{\bar{\beta}} + iX^0 Y^A Y^{\bar{B}} + \mathcal{A}^0 \mapsto (X^\alpha, X^{\bar{\alpha}}, X^0). \tag{5.3.30}$$

5.3.6 The Tractor Curvature

The curvature of the standard tractor connection agrees with the usual ($\mathfrak{g}$-valued) curvature of the canonical Cartan connection when the latter is thought of as an adjoint tractor ($\mathcal{A} = \mathcal{G} \times_P \mathfrak{g}$) valued two form. To normalise our conventions with the index notation we define the curvature of the tractor bundle by

$$\nabla_a \nabla_b v_C - \nabla_b \nabla_a v_C + T^{\nabla e}_{ab} \nabla_e v_C = -\kappa_{abC}{}^D v_D \tag{5.3.31}$$

for all sections v_C of $\mathcal{E}_C$, where ∇ denotes the tractor connection coupled with any connection on the tangent bundle and T^∇ is the torsion of that connection on the tangent bundle. Since we allow for the use of connections with torsion on the tangent bundle in the above, we may compute $\kappa_{abC}{}^D v_D$ explicitly in its decomposition with respect to any contact form θ using the weighted Tanaka-Webster calculus developed in §5.2. The resulting expressions were given in [79] (cf. [41]); we have

$$\kappa_{\alpha\beta C}{}^D = 0, \tag{5.3.32}$$

$$\kappa_{\alpha\bar{\beta}C}{}^D \overset{\theta}{=} \begin{pmatrix} 0 & 0 & 0 \\ iV_{\alpha\bar{\beta}\gamma} & S_{\alpha\bar{\beta}\gamma}{}^\delta & 0 \\ U_{\alpha\bar{\beta}} & -iV_{\alpha\bar{\beta}}{}^\delta & 0 \end{pmatrix}, \tag{5.3.33}$$

$$\kappa_{\alpha 0 C}{}^D \overset{\theta}{=} \begin{pmatrix} 0 & 0 & 0 \\ Q_{\alpha\gamma} & V_\alpha{}^\delta{}_\gamma & 0 \\ Y_\alpha & -iU_\alpha{}^\delta & 0 \end{pmatrix}, \tag{5.3.34}$$

where

$$V_{\alpha\bar{\beta}\gamma} = \nabla_{\bar{\beta}} A_{\alpha\gamma} + i\nabla_\gamma P_{\alpha\bar{\beta}} - iT_\gamma \boldsymbol{h}_{\alpha\bar{\beta}} - 2iT_\alpha \boldsymbol{h}_{\gamma\bar{\beta}}, \tag{5.3.35}$$

$$U_{\alpha\bar{\beta}} = \nabla_\alpha T_{\bar{\beta}} + \nabla_{\bar{\beta}} T_\alpha + P_\alpha{}^\gamma P_{\gamma\bar{\beta}} - A_{\alpha\gamma} A^\gamma_{\bar{\beta}} + S\boldsymbol{h}_{\alpha\bar{\beta}}, \tag{5.3.36}$$

$$Q_{\alpha\bar{\beta}} = i\nabla_0 A_{\alpha\beta} - 2i\nabla_\beta T_\alpha + 2P_\alpha{}^\gamma A_{\gamma\beta}, \tag{5.3.37}$$

$$Y_\alpha = \nabla_0 T_\alpha - i\nabla_\alpha S + 2iP_\alpha^\gamma T_\gamma - 3A_\alpha^{\bar{\gamma}} T_{\bar{\gamma}}. \tag{5.3.38}$$

Here the matrices appearing in (5.3.33) and (5.3.34) are arranged so that the action of $\kappa_{\alpha\bar{\beta}C}{}^D$ and $\kappa_{\alpha 0 C}{}^D$ on v_D is given (with respect to θ) by the action of the respective matrices on the column vector representing v_D. The remaining components of the tractor/Cartan curvature κ are determined by the obvious symmetries. We also have from [79] (again cf. [41]) that

$$V_{\alpha\bar{\beta}\gamma} = V_{\gamma\bar{\beta}\alpha}, \quad V_\alpha{}^\alpha{}_\gamma = 0,$$
$$U_{\alpha\bar{\beta}} = \overline{U_{\beta\bar{\alpha}}}, \quad U_\alpha{}^\alpha = 0,$$
$$\text{and } \; Q_{\alpha\beta} = Q_{\alpha\beta}.$$

The tractor connection on the CR sphere $\mathbb{S}^{2n+1}$ is flat, and for a general strictly pseudoconvex CR manifold the tractor curvature is precisely the obstruction to being locally CR equivalent to the sphere (see Theorem 6.5.1).

5.3.7 Invariant Tractor Operators

The tractor calculus can be used to give a uniform construction of curved analogues for almost all CR invariant differential operators between irreducible bundles on the model CR sphere. The key idea behind this is to apply Eastwood's 'curved translation principle' [52, 50] to the tractor covariant exterior derivative d^∇ using certain invariant *differential splitting operators* constructed via the 'BGG machinery' of [35, 25]. Important exceptional cases are dealt with in [79] where the authors construct CR invariant powers of the sublaplacian on curved CR manifolds using the tractor D-operator (which extends one of the BGG splitting operators to a family of operators parametrised by weight). Such invariant differential splitting operators are also very useful in the problem of constructing invariants of CR structures, since they allow the jets of the structure (or rather of some invariant curvature tensor) to be packaged in a tractorial object which can be further differentiated invariantly. In the following we present the most basic and important of these (families of) invariant operators.

5.3.7.1 The tractor D-operator(s)

Let $\mathcal{E}^\Phi$ denote any tractor bundle and let $\mathcal{E}^\Phi(w,w')$ denote the tensor product of $\mathcal{E}^\Phi$ with $\mathcal{E}(w,w')$.

Definition 5.3.1. The *tractor D-operator* of [79]

$$\mathsf{D}_A : \mathcal{E}^\Phi(w,w') \to \mathcal{E}_A \otimes \mathcal{E}^\Phi(w-1,w')$$

is defined by

$$\mathsf{D}_A f^\Phi \overset{\theta}{=} \begin{pmatrix} w(n+w+w')f^\Phi \\ (n+w+w')\nabla_\alpha f^\Phi \\ -\left(\nabla^\beta\nabla_\beta f^\Phi + iw\nabla_0 f^\Phi + w(1+\frac{w'-w}{n+2})Pf^\Phi\right) \end{pmatrix} \tag{5.3.39}$$

where ∇ denotes the tractor connection coupled to the Tanaka-Webster connection of θ.

One may easily check directly that D_A, as defined, does not depend on the choice of θ. This operator is an analogue of the Thomas tractor D-operator in conformal geometry [7]. Observe that D_A is a splitting operator (has a bundle map as left inverse) except at weights where $w(n+w+w')=0$.

Related to the tractor D-operator is the θ dependent operator $\tilde{\mathsf{D}}_A$ given by $wf^\Phi Y_A + (\nabla_\alpha f^\Phi)W^\alpha_A$ on a section f^Φ of $\mathcal{E}^\Phi(w,w')$. The operator $\mathbb{D}_{AB}$ defined by

$$\mathbb{D}_{AB} f^\Phi = 2Z_{[A}\tilde{\mathsf{D}}_{B]} f^\Phi \tag{5.3.40}$$

does not depend on the choice of θ. The operator $\mathbb{D}_{AB}$ has a partner $\mathbb{D}_{A\bar{B}}$ defined by

$$\mathbb{D}_{A\bar{B}} f^\Phi = Z_{\bar{B}}\tilde{\mathsf{D}}_A f^\Phi - Z_A\tilde{\mathsf{D}}_{\bar{B}} f^\Phi - Z_A Z_{\bar{B}}\left[i\nabla_0 f^\Phi + \tfrac{w'-w}{n+2}Pf^\Phi\right] \tag{5.3.41}$$

where $\tilde{\mathsf{D}}_{\bar{B}} f^\Phi = \overline{\tilde{\mathsf{D}}_B \overline{f^\Phi}}$. Note that if f^Φ has weight $(0,0)$ then

$$\mathbb{D}_{A\bar{B}} f^\Phi = Z_{\bar{B}} W^\alpha_A \nabla_\alpha f^\Phi - Z_A W^{\bar{\beta}}_{\bar{B}} \nabla_{\bar{\beta}} f^\Phi - iZ_A Z_{\bar{B}}\nabla_0 f^\Phi, \tag{5.3.42}$$

cf. (5.3.29), so that $\mathbb{D}_{A\bar{B}}$ takes sections of $\mathcal{E}^\Phi$ to sections of $\mathcal{A} \otimes \mathcal{E}^\Phi$. The pair of invariant operators $\mathbb{D}_{AB}$ and $\mathbb{D}_{A\bar{B}}$ acting on sections of $\mathcal{E}^\Phi(w, w')$ are called *double-D-operators* [74].

Remark 5.3.2. The less obvious operator $\mathbb{D}_{A\bar{B}}$ comes from coupling the fundamental derivative of [27] on densities with the tractor connection to give an operator on weighted tractors. The conformal double-D-operator on the Fefferman space, which comes from similarly twisting the fundamental derivative on conformal densities with the conformal tractor connection, can be seen to induce the pair of operators $\mathbb{D}_{A\bar{B}}$, $\mathbb{D}_{AB}$ (and the conjugate operator $\mathbb{D}_{\bar{A}\bar{B}}$) on the underlying CR manifold [29, Theorem 3.7]. ∎

5.3.7.2 Middle operators

One can also create CR invariant differential splitting operators which take weighted sections of tensor bundles of $\mathcal{E}^\alpha$ to weighted tractors. These are analogues of operators in conformal geometry used by Eastwood for 'curved translation' (see, e.g., [50]). We only construct the particular operators from this family that we will need in the following.

Definition 5.3.3. The *middle operator* acting on sections of $\mathcal{E}_\alpha(w, w')$ is the operator $\mathsf{M}_A^\alpha : \mathcal{E}_\alpha(w, w') \to \mathcal{E}_A(w-1, w')$ given with respect to a choice of contact form θ by

$$\mathsf{M}_A^\alpha \tau_\alpha = (n + w') W_A^\alpha \tau_\alpha - Z_A \nabla^\alpha \tau_\alpha. \tag{5.3.43}$$

To see that the operator defined by (5.3.43) is invariant one simply observes (by combining Proposition 5.2.8 with Proposition 5.2.9) that if $\hat{\theta} = e^\Upsilon \theta$ then

$$\hat{\nabla}^\alpha \tau_\alpha = \nabla^\alpha \tau_\alpha + (n + w') \Upsilon^\alpha \tau_\alpha \tag{5.3.44}$$

for τ_α of weight (w, w'), and on the other hand from (5.3.4) we have $\hat{W}_A^\alpha = W_A^\alpha + \Upsilon^\alpha Z_A$.

Remark 5.3.4. The operator M_A^α defined by (5.3.43) is a differential splitting operator, except when $w' = -n$, in which case $\tau_\alpha \mapsto \hat{\nabla}^\alpha \tau_\alpha$ is an invariant operator and M_A^α simply becomes (minus) the composition of this operator with the bundle map $\rho \mapsto \rho Z_A$ (for ρ of appropriate weight). ∎

In the same manner, by observing that when $\hat{\theta} = e^\Upsilon \theta$ we have

$$\hat{\nabla}^\alpha \tau_{\alpha\bar{\beta}} = \nabla^\alpha \tau_{\alpha\bar{\beta}} + (n + w' - 1) \Upsilon^\alpha \tau_{\alpha\bar{\beta}} - \Upsilon_{\bar{\beta}} \tau_\alpha{}^\alpha \tag{5.3.45}$$

for $\tau_{\alpha\bar{\beta}}$ of weight (w, w'), we see that there is an invariant operator on trace free sections of $\mathcal{E}_{\alpha\bar{\beta}}(w, w')$ given by

$$\mathsf{M}^{\alpha}_{A}\tau_{\alpha\bar{\beta}} = (n + w' - 1)W^{\alpha}_{A}\tau_{\alpha\bar{\beta}} - Z_A\nabla^{\alpha}\tau_{\alpha\bar{\beta}}. \tag{5.3.46}$$

Conjugating one obtains an operator $\mathsf{M}^{\bar{\beta}}_{\bar{B}}$ on trace free sections of $\mathcal{E}_{\alpha\bar{\beta}}(w, w')$ given by

$$\mathsf{M}^{\bar{\beta}}_{\bar{B}}\tau_{\alpha\bar{\beta}} = (n + w - 1)W^{\bar{\beta}}_{\bar{B}}\tau_{\alpha\bar{\beta}} - Z_{\bar{B}}\nabla^{\bar{\beta}}\tau_{\alpha\bar{\beta}}. \tag{5.3.47}$$

5.3.8 The Curvature Tractor

The operators M^{α}_{A} defined above are all first order, so in particular they are 'strongly invariant' meaning that we may couple the Tanaka-Webster connection ∇ used in their definitions with the tractor connection (on any tractor bundle $\mathcal{E}^{\Phi}$) to obtain invariant operators M^{α}_{A} on sections of $\mathcal{E}_{\alpha}{}^{\Phi}(w, w')$ and on trace free sections of $\mathcal{E}_{\alpha\bar{\beta}}{}^{\Phi}(w, w')$. We use these strongly invariant middle operators to define a CR analogue of the conformal 'W-tractor' of [74].

Definition 5.3.5. The *curvature tractor* of a CR manifold is the section of $\mathcal{E}_{A\bar{B}C\bar{D}}$ given by

$$\kappa_{A\bar{B}C\bar{D}} = \mathsf{M}^{\alpha}_{A}\mathsf{M}^{\bar{\beta}}_{\bar{B}}\kappa_{\alpha\bar{\beta}C\bar{D}} \tag{5.3.48}$$

where $\kappa_{\alpha\bar{\beta}C\bar{D}} = -\kappa_{\alpha\bar{\beta}\bar{D}C} = -\kappa_{\alpha\bar{\beta}}{}^{E}{}_{C}h_{E\bar{D}}$.

Remark 5.3.6. The expression for the curvature tractor $\kappa_{A\bar{B}C\bar{D}}$ does not involve the (θ-dependent) component $\kappa_{\alpha 0 C\bar{D}}$ of the tractor curvature $\kappa_{abC\bar{D}}$. One way to include this component in a CR invariant tractor is to define

$$\kappa_{AB\bar{B}'C\bar{D}} = \mathsf{M}^{\alpha}_{A}\left(\kappa_{\alpha\beta C\bar{D}}W^{\beta}_{B}Z_{\bar{B}'} - \kappa_{\alpha\bar{\beta}C\bar{D}}Z_B W^{\bar{\beta}}_{\bar{B}'} - i\kappa_{\alpha 0 C\bar{D}}Z_B Z_{\bar{B}'}\right), \tag{5.3.49}$$

where we have used the map $T^*M \hookrightarrow \mathcal{A}$, given explicitly by (5.3.29), on the 'b' index of $\kappa_{abC\bar{D}}$ to obtain $\kappa_{aB\bar{B}'C\bar{D}}$ and then applied M^{α}_{A} to $\kappa_{\alpha B\bar{B}'C\bar{D}}$. Alternatively one can apply the tensorial map $T^*M \hookrightarrow \mathcal{A}$ to both the 'a' and 'b' indices of $\kappa_{abC\bar{D}}$ to obtain $\kappa_{A\bar{A}'B\bar{B}'C\bar{D}}$ (as is done in §§6.4.2). ∎

5.3.9 Projecting Parts

If a standard tractor v lies in subbundle $\mathcal{T}^s$ of $\mathcal{T}$, s=1,0,-1 (see (5.3.1)), then the image of v under the projection

$$\mathcal{T}^s \to \mathcal{T}^s/\mathcal{T}^{s+1}$$

(where the subbundle $\mathcal{T}^2$ is the zero section) is called a *projecting part* of v. A projecting part may be zero. Since the filtration of the standard tractor bundle induces a filtration of all corresponding tensor bundles (and hence all tractor bundles), we may define a notion of projecting part(s) similarly for sections of any tractor bundle.

The notion can be easily formalised using the splitting tractors of §§5.3.2. The invariant 'top slot' $v^A Z_A$ of a standard tractor is always a projecting part. If this top slot vanishes, then the 'middle slot' $v^A W_A^\alpha$ is independent of the choice of θ by (5.3.4) and is a projecting part. If both $v^A Z_A = 0$ and $v^A W_A^\alpha = 0$, then the 'bottom slot' $v^A Y_A$ is independent of the choice of θ by (5.3.5) and is a projecting part of v^A.

To see how this works for higher valence tractors consider a tractor t^{AB} in $\mathcal{E}^{[AB]}$. Skewness implies $t^{AB} Z_A Z_B = 0$, so $t^{AB} Z_A W_B^\beta$ is independent of the choice of θ by (5.3.4) and is a projecting part. If $t^{AB} Z_A W_B^\beta = 0$ then both $t^{AB} W_A^\alpha W_B^\beta$ and $t^{AB} Z_A Y_B$ are independent of the choice of θ, and are both called projecting parts (the relevant composition factor of $\mathcal{E}^{[AB]}$ splits as a direct sum).

6 The Geometry of CR Embeddings

6.1 CR Embedded Submanifolds and Contact Forms

We now turn to the main subject of the article. We now suppose that $\iota : \Sigma \hookrightarrow M$ is a CR embedding of a nondegenerate CR manifold $(\Sigma^{2m+1}, H_\Sigma, J_\Sigma)$ into (M^{2n+1}, H, J), that is ι is an embedding for which $T\iota$ maps H_Σ into H and

$$J \circ T\iota = T\iota \circ J_\Sigma.$$

Equivalently (the complex linear extension of) $T\iota$ maps $T^{1,0}\Sigma$ into $T^{1,0}M$.

Suppose (M^{2n+1}, H, J) has signature (p, q). Without loss of generality $q \leq p$ (q is often alternatively called the signature). If $q < m$ (in particular, if M is strictly pseudoconvex) then $T_x\iota(T_x\Sigma) \not\subset H_x$ for all $x \in \Sigma$. In this case a choice of contact form θ for H induces a choice of contact form for H_Σ by pullback. If $q \geq m$ then we need to impose the condition $T_x\iota(T_x\Sigma) \not\subset H_x$ for all $x \in \Sigma$ as an additional assumption; such a CR embedding is said to be *transversal*. (Note that if $T_x\iota(T_x\Sigma) \subset H_x$ then $T_x\iota(T_x^{1,0}\Sigma) \subset T_x^{1,0}M$ is a totally isotropic subspace, but the maximum dimension of such a subspace is the signature q, so $q \geq m$.) We consider transversal CR embeddings in the following.

We will work in terms of a pair of pseudohermitian structures (M, H, J, θ) and $(\Sigma, H_\Sigma, J_\Sigma, \iota^*\theta)$ and aim for constructions which are invariant under ambient rescalings $\theta \to \hat{\theta} = e^\Upsilon \theta$. More precisely, our goal is to construct operators and quantities which may be expressed in terms of the Tanaka-Webster calculus of θ and of $\iota^*\theta$ which are invariant under the replacement of the pair $(\theta, \iota^*\theta)$ with $(e^\Upsilon\theta, \iota^*(e^\Upsilon\theta))$.

For simplicity we will initially restrict our attention to the case where $m = n-1$ ($m \geq 1$) and where both manifolds are strictly pseudoconvex (i.e. have positive definite Levi form for positively oriented contact forms). The general codimension (and signature) case is treated in §6.3, and much carries over directly.

6.1.1 Notation

We fix a bundle of $(1,0)$-densities on Σ, that is the dual of an $(m+2)^{th}$ root of $\mathscr{K}_\Sigma$, and denote it by $\mathcal{E}_\Sigma(1,0)$. The corresponding (w,w')-density bundles are denoted $\mathcal{E}_\Sigma(w,w')$. We use abstract index notation $\mathcal{E}^\mu$ for $T^{1,0}\Sigma$, and allow the use of Greek indices from the later part of the alphabet: μ, ν, λ, ρ, μ', and so on. Of course then $\mathcal{E}^{\bar{\mu}}$ denotes $T^{0,1}\Sigma$, $\mathcal{E}_\mu$ denotes $(T^{1,0}\Sigma)^*$, and so on. We denote the CR Levi form of Σ by $\boldsymbol{h}_{\mu\bar{\nu}}$ and its inverse by $\boldsymbol{h}^{\mu\bar{\nu}}$. We also occasionally use abstract index notation for $T\Sigma$, denoting it by $\mathcal{E}^i$ and allowing indices i, j, k, l, etcetera.

We identify Σ with its image under ι and write $\mathcal{E}^\alpha|_\Sigma \to \Sigma$ for the restriction of $\mathcal{E}^\alpha \to M$ to fibers over Σ (so $\mathcal{E}^\alpha|_\Sigma = \iota^*\mathcal{E}^\alpha$). We define the section Π^α_μ of $\mathcal{E}^\alpha|_\Sigma \otimes \mathcal{E}_\mu$ to be (the complex linear extension of) $T\iota$ as a map from $T^{1,0}\Sigma$ into $T^{1,0}M$, i.e. if $V \in T^{1,0}\Sigma$ and $U = T\iota(V)$ then $U^\alpha = \Pi^\alpha_\mu V^\mu$. We define the section Π^μ_α to be the map from $T^{1,0}M|_\Sigma$ onto $T^{1,0}\Sigma$ given by orthogonal projection with respect to the CR Levi form. Clearly $\Pi^\mu_\alpha \Pi^\alpha_\nu = \delta^\mu_\nu$, and $\Pi^\alpha_\mu \Pi^\mu_\beta$ is simply the orthogonal projection map from $T^{1,0}M|_\Sigma$ onto $T\iota(T^{1,0}\Sigma)$ given by the Levi form. It is also clear that

$$\boldsymbol{h}_{\mu\bar{\nu}} = \Pi^\alpha_\mu \Pi^{\bar{\beta}}_{\bar{\nu}} \boldsymbol{h}_{\alpha\bar{\beta}} \tag{6.1.1}$$

along Σ.

6.1.2 Compatible Scales

In developing the pseudohermitian and CR tractor calculus we have been making use of the fact that a choice of contact form θ for M gives us a direct sum decomposition of the complexified tangent bundle

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}T,$$

T being the Reeb vector field of θ. Now the contact form $\theta_\Sigma = \iota^*\theta$ for Σ also determines a direct sum decomposition

$$\mathbb{C}T\Sigma = T^{1,0}\Sigma \oplus T^{0,1}\Sigma \oplus \mathbb{C}T_\Sigma \tag{6.1.2}$$

where T_Σ is the Reeb vector field of θ_Σ. It is easy to see that in general these two Reeb vector fields will not agree along Σ. Clearly this will become a problem for us when we try to relate components of ambient tensor fields (decomposed w.r.t. θ) with components of submanifold tensor fields (decomposed w.r.t. θ_Σ). To remedy this problem we will make use of a basic lemma (cf. [54, Lemma 4.1]).

Lemma 6.1.1. *Let $\iota : \Sigma \hookrightarrow M$ be a CR embedding between nondegenerate CR manifolds. If θ_Σ is a contact form for Σ with Reeb vector field T_Σ, then there exists a contact form θ for M with $\iota^*\theta = \theta_\Sigma$ and whose Reeb vector field agrees with T_Σ along Σ. Moreover, the 1-jet of θ is uniquely determined along Σ.*

Proof. Fix a contact form θ' for M with $\iota^*\theta' = \theta_\Sigma$. Let f be an arbitrary smooth (real valued) function on M with $f|_\Sigma \equiv 0$, and consider the contact form $\theta = e^f\theta'$. First of all we have

$$(T\iota \cdot T_\Sigma) \lrcorner \mathrm{d}\theta = e^f (T\iota \cdot T_\Sigma) \lrcorner \mathrm{d}\theta' + e^f \mathrm{d}f$$

along Σ since $\theta'(T\iota \cdot T_\Sigma) = (\iota^*\theta')(T_\Sigma) = \theta_\Sigma(T_\Sigma) = 1$. Now since $\iota^*\theta = \theta_\Sigma$ we have

$$\iota^*((T\iota \cdot T_\Sigma) \lrcorner \mathrm{d}\theta) = T_\Sigma \lrcorner \mathrm{d}\theta_\Sigma = 0.$$

This means that $(T\iota \cdot T_\Sigma) \lrcorner \mathrm{d}\theta$ is zero when restricted to tangential directions. Consequently, we only need to see if we can make $(T\iota \cdot T_\Sigma) \lrcorner \mathrm{d}\theta$ zero on the quotient space $TM|_\Sigma / T\Sigma$. This requires choosing f such that along Σ

$$\mathrm{d}f = -(T\iota \cdot T_\Sigma) \lrcorner \mathrm{d}\theta'$$

on $TM|_\Sigma / T\Sigma$, which simply amounts to prescribing the normal derivatives of f off Σ. Choosing such an f we have that $(T\iota \cdot T_\Sigma) \lrcorner \mathrm{d}\theta = 0$ and $\theta(T\iota \cdot T_\Sigma) = \theta_\Sigma(T_\Sigma) = 1$ as required. □

Definition 6.1.2. A pair of contact forms θ, θ_Σ for M and Σ respectively will be called *compatible* if $\theta_\Sigma = \iota^*\theta$ and the Reeb vector field of θ restricts to the Reeb vector field of θ_Σ along Σ. A contact form θ which is compatible with $\iota^*\theta$, i.e. whose Reeb vector field is tangent to Σ, will be said to be *admissible* [54].

We will work primarily in terms of compatible contact forms in the following. When working in terms of compatible contact forms θ for M and θ_Σ for Σ we identify the density bundles $\mathcal{E}(1,1)|_\Sigma$ with $\mathcal{E}_\Sigma(1,1)$ using the trivialisations of these bundles induced by θ and θ_Σ respectively (in fact this identification is canonical, i.e. it is independent of the choice of compatible contact forms). We also identify the Reeb vector field T_Σ of θ_Σ with $T|_\Sigma$ where T is the ambient Reeb vector field. This means that the '0-component' of $X \in T\Sigma$ taken with respect to either θ_Σ or θ (identifying X with $T\iota \cdot X$) is the same, and that our ambient and intrinsic decompositions of tensors will always be nicely compatible.

Remark 6.1.3. Note that Lemma 6.1.1 holds for general codimension CR embeddings (with the same proof). We can therefore continue to work with compatible contact forms in the general codimension case discussed in §6.3. ∎

6.1.3 Normal Bundles

Clearly $T^{1,0}\Sigma$ has complex corank one inside $T^{1,0}M|_\Sigma$. The CR Levi form determines then a canonical complex line bundle $\mathcal{N}^\alpha \subset \mathcal{E}^\alpha|_\Sigma$ whose sections are those V^α for which $\Pi^\mu_\alpha V^\alpha \equiv 0$. There is also the corresponding dual complex line bundle $\mathcal{N}_\alpha \subset \mathcal{E}_\alpha|_\Sigma$ whose sections V_α satisfy $V_\alpha \Pi^\alpha_\mu \equiv 0$.

Remark 6.1.4. Given any choice of ambient contact form θ the manifold M gains a Riemannian structure from the Webster metric g_θ. One can therefore treat Σ as a Riemannian submanifold, in particular we have a Riemannian normal bundle to Σ. This Riemannian normal bundle will be the same for any admissible contact form θ, and we denote it by $N\Sigma$. Complexifying we see that $\mathbb{C}N\Sigma = \mathcal{N}^\alpha \oplus \mathcal{N}^{\bar\alpha}$ where $\mathcal{N}^\alpha$ is the i-eigenspace of J. ∎

6.1.3.1 Unit Normal Fields

Given a choice of ambient contact form θ, one may ask that a section N_α be unit with respect to the Levi form of θ. However, for CR geometry it is more natural to work with sections of the bundle $\mathcal{N}_\alpha(1,0) = \mathcal{N}_\alpha \otimes \mathcal{E}(1,0)|_\Sigma$, which is normed by the CR Levi form. Thus we make the following definition:

Definition 6.1.5. By a *(weighted) unit holomorphic conormal field* we mean a section N_α of $\mathcal{N}_\alpha(1,0)$ for which $\boldsymbol{h}^{\alpha\bar\beta} N_\alpha N_{\bar\beta} = 1$ where $N_{\bar\beta} = \overline{N_\beta}$. The field $N^\alpha = \boldsymbol{h}^{\alpha\bar\beta} N_{\bar\beta}$ obtained from such an N_α will be referred to as a *(weighted) unit holomorphic normal field.*

Remark 6.1.6. The bundles $\mathcal{N}_\alpha(w+1,-w)$ are also normed by the CR Levi form, but the natural weight for conormals is indeed $(1,0)$. The line bundle $\mathcal{N}_\alpha(1,0)$ plays an important role in the following since it relates ambient and intrinsic density bundles (see

(6.1.28) below). Moreover, $\mathcal{N}_\alpha(1,0)$ can be canonically identified with a non-null subbundle of the ambient cotractor bundle $\mathcal{E}_A|_\Sigma$, and hence carries a canonical CR invariant connection (see Proposition 6.2.3). ∎

If N_α is a unit holomorphic conormal then so is $N'_\alpha = e^{i\varphi}N_\alpha$ for any $\varphi \in C^\infty(\Sigma)$, and $N'^\alpha = e^{-i\varphi}N^\alpha$. However, the combinations $N^\alpha N_\beta$ and $N_\alpha N_{\bar\beta}$ are independent of the choice of holomorphic conormal, and these satisfy

$$\delta_\beta^\alpha = \Pi_\beta^\alpha + N^\alpha N_\beta \quad \text{and} \quad \boldsymbol{h}_{\alpha\bar\beta} = \boldsymbol{h}_{\mu\bar\nu}\Pi_\alpha^\mu \Pi_{\bar\beta}^{\bar\nu} + N_\alpha N_{\bar\beta} \tag{6.1.3}$$

along Σ, where Π_β^α is the tangential orthogonal projection $\Pi_\mu^\alpha \Pi_\beta^\mu$ and $\boldsymbol{h}_{\mu\bar\nu}$ is the CR Levi form of Σ.

6.1.4 Tangential Derivatives

Let θ be an admissible ambient contact form with Tanaka-Webster connection ∇. The pullback connection $\iota^*\nabla$ allows us to differentiate sections of ambient tensor bundles along Σ in directions tangential to Σ. Recall that we may think of the Tanaka-Webster connection ∇ as a triple of 'partial connections' $(\nabla_\alpha, \nabla_{\bar\alpha}, \nabla_0)$. Now suppose that the Reeb vector field T of θ is tangent to Σ, then θ and $\theta_\Sigma = \iota^*\theta$ are compatible. Then we can break up $\iota^*\nabla$ into a corresponding triple $(\nabla_\mu, \nabla_{\bar\mu}, \nabla_0)$. Precisely, ∇_μ is defined to act on sections of $\mathcal{E}^\alpha|_\Sigma$ according to the formula

$$\nabla_\mu \tau^\alpha = \Pi_\mu^\beta \nabla_\beta \tilde\tau^\alpha \tag{6.1.4}$$

where $\tilde\tau^\alpha$ is any extension of the section τ^α of $\mathcal{E}^\alpha|_\Sigma$ to a neighbourhd of Σ, and ∇_μ is defined similarly on sections of $\mathcal{E}^{\bar\alpha}|_\Sigma$, $\mathcal{E}_\alpha|_\Sigma$, and so on. We define $\nabla_{\bar\mu}$ similarly, and define ∇_0 on sections of $\mathcal{E}^\alpha|_\Sigma$ by the formula

$$\nabla_0 \tau^\alpha = \nabla_0 \tilde\tau^\alpha \tag{6.1.5}$$

along Σ, where $\tilde\tau^\alpha$ is any extension of τ^α, and similarly on sections of $\mathcal{E}^{\bar\alpha}|_\Sigma$, $\mathcal{E}_\alpha|_\Sigma$, and so on (note the independence of the choice of the extension relies on the fact that T is tangential to Σ).

Remark. We have identified $\mathcal{E}(1,1)|_\Sigma$ with $\mathcal{E}_\Sigma(1,1)$ and $T|_\Sigma$ and with the Reeb vector field T_Σ of θ_Σ, thus splitting $\iota^*\nabla$ up into $(\nabla_\mu, \nabla_{\bar\mu}, \nabla_0)$ corresponds precisely to restricting $\iota^*\nabla$ to the respective summands in the direct sum decomposition (6.1.2) induced by θ_Σ.

6.1.4.1 The normal Tanaka-Webster connection

The ambient Tanaka-Webster connection also induces a connection on the normal bundle.

Definition 6.1.7. Given an admissible ambient contact form θ, we define the *normal Tanaka-Webster connection* $\nabla^\perp$ on $\mathcal{N}_\alpha$ by differentiating tangentially using the Tanaka-Webster connection ∇ of θ and then projecting orthogonally onto $\mathcal{N}_\alpha$ using the Levi form.

6.1.5 The Submanifold Tanaka-Webster Connection

We may define a connection D on $T^{1,0}\Sigma = \mathcal{E}^\mu$ (which we identify with $T\iota(T^{1,0}\Sigma)$ in $T^{1,0}M|_\Sigma$) by differentiating in tangential directions using $\iota^*\nabla$ and projecting the result back onto $T^{1,0}\Sigma = \mathcal{E}^\mu$ orthogonally with respect to the Levi form. This means that if τ^μ is a section of $\mathcal{E}^\mu$ then we have

$$D_\nu \tau^\mu = \Pi^\mu_\alpha \nabla_\nu \tau^\alpha \tag{6.1.6}$$

where $\tau^\alpha = \Pi^\alpha_\lambda \tau^\lambda$. One may define D to act also on $T^{0,1}\Sigma = \mathcal{E}^{\bar{\mu}}$ by the analogous formula

$$D_\nu \tau^{\bar{\mu}} = \Pi^{\bar{\mu}}_{\bar{\alpha}} \nabla_\nu \tau^{\bar{\alpha}}. \tag{6.1.7}$$

Thus D may be thought of as a connection on H_Σ which preserves J_Σ. One may then extend D to a connection on $T\Sigma$ by requiring that T_Σ be parallel.

Remark. Equivalently one may define D as a connection on $T\Sigma$ from the start by differentiating tangent vectors to Σ in tangential directions using $\iota^*\nabla$ and projecting the result back onto $T\Sigma$ orthogonally with respect to the Webster metric of θ.

Provided θ and θ_Σ are compatible, the connection D constructed in this manner will be the Tanaka-Webster connection of θ_Σ (cf. [49, Theorem 6.4]):

Proposition 6.1.8. *If θ, θ_Σ are contact forms for M and Σ respectively which are compatible, that is, $\theta_\Sigma = \iota^*\theta$ and the Reeb vector field of θ is tangential to Σ, then the connection*

D on $T\Sigma$ induced by the Tanaka-Webster connection ∇ of θ (and projection with respect to the ambient Webster metric) is the Tanaka-Webster connection of θ_Σ.

Proof. We need to show that D preserves $(H_\Sigma, J_\Sigma, \theta_\Sigma)$ and satisfies the torsion conditions of §§5.2.1. It is clear that D preserves the decomposition (6.1.2) and gives a linear connection on each of the three direct summands. This implies that D preserves H and J in the appropriate senses. Since ∇ preserves the Reeb vector field T, $\iota^*\nabla$ preserves $T|_\Sigma = T_\Sigma$ and hence $DT_\Sigma = 0$. Since D preserves T_Σ and H it must also preserve θ_Σ.

Now let $f \in C^\infty(\Sigma)$ and choose an extension $\tilde{f}$ of f to M such that along Σ we have $\nabla_\alpha \tilde{f} = \Pi_\alpha^\mu \nabla_\mu f$, i.e. the derivative of $\tilde{f}$ vanishes in g_θ-normal directions along Σ (these directions don't depend on θ so long as we choose θ admissible). Then we have that $\nabla_{\bar\beta}\tilde{f} = \Pi_{\bar\beta}^{\bar\lambda}\nabla_{\bar\lambda} f$ along Σ and hence also that

$$\begin{aligned}
D_\mu D_{\bar\nu} f - D_{\bar\nu} D_\mu f &= D_\mu \nabla_{\bar\nu} f - D_{\bar\nu}\nabla_\mu f \\
&= \Pi_{\bar\nu}^{\bar\beta}\nabla_\mu(\Pi_{\bar\beta}^{\bar\lambda}\nabla_{\bar\lambda} f) - \Pi_\mu^\alpha \nabla_{\bar\nu}(\Pi_\alpha^\lambda \nabla_\lambda f) \\
&= \Pi_{\bar\nu}^{\bar\beta}\nabla_\mu\nabla_{\bar\beta}\tilde{f} - \Pi_\mu^\alpha\nabla_{\bar\nu}\nabla_\alpha\tilde{f} \\
&= \Pi_\mu^\alpha\Pi_{\bar\nu}^{\bar\beta}(\nabla_\alpha\nabla_{\bar\beta}\tilde{f} - \nabla_{\bar\beta}\nabla_\alpha\tilde{f}) \\
&= \Pi_\mu^\alpha\Pi_{\bar\nu}^{\bar\beta}(-i\boldsymbol{h}_{\alpha\beta}\nabla_0\tilde{f}) \\
&= -i\boldsymbol{h}_{\mu\bar\nu} D_0 f
\end{aligned}$$

where we have used that $D_\mu f = \nabla_\mu f$ and $D_{\bar\nu} f = \nabla_{\bar\nu} f$ as well as that $D_0 f = \nabla_0 f = \nabla_0 \tilde{f}$ along Σ . Similarly we may easily compute that

$$D_\mu D_\nu f - D_\nu D_\mu f = 0.$$

Finally we have

$$\begin{aligned}
D_\mu D_0 f - D_0 D_\mu f &= \nabla_\mu \nabla_0 f - \nabla_0 \nabla_\mu f \\
&= \Pi_\mu^\alpha(\nabla_\alpha\nabla_0\tilde{f} - \nabla_0\nabla_\alpha\tilde{f}) \\
&= \Pi_\mu^\alpha A^{\bar\gamma}{}_\alpha \nabla_{\bar\gamma}\tilde{f} \\
&= \Pi_\mu^\alpha \Pi_{\bar\gamma}^{\bar\lambda} A^{\bar\gamma}{}_\alpha \nabla_{\bar\lambda} f = A^{\bar\lambda}{}_\mu D_{\bar\lambda} f
\end{aligned}$$

where $A_{\mu\nu} = \Pi_\mu^\alpha \Pi_\nu^\beta A_{\alpha\beta}$. Since f was arbitrary, we conclude that D is the Tanaka-Webster connection of θ_Σ. □

Corollary 6.1.9. *Given an admissible ambient contact form θ with pseudohermitian torsion $A_{\alpha\beta}$, the pseudohermitian torsion of $\theta_\Sigma = \iota^*\theta$ is $A_{\mu\nu} = \Pi^\alpha_\mu \Pi^\beta_\nu A_{\alpha\beta}$.*

Remark 6.1.10. Note that Proposition 6.1.8 and Corollary 6.1.9 hold in the general codimension case by the same arguments. ∎

6.1.6 The Second Fundamental Form

We can now define a second fundamental form using an analogue of the Gauss formula from Riemannian submanifold geometry.

Definition 6.1.11. Given θ and θ_Σ compatible with respective Tanaka-Webster connections ∇ and D we define the *(pseudohermitian) second fundamental form* by

$$\nabla_X Y = D_X Y + II(X, Y), \tag{6.1.8}$$

for all $X, Y \in \mathfrak{X}(\Sigma)$, where we implicitly identify submanifold vector fields with tangential ambient vector fields along Σ and use the pullback connection $\iota^*\nabla$ on the left hand side.

Clearly $II(X, Y)$ is tensorial in X and Y, and is normal bundle ($N\Sigma$) valued. It is also clear from the definition that $II(\,\cdot\,, T_\Sigma) = 0$ and that $II(\,\cdot\,,\cdot\,)|_{H_\Sigma}$ is complex linear (with respect to J and J_Σ) in the second argument, that is

$$II(\,\cdot\,, J_\Sigma \cdot\,)|_{H_\Sigma} = J II(\,\cdot\,,\cdot\,)|_{H_\Sigma}.$$

In fact, these properties also hold for the first argument, II being symmetric.

Proposition 6.1.12. *The only nonzero components of the (pseudohermitian) second fundamental form II are $II_{\mu\nu}{}^\gamma$ and its conjugate. Moreover*

$$II_{\mu\nu}{}^\gamma = II_{\nu\mu}{}^\gamma, \tag{6.1.9}$$

so that II is symmetric.

Proof. Since $II(\,\cdot\,, T_\Sigma) = 0$ and $II(\,\cdot\,,\cdot\,)|_{H_\Sigma}$ is complex linear in the second argument, to prove the first claim it suffices to show that $II_{0\nu}{}^\gamma = 0$ and $II_{\bar{\mu}\nu}{}^\gamma = 0$.

Let N_α be a section of $\mathcal{N}_\alpha$ such that $h^{\alpha\bar{\beta}}N_\alpha N_{\bar{\beta}} = 1$. From the Gauss formula (6.1.8) we have that

$$\nabla_i V^\gamma = \Pi^\gamma_\lambda D_i V^\lambda + {I\!I_{i\nu}}^\gamma V^\nu \tag{6.1.10}$$

for any section V^λ of $\mathcal{E}^\lambda$, where $V^\gamma = \Pi^\gamma_\lambda V^\lambda$. Contracting the above display with N_γ and replacing the 'i' index with 'μ', '$\bar{\mu}$', and '0' respectively gives

$$N_\gamma {I\!I_{\mu\nu}}^\gamma = -\Pi^\gamma_\nu \nabla_\mu N_\gamma, \quad N_\gamma {I\!I_{\bar{\mu}\nu}}^\gamma = -\Pi^\gamma_\nu \nabla_{\bar{\mu}} N_\gamma,$$
$$\text{and} \quad N_\gamma {I\!I_{0\nu}}^\gamma = -\Pi^\gamma_\nu \nabla_0 N_\gamma,$$

since $N_\gamma V^\gamma = 0$ for all $V^\lambda \in \Gamma(\mathcal{E}^\lambda)$. By conjugating, one also has that $N_{\bar{\gamma}} {I\!I_{\nu\bar{\mu}}}^{\bar{\gamma}} = -\Pi^{\bar{\gamma}}_{\bar{\mu}} \nabla_\nu N_{\bar{\gamma}}$.

Now let f be a real valued function on M which vanishes on Σ and for which $(\nabla_\alpha f, \nabla_{\bar{\alpha}} f, \nabla_0 f)$ is equal to $(N_\alpha, N_{\bar{\alpha}}, 0)$ along Σ. (Note that we must require $\nabla_0 f$ to be zero along Σ since T is tangent to Σ and we ask that $f|_\Sigma \equiv 0$. Such an f exists because we are simply prescribing the normal derivatives of f off Σ. Any such f is, locally about Σ, a defining function for a real hypersurface in M containing Σ which is g_θ-orthogonal to the real part and tangent to the imaginary part of N^α.) From (5.2.1) and (5.2.2) we have that

$$\nabla_\alpha \nabla_\beta f = \nabla_\beta \nabla_\alpha f \quad \text{and} \quad \nabla_\alpha \nabla_{\bar{\beta}} f = \nabla_{\bar{\beta}} \nabla_\alpha f$$

along Σ. Projecting tangentially along Σ we immediately have that

$$N_\gamma {I\!I_{\mu\nu}}^\gamma = N_\gamma {I\!I_{\nu\mu}}^\gamma \quad \text{and} \quad N_{\bar{\gamma}} {I\!I_{\mu\bar{\nu}}}^{\bar{\gamma}} = N_\gamma {I\!I_{\bar{\nu}\mu}}^\gamma.$$

The first of these implies that ${I\!I_{\mu\nu}}^\gamma = {I\!I_{\nu\mu}}^\gamma$. Since N_γ was arbitrary the second implies that ${I\!I_{\mu\bar{\nu}}}^{\bar{\gamma}} = 0$ (replacing N_γ with iN_γ gives a minus sign).

Using the same function f, (5.2.3) states

$$\nabla_\alpha \nabla_0 f - \nabla_0 \nabla_\alpha f = {A^{\bar{\gamma}}}_\alpha \nabla_{\bar{\gamma}} f.$$

Applying Π^α_μ to both sides of the above display we get that

$$-\Pi^\alpha_\mu \nabla_0 \nabla_\alpha f = \Pi^\alpha_\mu {A^{\bar{\gamma}}}_\alpha \nabla_{\bar{\gamma}} f$$

along Σ (since $\Pi_\mu^\alpha \nabla_\alpha \nabla_0 f$ is zero along Σ). We conclude that

$$N_\gamma II_{0\mu}{}^\gamma = N_{\bar\gamma} \Pi_\mu^\alpha A_\alpha^{\bar\gamma}.$$

Again, since N_γ was arbitrary we must have

$$II_{0\mu}{}^\gamma = 0 \quad \text{and} \quad N_{\bar\gamma} \Pi_\mu^\alpha A_\alpha^{\bar\gamma} = 0. \tag{6.1.11}$$

□

The second of the expressions (6.1.11) should be seen as a constraint on the pseudohermitian torsion of an admissible contact form. We state this as a corollary:

Corollary 6.1.13. *If θ is an admissible ambient contact form then the pseudohermitian torsion of θ satisfies*

$$\Pi_\mu^\alpha A_{\alpha\beta} N^\beta = 0 \tag{6.1.12}$$

for any holomorphic normal field N^β.

Remark 6.1.14. Note that in the higher codimension case (of CR embeddings) if one defines the (pseudohermitian) second fundamental form of a pair of compatible contact forms as in Definition 6.1.11 then Proposition 6.1.12 holds with the proof unchanged (and consequently Corollary 6.1.13 also holds). ▮

Remark 6.1.15. Our claim that $II(T, \cdot) = 0$, and the above corollary, disagree with [47] and the book [49]. Our claim that $II(T, \cdot) = 0$ is confirmed however by the later article [48]. ▮

6.1.6.1 The CR second fundamental form

We shall now see that the component $II_{\mu\nu}{}^\gamma$ does not depend on the choice of compatible contact forms θ and θ_Σ.

Lemma 6.1.16. *Given compatible contact forms θ and θ_Σ one has*

$$II_{\mu\nu}{}^\gamma = -N^\gamma \Pi_\nu^\beta \nabla_\mu N_\beta \tag{6.1.13}$$

for any unit holomorphic conormal field N_α.

Proof. From the Gauss formula (cf. (6.1.10)) we have

$$\nabla_\mu V^\gamma = \Pi^\gamma_\lambda D_\mu V^\lambda + II_{\mu\nu}{}^\gamma V^\nu$$

for any section V^λ of $\mathcal{E}^\lambda$, where $V^\gamma = \Pi^\gamma_\lambda V^\lambda$. Contracting the above display with N_γ and using that $N_\gamma \nabla_\mu V^\gamma = -V^\gamma \nabla_\mu N_\gamma$ yields the result. □

Corollary 6.1.17. *The component $II_{\mu\nu}{}^\gamma$ of the pseudohermitian second fundamental form does not depend on the pair of compatible contact forms used to define it.*

Proof. Combining the Tanaka-Webster transformation laws of Proposition 5.2.8 and Proposition 5.2.9 we have that

$$\hat{\nabla}_\mu N_\beta = \nabla_\mu N_\beta - \Pi^\alpha_\mu \Upsilon_\beta N_\alpha - \Upsilon_\mu N_\beta + \Upsilon_\mu N_\beta = \nabla_\mu N_\beta$$

since N_β has weight $(1,0)$. The claim then follows from (6.1.13). □

We therefore term $II_{\mu\nu}{}^\gamma$ the *CR second fundamental form.*

Remark 6.1.18. The pseudohermitian second fundamental form II (of a pair of compatible contact forms) is not CR invariant, even though $II_{\mu\nu}{}^\gamma$ is, since the direct sum decompositions of $\mathbb{C}TM$ and $\mathbb{C}T^*\Sigma$ change under rescaling of the ambient and submanifold contact forms. ▮

Recall that we write Π^α_β for the tangential orthogonal projection $\Pi^\alpha_\mu \Pi^\mu_\beta$ on the ambient holomorphic tangent bundle along Σ. The following lemma will be useful in the derivations of §§6.1.7:

Lemma 6.1.19. *For any admissible ambient contact form we have*

$$\nabla_\mu \Pi^\gamma_\beta = II_{\mu\nu}{}^\gamma \Pi^\nu_\beta, \qquad \nabla_\mu \Pi^{\bar\beta}_{\bar\gamma} = II_\mu{}^{\bar\nu}{}_{\bar\gamma} \Pi^{\bar\beta}_{\bar\nu}, \tag{6.1.14}$$

$$\nabla_{\bar\mu} \Pi^{\bar\gamma}_{\bar\beta} = II_{\bar\mu\bar\nu}{}^{\bar\gamma} \Pi^{\bar\nu}_{\bar\beta}, \qquad \nabla_{\bar\mu} \Pi^{\beta}_{\gamma} = II_{\bar\mu}{}^{\nu}{}_{\gamma} \Pi^{\beta}_{\nu}, \tag{6.1.15}$$

and

$$\nabla_0 \Pi^\gamma_\beta = 0, \qquad \nabla_0 \Pi^{\bar\beta}_{\bar\gamma} = 0. \tag{6.1.16}$$

Proof. These follow immediately by differentiating $\delta^\gamma_\beta - N^\gamma N_\beta$, however we wish to give a proof that will also work in the higher codimension case. Pick a section V^ν and let $V^\beta = \Pi^\beta_\nu V^\nu$. Then

$$\nabla_\mu V^\gamma = \nabla_\mu(\Pi^\gamma_\beta V^\beta) = (\nabla_\mu \Pi^\gamma_\beta)V^\beta + \Pi^\gamma_\beta \nabla_\mu V^\beta.$$

Noting that $\Pi^\gamma_\beta \nabla_\mu V^\beta = \Pi^\gamma_\nu D_\mu V^\nu$, from the Gauss formula we have

$$\Pi^\beta_\nu \nabla_\mu \Pi^\gamma_\beta = II_{\mu\nu}{}^\gamma \tag{6.1.17}$$

since V^ν was arbitrary. Now on the other hand if N^α is any unit holomorphic normal then

$$N^\beta \nabla_\mu \Pi^\gamma_\beta = -(\nabla_\mu N^\beta)\Pi^\gamma_\beta = 0 \tag{6.1.18}$$

since $II_{\mu\bar{\nu}}{}^{\bar{\delta}} = 0$. The previous two displays imply the first equation of (6.1.14), and the second then follows by raising and lowering indices. Conjugating these gives (6.1.15). The expressions (6.1.16) are proved similarly using that $II_{0\nu}{}^\gamma = 0$ and $II_{0\bar{\nu}}{}^{\bar{\gamma}} = 0$. □

6.1.7 The Pseudohermitian Gauss, Codazzi, and Ricci Equations

Here we give pseudohermitian analogues of the Gauss, Codazzi, and Ricci equations from Riemannian submanifold theory. Real forms of these equations can be found in Ch. 6 of [49], note that $Q = 0$ in the pseudohermitian Codazzi equation they give (cf. Remark 6.1.15).

When working with compatible contact forms we denote the ambient and submanifold Tanaka-Webster connections by ∇ and D respectively. We write

$$\mathrm{N}^\alpha_\beta = \delta^\alpha_\beta - \Pi^\alpha_\beta \tag{6.1.19}$$

for the orthogonal projection onto $\mathcal{N}^\alpha \subset \mathcal{E}^\alpha|_\Sigma$. In this case $\mathrm{N}^\alpha_\beta = N^\alpha N_\beta$ for any unit holomorphic normal N^α. We adopt the convention of replacing uppercase root letters with lowercase root letters for submanifold curvature tensors, so the pseudohermitian curvature tensor of θ_Σ will be denoted by $r_{\mu\bar{\nu}\lambda\bar{\rho}}$, the pseudohermitian Ricci curvature by $r_{\mu\nu}$, and so on. For the ambient curvature tensors along Σ we will use submanifold abstract indices to denote tangential projections, for example

$$R_{\mu\bar{\nu}\lambda\bar{\rho}} = \Pi^\alpha_\mu \Pi^{\bar{\beta}}_{\bar{\nu}} \Pi^\gamma_\lambda \Pi^{\bar{\delta}}_{\bar{\rho}} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \quad \text{and} \quad R_{\mu\bar{\nu}\gamma\bar{\delta}} = \Pi^\alpha_\mu \Pi^{\bar{\beta}}_{\bar{\nu}} R_{\alpha\bar{\beta}\gamma\bar{\delta}}.$$

6.1.7.1 The pseudohermitian Gauss equation

Proposition 6.1.20. *Given compatible contact forms, the submanifold pseudohermitian curvature is related to the ambient curvature via*

$$R_{\mu\bar{\nu}\lambda\bar{\rho}} = r_{\mu\bar{\nu}\lambda\bar{\rho}} + \boldsymbol{h}_{\gamma\bar{\delta}} I\!I_{\mu\lambda}{}^{\gamma} I\!I_{\bar{\nu}\bar{\rho}}{}^{\bar{\delta}}. \tag{6.1.20}$$

Proof. Let V be a section of $T^{1,0}\Sigma$, let $V^{\bar{\gamma}} = \Pi^{\bar{\gamma}}_{\bar{\lambda}} V^{\bar{\lambda}}$, and let $\tilde{V}^{\bar{\gamma}}$ be a smooth extension of $V^{\bar{\gamma}}$ to a neighbourhood of Σ. Proposition 6.1.8 says that $D_{\bar{\nu}} V^{\bar{\lambda}} = \Pi^{\bar{\lambda}}_{\bar{\gamma}} \Pi^{\bar{\beta}}_{\bar{\nu}} \nabla_{\bar{\beta}} \tilde{V}^{\bar{\gamma}}$ and thus

$$\begin{aligned} D_\mu D_{\bar{\nu}} V^{\bar{\lambda}} &= \Pi^{\bar{\lambda}}_{\bar{\delta}} \Pi^{\bar{\epsilon}}_{\bar{\nu}} \nabla_\mu \left(\Pi^{\bar{\delta}}_{\bar{\gamma}} \Pi^{\bar{\beta}}_{\bar{\epsilon}} \nabla_{\bar{\beta}} \tilde{V}^{\bar{\gamma}} \right) \\ &= \Pi^{\bar{\lambda}}_{\bar{\delta}} \Pi^{\bar{\epsilon}}_{\bar{\nu}} (\nabla_\mu \Pi^{\bar{\delta}}_{\bar{\gamma}}) \Pi^{\bar{\beta}}_{\bar{\epsilon}} \nabla_{\bar{\beta}} \tilde{V}^{\bar{\gamma}} + \Pi^{\bar{\lambda}}_{\bar{\delta}} \Pi^{\bar{\epsilon}}_{\bar{\nu}} \Pi^{\bar{\delta}}_{\bar{\gamma}} (\nabla_\mu \Pi^{\bar{\beta}}_{\bar{\epsilon}}) \nabla_{\bar{\beta}} \tilde{V}^{\bar{\gamma}} \\ &\quad + \Pi^{\bar{\lambda}}_{\bar{\delta}} \Pi^{\bar{\epsilon}}_{\bar{\nu}} \Pi^{\bar{\delta}}_{\bar{\gamma}} \Pi^{\bar{\beta}}_{\bar{\epsilon}} \nabla_\mu \nabla_{\bar{\beta}} \tilde{V}^{\bar{\gamma}} \\ &= I\!I_\mu{}^{\bar{\lambda}}{}_{\bar{\gamma}} \nabla_{\bar{\nu}} V^{\bar{\gamma}} + \Pi^{\bar{\lambda}}_{\bar{\gamma}} \Pi^{\alpha}_{\mu} \Pi^{\bar{\beta}}_{\bar{\nu}} \nabla_\alpha \nabla_{\bar{\beta}} \tilde{V}^{\bar{\gamma}}, \end{aligned}$$

where we have used (6.1.14) of Lemma 6.1.19 in the final step. Since $\mathrm{N}^{\delta}_{\gamma} V^{\gamma} = 0$ we have $\mathrm{N}^{\bar{\delta}}_{\bar{\gamma}} \nabla_{\bar{\nu}} V^{\bar{\gamma}} = -V^{\bar{\gamma}} \nabla_{\bar{\nu}} \mathrm{N}^{\bar{\delta}}_{\bar{\gamma}} = V^{\bar{\rho}} I\!I_{\bar{\nu}\bar{\rho}}{}^{\bar{\delta}}$ using (6.1.15) of Lemma 6.1.19, and hence by writing $I\!I_\mu{}^{\bar{\lambda}}{}_{\bar{\gamma}}$ as $I\!I_\mu{}^{\bar{\lambda}}{}_{\bar{\delta}} \mathrm{N}^{\bar{\delta}}_{\bar{\gamma}}$ we obtain

$$D_\mu D_{\bar{\nu}} V^{\bar{\lambda}} = I\!I_\mu{}^{\bar{\lambda}}{}_{\bar{\delta}} I\!I_{\bar{\nu}\bar{\rho}}{}^{\bar{\delta}} V^{\bar{\rho}} + \Pi^{\bar{\lambda}}_{\bar{\gamma}} \Pi^{\alpha}_{\mu} \Pi^{\bar{\beta}}_{\bar{\nu}} \nabla_\alpha \nabla_{\bar{\beta}} \tilde{V}^{\bar{\gamma}}.$$

By a similar calculation with the roles of μ and ν interchanged we obtain

$$D_{\bar{\nu}} D_\mu V^{\bar{\lambda}} = \Pi^{\bar{\lambda}}_{\bar{\gamma}} \Pi^{\alpha}_{\mu} \Pi^{\bar{\beta}}_{\bar{\nu}} \nabla_{\bar{\beta}} \nabla_\alpha \tilde{V}^{\bar{\gamma}};$$

no second fundamental form terms arise since $I\!I_{\mu\bar{\nu}}{}^{\gamma} = 0$. Noting that $D_0 V^{\bar{\lambda}} = \Pi^{\bar{\lambda}}_{\bar{\gamma}} \nabla_0 V^{\bar{\gamma}}$ we have the result. □

Remark 6.1.21. The above proposition holds with the same proof in the general codimension setting. The equation (6.1.20) can also be found in [54] where it (or its trace free part) is the key to proving rigidity for CR embeddings into the sphere with sufficiently low codimension because it allows one to show that the intrinsic pseudohermitian curvature determines the second fundamental form $I\!I_{\mu\nu}{}^{\gamma}$. ∎

6.1.7.2 The pseudohermitian Codazzi equation

Proposition 6.1.22. *Given compatible contact forms,*

$$R_{\mu\bar{\nu}}{}^{\bar{\gamma}}{}_{\bar{\rho}}\mathrm{N}_{\bar{\gamma}}^{\bar{\delta}} = -D_\mu I\!I_{\bar{\nu}\bar{\rho}}{}^{\bar{\delta}} \tag{6.1.21}$$

where the submanifold Tanaka-Webster connection D is coupled with the normal Tanaka-Webster connection $\nabla^\perp$.

Proof. Let N^γ be an unweighted unit normal field and let $\tilde{N}^\gamma$ be an extension of N^γ to all of M such that, along Σ, $\mathrm{N}_\beta^\alpha\nabla_\alpha\tilde{N}^\gamma = 0$ and $\mathrm{N}_{\bar{\beta}}^{\bar{\alpha}}\nabla_{\bar{\alpha}}\tilde{N}^\gamma = 0$. Then along Σ we have

$$\nabla_{\bar{\beta}}\tilde{N}^\gamma = -I\!I_{\bar{\beta}}{}^\gamma{}_\delta N^\delta + \mathrm{N}_\delta^\gamma\nabla_{\bar{\beta}}\tilde{N}^\delta$$

where $I\!I_{\bar{\beta}}{}^\gamma{}_\delta := \Pi_{\bar{\beta}}^{\bar{\nu}}\Pi_\lambda^\gamma I\!I_{\bar{\nu}}{}^\lambda{}_\delta$, using that $I\!I_{\bar{\nu}}{}^\lambda{}_\delta N^\delta = -\Pi_\gamma^\lambda\nabla_{\bar{\nu}}N^\gamma$. Thus we compute that

$$\begin{aligned}\Pi_\mu^\alpha\Pi_{\bar{\nu}}^{\bar{\beta}}\Pi_\gamma^\lambda\nabla_\alpha\nabla_{\bar{\beta}}\tilde{N}^\gamma &= \Pi_{\bar{\nu}}^{\bar{\beta}}\Pi_\gamma^\lambda\left(-\nabla_\mu(I\!I_{\bar{\beta}}{}^\gamma{}_\delta N^\delta) + (\nabla_\mu\mathrm{N}_\delta^\gamma)\nabla_{\bar{\beta}}N^\delta\right)\\ &= -D_\mu(I\!I_{\bar{\nu}}{}^\lambda{}_\delta N^\delta)\end{aligned}$$

along Σ, where in the first step we used that $\Pi_\gamma^\lambda\mathrm{N}_\delta^\gamma = 0$ and in the second step we used (6.1.14) to show that $\Pi_\gamma^\lambda\nabla_\mu\mathrm{N}_\delta^\gamma = 0$ and Proposition 6.1.8. Now on the other hand (since $I\!I_{\mu\bar{\nu}}{}^{\bar{\gamma}} = 0$) we have

$$\nabla_\alpha\tilde{N}^\gamma = \mathrm{N}_\delta^\gamma\nabla_\alpha\tilde{N}^\delta$$

along Σ, and this time we compute that

$$\Pi_\mu^\alpha\Pi_{\bar{\nu}}^{\bar{\beta}}\Pi_\gamma^\lambda\nabla_{\bar{\beta}}\nabla_\alpha\tilde{N}^\gamma = -I\!I_{\bar{\nu}}{}^\lambda{}_\delta\nabla_\mu N^\delta = -I\!I_{\bar{\nu}}{}^\lambda{}_\delta\nabla_\mu^\perp N^\delta$$

since $\nabla_{\bar{\nu}}\mathrm{N}_\delta^\gamma = -I\!I_{\bar{\nu}}{}^\rho{}_\delta\Pi_\rho^\gamma$ by (6.1.15). Putting these together we get

$$\Pi_\mu^\alpha\Pi_{\bar{\nu}}^{\bar{\beta}}\Pi_\gamma^\lambda\left(\nabla_\alpha\nabla_{\bar{\beta}} - \nabla_{\bar{\beta}}\nabla_\alpha\right)\tilde{N}^\gamma = -(D_\mu I\!I_{\bar{\nu}}{}^\lambda{}_{\bar{\delta}})N^{\bar{\delta}}$$

along Σ. Since $I\!I_{0\mu}{}^\gamma = 0$ we have $\Pi_\gamma^\lambda\nabla_0 N^\gamma = 0$ and hence from (5.2.5) we obtain

$$R_{\mu\bar{\nu}}{}^\lambda{}_\delta N^\delta = (D_\mu I\!I_{\bar{\nu}}{}^\lambda{}_{\bar{\delta}})N^{\bar{\delta}}. \tag{6.1.22}$$

Noting that $R_{\mu\bar{\nu}\bar{\lambda}\delta} = -R_{\mu\bar{\nu}\delta\bar{\lambda}}$ then gives the result. □

6.1.7.3 The pseudohermitian Ricci equation

Given a compatible pair of contact forms we let $R^{\mathcal{N}^{\bar{\alpha}}}$ denote the curvature of the normal Tanaka-Webster connection $\nabla^\perp$ on the antiholomorphic normal bundle $\mathcal{N}^{\bar{\alpha}}$. With our conventions we have

$$\left(\nabla^\perp_\mu \nabla^\perp_{\bar{\nu}} N^{\bar{\gamma}} - \nabla^\perp_{\bar{\nu}} \nabla^\perp_\mu N^{\bar{\gamma}} + i\boldsymbol{h}_{\mu\bar{\nu}} \nabla^\perp_0 N^{\bar{\gamma}}\right) = -R^{\mathcal{N}^{\bar{\alpha}}}_{\mu\bar{\nu}}{}^{\bar{\gamma}}{}_{\bar{\delta}} N^{\bar{\delta}} \tag{6.1.23}$$

for any section $N^{\bar{\alpha}}$ of $\mathcal{N}^{\bar{\alpha}}$, where we have coupled the normal Tanaka-Webster connection $\nabla^\perp$ with the submanifold Tanaka-Webster connection D. The *pseudohermitian Ricci equation* relates the component $R^{\mathcal{N}^{\bar{\alpha}}}_{\mu\bar{\nu}}{}^{\bar{\gamma}}{}_{\bar{\delta}}$ of $R^{\mathcal{N}^{\bar{\alpha}}}$ to the component $R_{\mu\bar{\nu}}{}^{\bar{\gamma}'}{}_{\bar{\delta}'} \mathrm{N}^{\bar{\gamma}}_{\bar{\gamma}'} \mathrm{N}^{\bar{\delta}'}_{\bar{\delta}}$ of the ambient pseudohermitian curvature tensor:

Proposition 6.1.23. *Given compatible contact forms,*

$$R^{\mathcal{N}^{\bar{\alpha}}}_{\mu\bar{\nu}}{}^{\bar{\gamma}}{}_{\bar{\delta}} = R_{\mu\bar{\nu}}{}^{\bar{\gamma}'}{}_{\bar{\delta}'} \mathrm{N}^{\bar{\gamma}}_{\bar{\gamma}'} \mathrm{N}^{\bar{\delta}'}_{\bar{\delta}} + \boldsymbol{h}^{\lambda\bar{\rho}} I\!I_{\mu\lambda\bar{\delta}} I\!I_{\bar{\nu}\bar{\rho}}{}^{\bar{\gamma}}. \tag{6.1.24}$$

Proof. To facilitate calculation we couple the connection $\nabla^\perp$ with the submanifold Tanaka-Webster connection D; we also couple ∇ with D. If $N^{\bar{\gamma}}$ is a holomorphic normal field then

$$\begin{aligned}
\nabla^\perp_\mu \nabla^\perp_{\bar{\nu}} N^{\bar{\gamma}} &= \nabla^\perp_\mu (\mathrm{N}^{\bar{\gamma}}_{\bar{\delta}} \nabla_{\bar{\nu}} N^{\bar{\delta}}) \\
&= \mathrm{N}^{\bar{\gamma}}_{\bar{\epsilon}} \nabla_\mu (\mathrm{N}^{\bar{\epsilon}}_{\bar{\delta}} \nabla_{\bar{\nu}} N^{\bar{\delta}}) \\
&= \mathrm{N}^{\bar{\gamma}}_{\bar{\epsilon}} \left(-I\!I_\mu{}^{\bar{\lambda}}{}_{\bar{\delta}} \Pi^{\bar{\epsilon}}_{\bar{\lambda}} \nabla_{\bar{\nu}} N^{\bar{\delta}} + \mathrm{N}^{\bar{\epsilon}}_{\bar{\delta}} \nabla_\mu \nabla_{\bar{\nu}} N^{\bar{\delta}}\right) \\
&= \mathrm{N}^{\bar{\gamma}}_{\bar{\delta}} \nabla_\mu \nabla_{\bar{\nu}} N^{\bar{\delta}}
\end{aligned}$$

On the other hand, when we interchange the roles of μ and ν we obtain

$$\begin{aligned}
\nabla^\perp_{\bar{\nu}} \nabla^\perp_\mu N^{\bar{\gamma}} &= \mathrm{N}^{\bar{\gamma}}_{\bar{\epsilon}} \left(-I\!I_{\bar{\nu}\bar{\lambda}}{}^{\bar{\epsilon}} \Pi^{\bar{\lambda}}_{\bar{\delta}} \nabla_\mu N^{\bar{\delta}} + \mathrm{N}^{\bar{\epsilon}}_{\bar{\delta}} \nabla_{\bar{\nu}} \nabla_\mu N^{\bar{\delta}}\right) \\
&= I\!I_{\bar{\nu}\bar{\lambda}}{}^{\bar{\gamma}} I\!I_\mu{}^{\bar{\lambda}}{}_{\bar{\delta}} N^{\bar{\delta}} + \mathrm{N}^{\bar{\gamma}}_{\bar{\delta}} \nabla_{\bar{\nu}} \nabla_\mu N^{\bar{\delta}}
\end{aligned}$$

Now observe that if one extends $N^{\bar{\gamma}}$ off Σ such that $N^\alpha \nabla_\alpha \tilde{N}^{\bar{\gamma}} = 0$ and $N^{\bar{\alpha}} \nabla_{\bar{\alpha}} \tilde{N}^{\bar{\gamma}} = 0$, then

$$\Pi^\alpha_\mu \Pi^{\bar{\beta}}_{\bar{\nu}} \nabla_\alpha \nabla_{\bar{\beta}} \tilde{N}^{\bar{\delta}} = \nabla_\mu \nabla_{\bar{\nu}} N^{\bar{\delta}} \quad \text{and} \quad \Pi^\alpha_\mu \Pi^{\bar{\beta}}_{\bar{\nu}} \nabla_{\bar{\beta}} \nabla_\alpha \tilde{N}^{\bar{\delta}} = \nabla_{\bar{\nu}} \nabla_\mu N^{\bar{\delta}}.$$

Thus by (6.1.23) and (5.2.5) (noting that $\nabla^\perp_0 N^{\bar{\gamma}} = \nabla_0 N^{\bar{\gamma}}$) one has the result. □

Remark 6.1.24. Since $\mathcal{N}^{\bar{\alpha}}$ is a line bundle we may think of the curvature $R^{\mathcal{N}^{\bar{\alpha}}}$ instead as a two form. By convention we take $R^{\mathcal{N}^{\bar{\alpha}}}_{\mu\bar{\nu}}$ to be $R^{\mathcal{N}^{\bar{\alpha}}}_{\mu\bar{\nu}}{}^{\bar{\gamma}}{}_{\bar{\gamma}}$, which means that $R^{\mathcal{N}^{\bar{\alpha}}}$ is minus the usual curvature two form of the connection $\nabla^{\perp}$ on the line bundle $\mathcal{N}^{\bar{\alpha}}$. With this convention we may write

$$R^{\mathcal{N}^{\bar{\alpha}}}_{\mu\bar{\nu}} = R_{\mu\bar{\nu}N\bar{N}} + \boldsymbol{h}_{\gamma\bar{\delta}}\boldsymbol{h}^{\lambda\bar{\rho}} I\!I_{\mu\lambda}{}^{\gamma} I\!I_{\bar{\nu}\bar{\rho}}{}^{\bar{\delta}} \tag{6.1.25}$$

where $R_{\mu\bar{\nu}N\bar{N}} = R_{\mu\bar{\nu}\gamma\bar{\delta}}N^{\gamma}N^{\bar{\delta}}$ for any weight $(1,0)$ unit normal field N^{α}. Also, since $\nabla^{\perp}$ is Hermitian with respect to the Levi form of θ (on $\mathcal{N}^{\bar{\alpha}}$), one has that $R^{\mathcal{N}^{\bar{\alpha}}} = R^{\mathcal{N}_{\alpha}}$ as two forms. Moreover, by duality one has that $R^{\mathcal{N}_{\alpha}} = -R^{\mathcal{N}^{\alpha}}$. ∎

6.1.8 Relating Density Bundles

We have already been using compatible contact forms to identify the density bundles $\mathcal{E}(1,1)|_{\Sigma}$ and $\mathcal{E}_{\Sigma}(1,1)$, and have commented in passing that this identification does not in fact depend on any choice of (compatible) contact forms. Let ς be a positive real element of $\mathcal{E}(1,1)|_{\Sigma}$, then there is a unique real element ς_{Σ} in $\mathcal{E}_{\Sigma}(1,1)$ such that $\varsigma^{-1}\boldsymbol{\theta}$ pulls back to $\varsigma_{\Sigma}\boldsymbol{\theta}_{\Sigma}$ under ι. This correspondence induces an isomorphism of complex line bundles. In this way we obtain canonical identifications between all diagonal density bundles $\mathcal{E}(w,w)|_{\Sigma}$ and $\mathcal{E}_{\Sigma}(w,w)$. These identifications also agree with those induced by trivialising the ambient and intrinsic (diagonal) density bundles using an ambient contact form θ and its pullback $\iota^*\theta$ respectively.

On the other hand it is not *a priori* obvious whether one may canonically identify the density bundles $\mathcal{E}(1,0)|_{\Sigma}$ and $\mathcal{E}_{\Sigma}(1,0)$, and therefore identify all corresponding density bundles $\mathcal{E}(w,w')|_{\Sigma}$ and $\mathcal{E}_{\Sigma}(w,w')$. We require that any isomorphism of $\mathcal{E}(1,0)|_{\Sigma}$ with $\mathcal{E}_{\Sigma}(1,0)$ should be compatible with the identification of $\mathcal{E}(1,1)|_{\Sigma}$ with $\mathcal{E}_{\Sigma}(1,1)$ already defined. Any two such isomorphisms of $\mathcal{E}(1,0)|_{\Sigma}$ with $\mathcal{E}_{\Sigma}(1,0)$ are related by an automorphism of $\mathcal{E}(1,0)|_{\Sigma}$ given by multiplication by $e^{i\varphi}$ for some $\varphi \in C^{\infty}(\Sigma)$. This is precisely the same freedom as in the choice of a unit holomorphic conormal, in fact, we shall see below that these two choices are intrinsically connected.

6.1.8.1 Densities and holomorphic conormals

Let $\Lambda^{1,0}_{\perp}\Sigma$ denote the subbundle of $\Lambda^{1,0}M|_{\Sigma}$ consisting of all forms N which vanish on the tangent space of Σ. The bundle $\Lambda^{1,0}_{\perp}\Sigma$ may be canonically identified with $\mathcal{N}_{\alpha}$ by restriction to $T^{1,0}M$.

Lemma 6.1.25. *Along Σ the ambient and submanifold canonical bundles are related by the canonical isomorphism*

$$\mathscr{K}|_{\Sigma} \cong \mathscr{K}_{\Sigma} \otimes \Lambda^{1,0}_{\perp}\Sigma. \tag{6.1.26}$$

Proof. The map from $\Lambda^n(\Lambda^{1,0}\Sigma) \otimes \Lambda^{1,0}_{\perp}\Sigma$ to $\Lambda^{n+1}(\Lambda^{1,0}M|_{\Sigma})$ is given by

$$\zeta_{\Sigma} \otimes N \mapsto \eta \wedge N \tag{6.1.27}$$

where η is any element of $\Lambda^n(\Lambda^{1,0}M|_{\Sigma})$ with $\iota^*\eta = \zeta_{\Sigma}$. □

Given a section $\zeta_{\Sigma} \otimes N$ of $\mathscr{K}_{\Sigma} \otimes \Lambda^{1,0}_{\perp}\Sigma$ we write $\zeta_{\Sigma} \wedge N$ for the corresponding section of $\mathscr{K}|_{\Sigma}$. The above lemma is the key to relating ambient and submanifold densities:

Corollary 6.1.26. *The ambient and submanifold density bundles are related via the canonical isomorphism*

$$\mathcal{E}(-n-1,0)|_{\Sigma} \cong \mathcal{E}_{\Sigma}(-n-1,0) \otimes \mathcal{N}_{\alpha}(1,0). \tag{6.1.28}$$

Proof. By definition $\mathcal{E}(-n-2,0) = \mathscr{K}$ and $\mathcal{E}_{\Sigma}(-n-1,0) = \mathscr{K}_{\Sigma}$. Using this in (6.1.26), tensoring both sides with $\mathcal{E}(1,0)|_{\Sigma}$, and identifying $\Lambda^{1,0}_{\perp}\Sigma$ with $\mathcal{N}_{\alpha}$ gives the result. □

Thus any trivialisation of the line bundle $\mathcal{N}_{\alpha}(1,0)$ gives an identification of the corresponding ambient and submanifold density bundles along Σ. One can check that if the trivialisation of $\mathcal{N}_{\alpha}(1,0)$ is given by a unit holomorphic conormal then the resulting identification of density bundles will be compatible with the usual identification of $\mathcal{E}(w,w)|_{\Sigma}$ with $\mathcal{E}_{\Sigma}(w,w)$; this amounts to the claim that, given compatible contact forms θ and θ_{Σ}, if ζ_{Σ} is a section of $\mathscr{K}_{\Sigma}$ volume normalised for θ_{Σ} and N is a section of $\Lambda^{1,0}_{\perp}\Sigma$ which is normalised with respect to the Levi form of θ (i.e. satisfies $h^{\alpha\bar{\beta}}N_{\alpha}\overline{N_{\beta}} = 1$) then the section $\zeta_{\Sigma} \wedge N$ of $\mathscr{K}|_{\Sigma}$ is volume normalised for θ.

Remark 6.1.27. The preceding observation motivates the search for a canonical unit holomorphic conormal. One way to approach this search is to observe that for any unit holomorphic conormal N_{α} the field $\varpi_{\bar{\mu}} := N^{\alpha}\nabla_{\bar{\mu}}N_{\alpha} = -N^{\bar{\alpha}}\nabla_{\bar{\mu}}N_{\bar{\alpha}}$ does not depend on the choice of admissible ambient contact form used to define $\nabla_{\bar{\mu}}$ and a calculation shows that $\varpi_{\bar{\mu}}$ satisfies $\nabla_{[\bar{\mu}}\varpi_{\bar{\nu}]} = 0$. In the case where one has local exactness of the tangential Cauchy-Riemann complex of Σ at $(0,1)$-forms one can then (locally) define a canonical unit holomorphic conormal N_{α} for which $\varpi_{\bar{\mu}} = \nabla_{\bar{\mu}}f$ with f a real valued function; the *a priori* phase freedom in the unit normal is used to eliminate the imaginary part of f, leaving no further freedom. However, for smooth (rather than real analytic) embeddings the required local exactness may not hold, as was famously demonstrated by Lewy for the three dimensional Heisenberg group [107]. In the following it will become plain that we should keep $\mathcal{N}_\alpha(1,0)$ in the picture, rather than trivialise it, and thus we have not pursued this direction further. ∎

6.1.9 Relating Connections on Density Bundles

Given an admissible ambient contact form θ, the normal Tanaka-Webster connection $\nabla^\perp$ on $\mathcal{N}_\alpha$ can equivalently be thought of as the connection on $\Lambda^{1,0}_\perp\Sigma$ defined by differentiating tangentially using the Tanaka-Webster connection ∇ and then projecting using the Webster metric g_θ.

Lemma 6.1.28. *Given any pair of compatible contact forms the isomorphism* (6.1.26) *of Lemma 6.1.25 intertwines the respective Tanaka-Webster connections:*

$$\begin{array}{ccccc} \mathscr{K}|_\Sigma & \cong & \mathscr{K}_\Sigma & \otimes & \Lambda^{1,0}_\perp\Sigma \\ \iota^*\nabla & \cong & D & \otimes & \nabla^\perp. \end{array}$$

Proof. Let $\zeta_\Sigma \otimes N$ be a section of $\mathscr{K}_\Sigma \otimes \Lambda^{1,0}_\perp\Sigma$. Let η be any section of $\Lambda^n(\Lambda^{1,0}M|_\Sigma)$ which pulls back to ζ_Σ. Then $\zeta_\Sigma \wedge N := \eta \wedge N$. If $X \in T\Sigma$ then

$$\nabla_X(\zeta_\Sigma \wedge N) = (\nabla_X \eta) \wedge N + \eta \wedge (\nabla_X N),$$

but $(\nabla_X\eta)\wedge N = (\Pi_\Sigma \nabla_X \eta)\wedge N$ which is the section of $\mathscr{K}|_\Sigma$ corresponding to $(D_X\zeta_\Sigma)\otimes N$ (here Π_Σ denotes submanifold tangential projection with respect to g_θ), and $\eta \wedge (\nabla_X N) = \eta \wedge (\nabla^\perp_X N)$. □

Observing that the connection $\nabla^\perp$ on $\mathcal{N}_\alpha(1,0)$ agrees with the coupling of $\nabla^\perp$ on $\mathcal{N}_\alpha$ with $\iota^*\nabla$ on $\mathcal{E}(1,0)|_\Sigma$ we have the following corollary:

Corollary 6.1.29. *Given any pair of compatible contact forms the (canonical) isomorphism* (6.1.28) *of Corollary 6.1.26 intertwines the respective Tanaka-Webster connections:*

$$\begin{array}{ccccc} \mathcal{E}(-n-1,0)|_\Sigma & \cong & \mathcal{E}_\Sigma(-n-1,0) & \otimes & \mathcal{N}_\alpha(1,0) \\ \iota^*\nabla & \cong & D & \otimes & \nabla^\perp. \end{array}$$

This means that if we want to identify corresponding ambient and submanifold density bundles (along Σ) in such a way that the ambient and submanifold Tanaka-Webster connections of a pair of compatible contact forms agree (in the sense that $\iota^*\nabla = D$), then we must trivialise $\mathcal{N}_\alpha(1,0)$ using a section which is parallel for the normal Tanaka-Webster connection $\nabla^\perp$. This is not a CR invariant condition on the section of $\mathcal{N}_\alpha(1,0)$, and the following lemma shows that it is not possible to find a parallel section in general because of curvature:

Lemma 6.1.30. *Let θ and θ_Σ be compatible contact forms and let $R^{\mathcal{N}_\alpha(1,0)}$ denote the curvature of $\nabla^\perp$ on the bundle $\mathcal{N}_\alpha(1,0)$, then the $(1,1)$-component of $R^{\mathcal{N}_\alpha(1,0)}|_{H_\Sigma}$ satisfies*

$$R_{\mu\bar{\nu}}^{\mathcal{N}_\alpha(1,0)} = \frac{n+1}{n+2} R_{\mu\bar{\nu}} - r_{\mu\bar{\nu}}, \tag{6.1.29}$$

where $R_{\mu\bar{\nu}} = \Pi_\mu^\alpha \Pi_{\bar{\nu}}^{\bar{\beta}} R_{\alpha\bar{\beta}}$.

Proof. By Proposition 5.2.5 the $(1,1)$-component of the restriction to H of the curvature of the Tanaka-Webster connection on the line bundle $\mathcal{E}(1,0)$ is $\frac{1}{n+2}R_{\alpha\bar{\beta}}$. Thus the $(1,1)$-component of the the restriction to H_Σ of the curvature of $\iota^*\nabla$ on $\mathcal{E}(1,0)|_\Sigma$ is $\frac{1}{n+2}R_{\mu\bar{\nu}}$. Combining this with the Ricci equation (6.1.25) for $R_{\mu\bar{\nu}}^{\mathcal{N}^{\bar{\alpha}}} = R_{\mu\bar{\nu}}^{\mathcal{N}_\alpha}$ we have

$$R_{\mu\bar{\nu}}^{\mathcal{N}_\alpha(1,0)} = R_{\mu\bar{\nu}N\bar{N}} + \boldsymbol{h}_{\gamma\bar{\delta}} I\!I_{\mu\lambda}{}^{\gamma} I\!I_{\bar{\nu}}{}^{\lambda\bar{\delta}} - \frac{1}{n+2} R_{\mu\bar{\nu}}.$$

Using the once contracted Gauss equation

$$R_{\mu\bar{\nu}} - R_{\mu\bar{\nu}N\bar{N}} = r_{\mu\bar{\nu}} + \boldsymbol{h}_{\gamma\bar{\delta}} I\!I_{\mu\lambda}{}^{\gamma} I\!I_{\bar{\nu}}{}^{\lambda\bar{\delta}}$$

obtained from (6.1.20) we have the result. □

Remark 6.1.31. Here, because of our conventions (cf. Remark 6.1.24), we take $R^{\mathcal{N}_\alpha(1,0)}$ to be minus the usual curvature of $\mathcal{N}_\alpha(1,0)$ as a line bundle. ▮

6.1.10 The Ratio Bundle of Densities

The observations of §§6.1.8 and §§6.1.9 motivate us to look at the relationship between corresponding ambient and submanifold density bundles rather than seeking to identify them (along Σ). We therefore make the following definition:

Definition 6.1.32. The *ratio bundle of densities* of weight (w, w') is the complex line bundle

$$\mathcal{R}(w, w') := \mathcal{E}(w, w')|_\Sigma \otimes \mathcal{E}_\Sigma(-w, -w') \tag{6.1.30}$$

on the submanifold Σ. Equivalently $\mathcal{R}(w, w')$ is the bundle whose sections are endomorphisms from $\mathcal{E}_\Sigma(w, w')$ to $\mathcal{E}(w, w')|_\Sigma$.

Note that the bundles $\mathcal{R}(w, w)$ are canonically trivial, and therefore $\mathcal{R}(w, w')$ is canonically isomorphic to $\mathcal{R}(w - w', 0)$. Also by definition $\mathcal{R}(-n - 1, 0)$ is canonically isomorphic to $\mathcal{N}_\alpha(1, 0)$, and we make this into an identification

$$\mathcal{R}(-n - 1, 0) = \mathcal{N}_\alpha(1, 0). \tag{6.1.31}$$

6.1.11 The Canonical Connection on the Ratio Bundles

Borrowing insight from §6.2 below we observe that the bundle $\mathcal{N}_\alpha(1, 0)$ carries a natural CR invariant connection, which induces connections on the density ratio bundles $\mathcal{R}(w, w')$. The reason is that $\mathcal{N}_\alpha(1, 0)$ is canonically isomorphic to a subbundle $\mathcal{N}_A$ of the ambient cotractor bundle $\mathcal{E}_A$ along Σ which has an invariant connection induced by the ambient tractor connection (Proposition 6.2.3). We denote this canonical invariant connection on $\mathcal{R}(w, w')$ by $\nabla^{\mathcal{R}}$. It turns out to be very naturally expressed in terms of Weyl connections (recall §§§5.3.4.1). Hence we make the following definition:

Definition 6.1.33. Given an admissible ambient contact form θ, the *normal Weyl connection* $\nabla^{W,\perp}$ on $\mathcal{N}^\alpha(w, w')$ is the connection induced by ∇^W (projecting tangential derivatives of sections back into $\mathcal{N}^\alpha$ using the Levi form). Dually, the connection $\nabla^{W,\perp}$ acts on $\mathcal{N}_\alpha(-w, -w')$.

For calculational purposes we will need the following lemma:

Lemma 6.1.34. *Given an admissible ambient contact form θ the connection $\nabla^{W,\perp}$ on $\mathcal{N}_\alpha(1, 0)$ acts on a section τ_α by*

$$\nabla^{W,\perp}_\mu \tau_\alpha = \nabla^\perp_\mu \tau_\alpha, \quad \nabla^{W,\perp}_{\bar\mu} \tau_\alpha = \nabla^\perp_{\bar\mu} \tau_\alpha \tag{6.1.32}$$

and

$$\nabla^{W,\perp}_0 \tau_\alpha = \nabla^\perp_0 \tau_\alpha - iP_{N\bar N}\tau_\alpha + \frac{i}{n+2} P\tau_\alpha \tag{6.1.33}$$

where $P_{N\bar{N}} = P_{\alpha\bar{\beta}}N^{\alpha}N^{\bar{\beta}}$ *for any (weighted) unit holomorphic normal* N^{α} *and* $P = P_{\alpha}{}^{\alpha}$.

Proof. This follows immediately from the definitions of the Weyl and normal Weyl connections and the formula (5.3.26). □

The connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}(-n-1,0) = \mathcal{N}_{\alpha}(1,0)$ turns out to agree precisely with the normal Weyl connection of *any* admissible contact form. In particular the normal Weyl connection $\nabla^{W,\perp}$ on the bundle $\mathcal{N}_{\alpha}(1,0)$ does not depend on the choice of admissible ambient contact form. This follows from Proposition 6.2.3 below, but here we give a direct proof. Before we prove this we make an important technical observation, stated in the following lemma:

Lemma 6.1.35. *Let* θ *be an admissible ambient contact form. The contact form* $\hat{\theta} = e^{\Upsilon}\theta$ *is admissible if and only if*

$$\Upsilon_{\alpha} = \Pi_{\alpha}^{\beta}\Upsilon_{\beta} \tag{6.1.34}$$

along Σ.

Proof. This follows immediately from the transformation law for the Reeb vector field given in Lemma 5.2.6 since both T and $\hat{T}$ must be tangent to Σ. □

Proposition 6.1.36. *The normal Weyl connection* $\nabla^{W,\perp}$ *on the bundle* $\mathcal{N}_{\alpha}(1,0)$ *does not depend on the choice of admissible contact form.*

Proof. Fix a pair of compatible contact forms θ, θ_{Σ} and suppose $\hat{\theta} = e^{\Upsilon}\theta$ is any other admissible ambient contact form. Let τ_{α} be a section of $\mathcal{N}_{\alpha}(1,0)$. Extend τ_{α} arbitrarily off Σ. When differentiating in contact directions the connections $\nabla^{W,\perp}$ and $\nabla^{\perp}$ agree, so from (5.2.30) and Proposition 5.2.9 we have

$$\begin{aligned}
\hat{\nabla}^{W,\perp}_{\mu}\tau_{\delta} &= \mathrm{N}_{\delta}^{\beta}\Pi_{\mu}^{\alpha}\hat{\nabla}_{\alpha}\tau_{\beta} \\
&= \mathrm{N}_{\delta}^{\beta}\Pi_{\mu}^{\alpha}(\nabla_{\alpha}\tau_{\beta} - \Upsilon_{\beta}\tau_{\alpha} - \Upsilon_{\alpha}\tau_{\beta} + \Upsilon_{\alpha}\tau_{\beta}) \\
&= \mathrm{N}_{\delta}^{\beta}\Pi_{\mu}^{\alpha}\nabla_{\alpha}\tau_{\beta}
\end{aligned}$$

since $\Pi_{\mu}^{\alpha}\tau_{\alpha} = 0$ (note that $\mathrm{N}_{\gamma}^{\beta}\Upsilon_{\beta}$ also vanishes since θ and $\hat{\theta}$ are admissible). Similarly, from (5.2.31) and Proposition 5.2.9 we have

$$\begin{aligned}\hat{\nabla}^{W,\perp}_{\bar{\mu}}\tau_\delta &= \mathrm{N}^\beta_\delta \Pi^{\bar{\alpha}}_{\bar{\mu}} \hat{\nabla}_{\bar{\alpha}}\tau_\beta \\ &= \mathrm{N}^\beta_\delta \Pi^{\bar{\alpha}}_{\bar{\mu}} (\nabla_{\bar{\alpha}}\tau_\beta + \boldsymbol{h}_{\beta\bar{\alpha}}\Upsilon^\gamma\tau_\gamma) \\ &= \mathrm{N}^\beta_\delta \Pi^{\bar{\alpha}}_{\bar{\mu}} \nabla_{\bar{\alpha}}\tau_\beta\end{aligned}$$

since $\mathrm{N}^\beta_\delta \Pi^\alpha_\mu \boldsymbol{h}_{\beta\bar{\alpha}} = 0$.

The operators ∇^W_0 and ∇_0 acting on τ_α are related by

$$\nabla^W_0\tau_\alpha = \nabla_0\tau_\alpha - iP_\alpha{}^\beta\tau_\beta + \frac{i}{n+2}P\tau_\alpha.$$

Now, on the one hand, by (5.2.32) and Proposition 5.2.9, noting that $\mathrm{N}^\beta_\delta\Upsilon_\beta = 0$ by Lemma 6.1.35, we have

$$\begin{aligned}\hat{\nabla}^\perp_0\tau_\delta = \mathrm{N}^\beta_\delta[&\nabla_0\tau_\beta + i\Upsilon^{\bar{\gamma}}\nabla_{\bar{\gamma}}\tau_\beta - i\Upsilon^\gamma\nabla_\gamma\tau_\beta - i\Upsilon^\gamma{}_\beta\tau_\gamma \\ &+ \tfrac{1}{n+2}(\Upsilon_0 + i\Upsilon^\gamma{}_\gamma - i\Upsilon^\gamma\Upsilon_\gamma)\tau_\beta].\end{aligned}$$

On the other hand from (5.2.37) and (5.2.38), noting that $\Upsilon_{\alpha\bar{\beta}} + \Upsilon_{\bar{\beta}\alpha} = 2\Upsilon_{\bar{\beta}\alpha} - i\boldsymbol{h}_{\alpha\bar{\beta}}\Upsilon_0$ by (5.2.1), we have

$$\begin{aligned}\mathrm{N}^\beta_\delta[i\hat{P}_\beta{}^\gamma\tau_\gamma - \tfrac{i}{n+2}\hat{P}\tau_\beta] = \mathrm{N}^\beta_\delta[&i(P_\beta{}^\gamma - \Upsilon^\gamma{}_\beta + \tfrac{i}{2}\Upsilon_0\delta^\gamma_\beta - \tfrac{1}{2}\Upsilon^\epsilon\Upsilon_\epsilon\delta^\gamma_\beta)\tau_\gamma \\ &- \tfrac{i}{n+2}(P - \Upsilon^\gamma{}_\gamma + \tfrac{in}{2}\Upsilon_0 - \tfrac{n}{2}\Upsilon^\epsilon\Upsilon_\epsilon)\tau_\beta].\end{aligned}$$

Since $\frac{1}{2} - \frac{n}{2(n+2)} = \frac{1}{n+2}$ we obtain that

$$\hat{\nabla}^{W,\perp}_0\tau_\delta = \nabla^{W,\perp}_0\tau_\beta + i\Upsilon^{\bar{\mu}}\nabla^{W,\perp}_{\bar{\mu}}\tau_\beta - i\Upsilon^\mu\nabla^{W,\perp}_\mu\tau_\beta,$$

as required (recall that the '0-direction' has a different meaning on the left and right hand sides of the above display, cf. Lemma 5.2.7). □

Remark 6.1.37. Both Lemma 6.1.35 and Proposition 6.1.36 hold in the general codimension case with the same proof (as does Proposition 6.2.3). ▮

We therefore take $\nabla^{\mathcal{R}}$ to be the connection induced on the ratio bundles by the normal Weyl connection of an admissible contact form on $\mathcal{N}_\alpha(1,0)$, and give later in Proposition 6.2.3 of §§6.2.1 the tractor explanation for this invariant connection. In order to compute with $\nabla^{\mathcal{R}}$ we will need the following lemma:

Lemma 6.1.38. *In terms of a compatible pair of contact forms, θ, θ_Σ, the connection $\nabla^{\mathcal{R}}$ on a section $\phi \otimes \sigma$ of $\mathcal{E}(w, w')|_\Sigma \otimes \mathcal{E}_\Sigma(-w, -w')$ is given by*

$$\nabla^{\mathcal{R}}_\mu(\phi \otimes \sigma) = (\nabla_\mu \phi) \otimes \sigma + \phi \otimes (D_\mu \sigma), \tag{6.1.35}$$

$$\nabla^{\mathcal{R}}_{\bar\mu}(\phi \otimes \sigma) = (\nabla_{\bar\mu} \phi) \otimes \sigma + \phi \otimes (D_{\bar\mu} \sigma), \tag{6.1.36}$$

and

$$\nabla^{\mathcal{R}}_0(\phi \otimes \sigma) = (\nabla_0 \phi) \otimes \sigma + \phi \otimes (D_0 \sigma) + \tfrac{w-w'}{n+1}(iP_{N\bar N} - \tfrac{i}{n+2}P)\phi \otimes \sigma. \tag{6.1.37}$$

Proof. This follows from Lemma 6.1.34 combined with Corollary 6.1.29. □

Corollary 6.1.39. *The connection $\nabla^{\mathcal{R}}$ on the diagonal bundles $\mathcal{R}(w, w)$ is flat and agrees with the exterior derivative of sections in the canonical trivialisation.*

Proof. This follows from Lemma 6.1.38 combined with Lemma 5.2.1. □

Remark 6.1.40. By coupling with the connection $\nabla^{\mathcal{R}}$ we can invariantly convert connections (and hence other operators) acting on intrinsic densities to ones on ambient densities, and vice versa. This will allow us to relate the intrinsic and ambient tractor connections, their difference giving rise to the basic CR invariants of the embedding. ▮

6.1.11.1 The curvature of the canonical ratio bundle connection

We shall see that the connection $\nabla^{\mathcal{R}}$ is not flat in general, making it unnatural to identify the ambient and submanifold density bundles along Σ.

Let $\kappa^{\mathcal{R}(w,w')}$ denote the curvature of $\nabla^{\mathcal{R}}$ on the line bundle $\mathcal{R}(w, w')$, and let $R^{\mathcal{N}^*}$ denote the curvature of $\nabla^{W,\perp}$ on $\mathcal{N}_\alpha(1, 0)$ for any admissible contact form θ. By convention $R^{\mathcal{N}^*}$ has the opposite sign to the usual line bundle curvature $\kappa^{\mathcal{R}(-n-1,0)}$. Clearly the curvatures $\kappa^{\mathcal{R}(w,w')}$ are determined by $R^{\mathcal{N}^*}$, in particular

$$\kappa^{\mathcal{R}(1,0)} = \frac{1}{n+1} R^{\mathcal{N}^*}.$$

Here we give expressions for the components of $R^{\mathcal{N}^*}$. Note that the components of the restriction $R^{\mathcal{N}^*}|_{H_\Sigma}$ must be invariant.

Proposition 6.1.41. *The* $(1,1)$*-part of* $R^{\mathcal{N}^*}|_{H_\Sigma}$ *satisfies*

$$R^{\mathcal{N}^*}_{\mu\bar{\nu}} = (n+1)(P_{\mu\bar{\nu}} - p_{\mu\bar{\nu}}) + (P - P_{N\bar{N}} - p)\boldsymbol{h}_{\mu\bar{\nu}}, \tag{6.1.38}$$

where $P_{N\bar{N}} = P_{\alpha\bar{\beta}}N^\alpha N^{\bar{\beta}}$ *for any (weighted) unit holomorphic normal* N^α, $P = P_\alpha{}^\alpha$, *and* $p = p_\mu{}^\mu$.

Proof. Recall that $\nabla^{\mathcal{R}} = \nabla^{W,\perp}$ on $\mathcal{R}(-n-1,0) = \mathcal{N}_\alpha(1,0)$ for any admissible ambient contact form θ. Fixing θ admissible we have

$$\begin{aligned} -R^{\mathcal{N}^*}_{\mu\bar{\nu}}\tau_\alpha &= \left(\nabla^{W,\perp}_\mu\nabla^{W,\perp}_{\bar{\nu}} - \nabla^{W,\perp}_{\bar{\nu}}\nabla^{W,\perp}_\mu + i\boldsymbol{h}_{\mu\bar{\nu}}\nabla^{W,\perp}_0\right)\tau_\alpha \\ &= \left(\nabla^\perp_\mu\nabla^\perp_{\bar{\nu}} - \nabla^\perp_{\bar{\nu}}\nabla^\perp_\mu + i\boldsymbol{h}_{\mu\bar{\nu}}\nabla^\perp_0\right)\tau_\alpha \\ &\quad + (P_{N\bar{N}} - \frac{1}{n+2}P)\boldsymbol{h}_{\mu\bar{\nu}}\tau_\alpha \end{aligned}$$

for any section τ_α of $\mathcal{N}_\alpha(1,0)$, using Lemma 6.1.34. Thus from (6.1.29) of Lemma 6.1.30 we have that

$$R^{\mathcal{N}^*}_{\mu\bar{\nu}} = \frac{n+1}{n+2}R_{\mu\bar{\nu}} - r_{\mu\bar{\nu}} - (P_{N\bar{N}} - \frac{1}{n+2}P)\boldsymbol{h}_{\mu\bar{\nu}}.$$

Now using that $R_{\alpha\bar{\beta}} = (n+2)P_{\alpha\bar{\beta}} + P\boldsymbol{h}_{\alpha\bar{\beta}}$, from the definition of $P_{\alpha\bar{\beta}}$, and using the corresponding expression for $r_{\mu\bar{\nu}}$, we have the result. □

Note that $P - P_{N\bar{N}} - p$ is the trace of $P_{\mu\bar{\nu}} - p_{\mu\bar{\nu}}$, with respect to $\boldsymbol{h}_{\mu\bar{\nu}}$. The following lemma therefore manifests the CR invariance of $R^{\mathcal{N}^*}_{\mu\bar{\nu}}$.

Lemma 6.1.42. *Given any pair of compatible contact forms, the difference* $P_{\mu\bar{\nu}} - p_{\mu\bar{\nu}}$ *satisfies*

$$\begin{aligned} P_{\mu\bar{\nu}} - p_{\mu\bar{\nu}} &= \frac{1}{n+1}(S_{\mu\bar{\nu}N\bar{N}} + \tfrac{1}{2n}S_{N\bar{N}N\bar{N}}\boldsymbol{h}_{\mu\bar{\nu}}) \\ &\quad + \frac{1}{n+1}(I\!I_{\mu\lambda\bar{\gamma}}I\!I_{\bar{\nu}}{}^{\lambda\bar{\gamma}} + \tfrac{1}{2n}I\!I_{\rho\lambda\bar{\gamma}}I\!I^{\rho\lambda\bar{\gamma}}\boldsymbol{h}_{\mu\bar{\nu}}), \end{aligned} \tag{6.1.39}$$

where $S_{\mu\bar{\nu}N\bar{N}} = \Pi^\alpha_\mu\Pi^{\bar{\beta}}_{\bar{\nu}}S_{\alpha\bar{\beta}\gamma\bar{\delta}}N^\gamma N^{\bar{\delta}}$ *and* $S_{N\bar{N}N\bar{N}} = S_{\alpha\bar{\beta}\gamma\bar{\delta}}N^\alpha N^{\bar{\beta}}N^\gamma N^{\bar{\delta}}$ *for any (weighted) unit holomorphic normal* N^α.

Proof. Taking the trace free part of the Gauss equation (6.1.20) one has

$$\tfrac{1}{n+1}(S_{\mu\bar{\nu}\lambda}{}^\lambda - \tfrac{1}{n}S_\rho{}^\rho{}_\lambda{}^\lambda\boldsymbol{h}_{\mu\bar{\nu}}) + P_{\mu\bar{\nu}} = p_{\mu\bar{\nu}} + \tfrac{1}{n+1}(I\!I_{\mu\lambda\bar{\gamma}}I\!I_{\bar{\nu}}{}^{\lambda\bar{\gamma}} + \tfrac{1}{2n}I\!I_{\rho\lambda\bar{\gamma}}I\!I^{\rho\lambda\bar{\gamma}}\boldsymbol{h}_{\mu\bar{\nu}})$$

and noting that $S_{\mu\bar{\nu}\lambda}{}^{\lambda} = S_{\mu\bar{\nu}\gamma\bar{\delta}}(\boldsymbol{h}^{\gamma\bar{\delta}} - N^{\gamma}N^{\bar{\delta}}) = -S_{\mu\bar{\nu}N\bar{N}}$ and similarly that $S_{\rho}{}^{\rho}{}_{\lambda}{}^{\lambda} = S_{N\bar{N}N\bar{N}}$ one has the result. □

Remark 6.1.43. The difference $P_{\mu\bar{\nu}} - p_{\mu\bar{\nu}}$ is the CR analogue of the so called Fialkow tensor [44, 141] in conformal submanifold geometry, though here it is showing up in a completely new role. ▮

Proposition 6.1.44. *We have*

$$R^{\mathcal{N}^*}_{\mu\nu} = 0 \quad \textit{and} \quad R^{\mathcal{N}^*}_{\bar{\mu}\bar{\nu}} = 0. \tag{6.1.40}$$

Proof. By a straightforward calculation along the lines of the proof of Proposition 6.1.23 we have, given compatible contact forms, that

$$\nabla^{\perp}_{\mu}\nabla^{\perp}_{\nu}N_{\alpha} - \nabla^{\perp}_{\nu}\nabla^{\perp}_{\mu}N_{\alpha} = N_{\alpha}N^{\beta}(\nabla_{\mu}\nabla_{\nu}N_{\beta} - \nabla_{\nu}\nabla_{\mu}N_{\beta})$$

for any unit holomorphic conormal field N_{α} (where both $\iota^*\nabla$ and $\nabla^{\perp}$ are coupled with the submanifold Tanaka-Webster connection D). Noting that $\nabla_{\mu}\nabla_{\nu} = \nabla_{\nu}\nabla_{\mu}$ on densities by Proposition 5.2.5, we get that $-R^{\mathcal{N}^*}_{\mu\nu} = \Pi^{\alpha}_{\mu}\Pi^{\beta}_{\nu}R_{\alpha\beta}{}^{\gamma}{}_{\delta}N_{\gamma}N^{\delta}$; this is zero by (5.2.19), noting that $R_{\alpha\beta\bar{\gamma}\delta} = -R_{\alpha\beta\delta\bar{\gamma}}$. In a similar manner one shows that $R^{\mathcal{N}^*}_{\bar{\mu}\bar{\nu}}$ also vanishes. □

Given compatible contact forms, one also has the component $R^{\mathcal{N}^*}_{\mu 0}$. By a similar but more tedious calculation one arrives at the expression

$$R^{\mathcal{N}^*}_{\mu 0} = -V_{\mu\bar{N}N} - iT_{\mu} \tag{6.1.41}$$

where $T_{\mu} = \Pi^{\alpha}_{\mu}T_{\alpha}$, $V_{\mu\bar{N}N} = \Pi^{\alpha}_{\mu}V_{\alpha\bar{\beta}\gamma}N^{\bar{\beta}}N^{\gamma}$ for any (weighted) holomorphic normal field N^{α}, and the tensors T_{α} and $V_{\alpha\bar{\beta}\gamma}$ are defined by (5.3.11) and (5.3.35) respectively. One can obtain this expression more easily using the description of the canonical connection on $\mathcal{N}_{\alpha}(1,0)$ in terms of the ambient tractor connection given below.

6.2 CR Embedded Submanifolds and Tractors

Here we continue to work in the setting where $\iota : \Sigma \hookrightarrow M$ is a CR embedding of a hypersurface type CR manifold $(\Sigma^{2m+1}, H_{\Sigma}, J_{\Sigma})$ into a strictly pseudoconvex CR manifold

(M^{2n+1}, H, J) with $m = n - 1$. We adopt the notation $\mathcal{T}M$ rather than $\mathcal{T}$ for the standard tractor bundle of M, and write $\mathcal{T}\Sigma$ for the standard tractor bundle of Σ. Similarly we will denote the adjoint tractor bundles of M and Σ by $\mathcal{A}M$ and $\mathcal{A}\Sigma$ respectively. We will also use the abstract index notation $\mathcal{E}^I$ for $\mathcal{T}\Sigma$ and allow the use of indices I, J, K, L, I', and so on.

6.2.1 Normal Tractors

Given any unit section N_α of $\mathcal{N}_\alpha(1,0)$ we define the corresponding *(unit) normal (co)tractor* N_A to be the section of $\mathcal{E}_A|_\Sigma$, the ambient tractor bundle restricted to fibers over Σ, given by

$$N_A \overset{\theta}{=} \begin{pmatrix} 0 \\ N_\alpha \\ -\mathsf{H} \end{pmatrix} \tag{6.2.1}$$

where $\mathsf{H} = \frac{1}{n-1}\boldsymbol{h}^{\mu\bar{\nu}}\Pi_\mu^\alpha \nabla_{\bar{\nu}} N_\alpha$ and $\nabla_{\bar{\nu}}$ denotes the Tanaka-Webster connection of θ acting in tangential antiholomorphic directions along Σ; the tractor field N_A does not depend on the choice of ambient contact form θ since from (5.2.31) of Proposition 5.2.8 combined with Proposition 5.2.9 we have that

$$\hat{\mathsf{H}} = \mathsf{H} + \Upsilon^\alpha N_\alpha$$

when $\hat{\theta} = e^\Upsilon \theta$ (with $\Upsilon^\alpha = \nabla^\alpha \Upsilon$), as required by (5.3.3). If θ is admissible for the submanifold Σ then $\mathsf{H} = 0$ (since $II_{\bar{\nu}\mu}{}^\gamma = 0$) and

$$N_A \overset{\theta}{=} \begin{pmatrix} 0 \\ N_\alpha \\ 0 \end{pmatrix}. \tag{6.2.2}$$

Remark 6.2.1. The normal tractor N_A associated to a unit holomorphic conormal N_α is an analogue of the normal tractor associated to a weighted unit (co)normal field in conformal hypersurface geometry defined first in [7]. ∎

Definition 6.2.2. The *normal cotractor bundle* $\mathcal{N}_A$ is the subbundle of $\mathcal{E}_A|_\Sigma$, the ambient cotractor bundle along Σ, spanned by the normal tractor N_A given any unit holomorphic conormal field N_α. The *normal tractor bundle* $\mathcal{N}^A$ is the dual line subbundle of

$\mathcal{E}^A|_\Sigma$ spanned by $N^A = h^{A\bar{B}}\overline{N_B}$. We alternatively denote $\mathcal{N}^A$ and $\mathcal{N}_A$ by $\mathcal{N}$ and $\mathcal{N}^*$ respectively.

Since the ambient tractor bundle carries a parallel Hermitian bundle metric the ambient tractor connection induces a connection $\nabla^{\mathcal{N}}$ on the non-null subbundle $\mathcal{N}_A$ of $\mathcal{E}_A|_\Sigma$. Explicitly, if N_B^A is the orthogonal projection from $\mathcal{E}_A|_\Sigma$ onto $\mathcal{N}_B$ then we have

$$\nabla_i^{\mathcal{N}} v_B = \mathrm{N}_B^A \nabla_i v_A \tag{6.2.3}$$

for any section v_B of $\mathcal{N}_B$, where ∇_i is the ambient standard tractor connection (pulled back via ι). We can now explain the origin of the canonical connection on $\mathcal{N}_\alpha(1,0)$.

Proposition 6.2.3. *The weighted conormal bundle $\mathcal{N}_\alpha(1,0)$ is canonically isomorphic to the normal cotractor bundle $\mathcal{N}_A$ via the map*

$$\tau_\alpha \mapsto \tau_A \overset{\theta}{=} \begin{pmatrix} 0 \\ \tau_\alpha \\ 0 \end{pmatrix} \tag{6.2.4}$$

defined with respect to any admissible ambient contact form θ. Moreover, the above isomorphism intertwines the normal tractor connection $\nabla^{\mathcal{N}}$ on $\mathcal{N}_A$ with the normal Weyl connection on $\mathcal{N}_\alpha(1,0)$ of any admissible θ.

Proof. The first part follows from the fact that if θ is admissible then $\hat{\theta} = e^{\Upsilon}\theta$ is admissible if and only if $\Upsilon_\alpha N^\alpha = 0$, where N^α is a holomorphic normal field (a consequence of Lemma 5.2.6). The second part follows from the explicit formulae for the tractor connection given in §§5.3.4 (noting in particular (5.3.27)) and the observation that the orthogonal projection $\mathcal{E}_A|_\Sigma \to \mathcal{N}_A$ is given, with respect to any admissible ambient contact form, by

$$\begin{pmatrix} \sigma \\ \tau_\alpha \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \mathrm{N}_\alpha^\beta \tau_\beta \\ 0 \end{pmatrix}. \tag{6.2.5}$$

□

Remark 6.2.4. Clearly the isomorphism of Proposition 6.2.3 is Hermitian; in particular if N_A is the normal tractor corresponding to a unit normal field N_α then

$$N^A N_A = N^\alpha N_\alpha = 1,$$

so N_A is indeed a unit normal tractor. Although a unit normal tractor is determined only up to phase, the tractors

$$N_A N_{\bar{B}} \quad \text{and} \quad N^A N_B,$$

are independent of the choice of unit length section of $\mathcal{N}_A$. Indeed, $N^A N_B = \mathrm{N}^A_B$ and the section

$$\Pi^A_B = \delta^A_B - N^A N_B$$

projects orthogonally from $\mathcal{E}_A|_\Sigma$ onto the orthogonal complement $\mathcal{N}_A^\perp$ of $\mathcal{N}_A$ in $\mathcal{E}_A|_\Sigma$. ∎

6.2.2 Tractor Bundles and Densities

Clearly $\mathcal{N}_A^\perp$ has the same rank as $\mathcal{E}_I$; they also have the same rank subbundles in their canonical filtration structures. Moreover, both $\mathcal{N}_A^\perp$ and $\mathcal{E}_I$ carry canonical Hermitian bundle metrics (and Hermitian connections). On the other hand we note that for $\mathcal{N}_A^\perp$ we have the canonical map

$$\begin{array}{ccc} \mathcal{N}_A^\perp & \to & \mathcal{E}(1,0)|_\Sigma \\ v_A & \mapsto & Z^A v_A \end{array}$$

where Z^A is the ambient canonical tractor, whereas for $\mathcal{E}_I$ we have the canonical map

$$\begin{array}{ccc} \mathcal{E}_I & \to & \mathcal{E}_\Sigma(1,0) \\ v_I & \mapsto & Z^I v_I \end{array}$$

where Z^I is the canonical tractor of Σ. It seems natural that we should look to identify these bundles (canonically), but doing so clearly also involves identifying the density bundles $\mathcal{E}(1,0)|_\Sigma$ and $\mathcal{E}_\Sigma(1,0)$ (also canonically). The following lemma shows us that this is the only thing stopping us from identifying $\mathcal{E}_I$ with $\mathcal{N}_A^\perp$:

Lemma 6.2.5. *Fix a local isomorphism $\psi : \mathcal{E}_\Sigma(1,0) \to \mathcal{E}(1,0)|_\Sigma$ (compatible with the canonical identification of $\mathcal{E}_\Sigma(1,1)$ with $\mathcal{E}(1,1)|_\Sigma$) and identify all corresponding density bundles $\mathcal{E}_\Sigma(w,w')$ and $\mathcal{E}(w,w')|_\Sigma$ using ψ. Then locally there is a canonically induced map from $\mathcal{E}_I$ to $\mathcal{N}_A^\perp$, given with respect to any pair θ, θ_Σ of compatible contact forms by*

$$v_I \stackrel{\theta_\Sigma}{=} \begin{pmatrix} \sigma \\ \tau_\mu \\ \rho \end{pmatrix} \mapsto v_A \stackrel{\theta}{=} \begin{pmatrix} \sigma \\ \tau_\alpha \\ \rho \end{pmatrix} \tag{6.2.6}$$

where $\tau_\alpha = \Pi_\alpha^\mu \tau_\mu$, which is a filtration preserving isomorphism of Hermitian vector bundles.

Proof. Let us start by fixing θ and θ_Σ. That the map described above is a filtration preserving bundle isomorphism is obvious. That the map pulls back the Hermitian bundle metric of $\mathcal{N}_A^\perp$ to that of $\mathcal{E}_I$ is also obvious. It remains to show that the map is independent of the choice of compatible contact forms. To see this we suppose that $\hat{\theta} = e^\Upsilon \theta$ is any other admissible contact form and let $\hat{\theta}_\Sigma = \iota^*\hat{\theta} = e^\Upsilon \theta_\Sigma$. We need to compare the submanifold and ambient versions of the tractor transformation law (5.3.3). By the compatibility of θ and θ_Σ we have $\nabla_0 \Upsilon = D_0 \Upsilon$ along Σ, and by Lemma 6.1.35 we also have $\Upsilon_\alpha = \Pi_\alpha^\mu \Upsilon_\mu$ where $\Upsilon_\mu = D_\mu \Upsilon$. These observations ensure that the map is well-defined. □

The local bundle isomorphism $\psi : \mathcal{E}_\Sigma(1,0) \to \mathcal{E}(1,0)|_\Sigma$ in the above lemma can also be thought of as a nonvanishing local section (or local trivialisation) of the ratio bundle $\mathcal{R}(1,0)$. The bundle $\mathcal{R}(1,1)$ is canonically trivial because of the canonical isomorphism of $\mathcal{E}_\Sigma(1,1)$ with $\mathcal{E}(1,1)|_\Sigma$, so that $\mathcal{R}(1,0)$ carries a natural Hermitian bundle metric (i.e. is a $\mathrm{U}(1)$-bundle) and the compatibility of ψ with the identification $\mathcal{E}_\Sigma(1,1) = \mathcal{E}(1,1)|_\Sigma$ is equivalent to ψ being a unit section of $\mathcal{R}(1,0)$. The ratio bundle $\mathcal{R}(1,0)$ will prove to be the key to relating the tractor bundles (globally) without making an unnatural (local) identification of density bundles.

6.2.3 Relating Tractor Bundles

If we tensor $\mathcal{E}_I$ with $\mathcal{E}_\Sigma(0,1)$ then choosing a submanifold contact form θ_Σ identifies this bundle with

$$[\mathcal{E}_I]_{\theta_\Sigma} \otimes \mathcal{E}_\Sigma(0,1) = \mathcal{E}_\Sigma(1,1) \oplus \mathcal{E}_\mu(1,1) \oplus \mathcal{E}_\Sigma(0,0)$$

where $\mathcal{E}_\Sigma(0,0)$ is the trivial bundle $\Sigma \times \mathbb{C}$. Similarly, given an ambient contact form θ we may identify the $\mathcal{N}_A^\perp \otimes \mathcal{E}(0,1)|_\Sigma$ with

$$[\mathcal{N}_A^\perp]_\theta \otimes \mathcal{E}(0,1)|_\Sigma = \mathcal{E}(1,1)|_\Sigma \oplus \mathcal{N}_\alpha^\perp(1,1) \oplus \mathcal{E}(0,0)|_\Sigma$$

where $\mathcal{E}(0,0)$ is the trivial bundle $M \times \mathbb{C}$ and $\mathcal{N}_\alpha^\perp$ denotes the orthogonal complement to $\mathcal{N}_\alpha$ in $\mathcal{E}_\alpha|_\Sigma$. Since $\mathcal{E}_\Sigma(w,w)$ is canonically identified with $\mathcal{E}(w,w)|_\Sigma$ we have the following theorem:

Theorem 6.2.6. *There is a canonical filtration preserving bundle isomorphism*

$$\Pi_A^I : \mathcal{E}_I \otimes \mathcal{E}_\Sigma(0,1) \to \mathcal{N}_A^\perp \otimes \mathcal{E}(0,1)|_\Sigma$$

given with respect to a pair of compatible contact forms θ, θ_Σ by

$$[\mathcal{E}_I]_{\theta_\Sigma} \otimes \mathcal{E}_\Sigma(0,1) \ni \begin{pmatrix} \sigma \\ \tau_\mu \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \tau_\alpha \\ \rho \end{pmatrix} \in [\mathcal{N}_A^\perp]_\theta \otimes \mathcal{E}(0,1)|_\Sigma$$

where $\tau_\alpha = \Pi_\alpha^\mu \tau_\mu$.

Proof. We only need to establish that the map is independent of the choice of compatible contact forms, and this follows from comparing the submanifold and ambient versions of (5.3.3) noting that $\nabla_0 \Upsilon = D_0 \Upsilon$ and $\Upsilon_\alpha = \Pi_\alpha^\mu \Upsilon_\mu$ as in the proof of Lemma 6.2.5. □

Remark 6.2.7. If $\psi : \mathcal{E}_\Sigma(1,0) \to \mathcal{E}(1,0)|_\Sigma$ is a local bundle isomorphism (unit as a local section of $\mathcal{R}(1,0)$) and we denote by $T_\psi \iota$ the local isomorphism $\mathcal{E}_I \to \mathcal{N}_A^\perp$ given by Lemma 6.2.5 then isomorphism of Proposition 6.2.6 agrees with $T_\psi \iota \otimes \overline{\psi}$ where this is defined. ▮

Conjugating the map (6.2.6) and raising tractor indices one gets an isomorphism

$$\Pi_I^A : \mathcal{E}^I \otimes \mathcal{E}_\Sigma(1,0) \to (\mathcal{N}^A)^\perp \otimes \mathcal{E}(1,0)|_\Sigma$$

and tensoring both sides with $\mathcal{E}_\Sigma(-1,0)$ one gets another isomorphism

$$\mathcal{E}^I \to (\mathcal{N}^A)^\perp \otimes \underbrace{\mathcal{E}(1,0)|_\Sigma \otimes \mathcal{E}_\Sigma(-1,0)}_{\mathcal{R}(1,0)} \tag{6.2.7}$$

which we may also denote by Π_I^A. We think of Π_I^A as a section of $\mathcal{E}_I \otimes \mathcal{E}^A|_\Sigma \otimes \mathcal{R}(1,0)$ and Π_A^I as a section of $\mathcal{E}^I \otimes \mathcal{E}_A|_\Sigma \otimes \mathcal{R}(0,1)$. Thinking about these objects as sections emphasises that they can be interpreted as maps in a variety of ways.

Definition 6.2.8. The isomorphism (6.2.7) gives an injective bundle map

$$\mathcal{T}^{\mathcal{R}}\iota : \mathcal{T}\Sigma \to \mathcal{T}M|_\Sigma \otimes \mathcal{R}(1,0) \tag{6.2.8}$$

which we term the *twisted tractor map.*

The twisted tractor map is clearly filtration preserving, and restricts to an isomorphism $\mathcal{T}^1\Sigma \to \mathcal{T}^1 M|_\Sigma \otimes \mathcal{R}(1,0)$. This is just the trivial isomorphism

$$\mathcal{E}_\Sigma(-1,0) \cong \mathcal{E}(-1,0)|_\Sigma \otimes \mathcal{E}(1,0)|_\Sigma \otimes \mathcal{E}_\Sigma(-1,0). \tag{6.2.9}$$

Since it is filtration preserving $\mathcal{T}^{\mathcal{R}}\iota$ also induces an injective bundle map $\mathcal{T}^0\Sigma/\mathcal{T}^1\Sigma \to (\mathcal{T}^0 M|_\Sigma/\mathcal{T}^1 M|_\Sigma) \otimes \mathcal{R}(1,0)$ and this is simply the tangent map $\mathcal{E}^\mu \to \mathcal{E}^\alpha|_\Sigma$ tensored with the isomorphism (6.2.9). The map $\mathcal{T}\Sigma/\mathcal{T}^0\Sigma \to (\mathcal{T}M|_\Sigma/\mathcal{T}^0 M|_\Sigma)\otimes\mathcal{R}(1,0)$ induced by the twisted tractor map is the isomorphism

$$\mathcal{E}_\Sigma(0,1) \cong \mathcal{E}(0,1)|_\Sigma \otimes \mathcal{R}(1,0)$$

which simply comes from noting that $\mathcal{R}(1,0) = \mathcal{R}(0,-1)$ since $\mathcal{E}_\Sigma(1,1) = \mathcal{E}(1,1)|_\Sigma$. Note that since $\mathcal{R}(1,0)$ is Hermitian, so is $\mathcal{T}M|_\Sigma \otimes \mathcal{R}(1,0)$, and $\mathcal{T}^{\mathcal{R}}\iota$ is clearly a Hermitian bundle map. These properties characterise the twisted tractor map.

6.2.3.1 The adjoint tractor map

Since $\mathcal{R}(1,1)$ is canonically trivial the section $\Pi_I^A \Pi_B^J$ gives us a canonical bundle map

$$\mathrm{End}(\mathcal{T}\Sigma) \to \mathrm{End}(\mathcal{T}M).$$

Since the twisted tractor map is metric preserving by restricting to skew-Hermitian endomorphisms we get a map

$$\mathcal{A}\iota : \mathcal{A}\Sigma \to \mathcal{A}M$$

which we term the *adjoint tractor map*. Recalling the projection $\mathcal{A}M \to TM$ given by (5.3.30) we note that the diagram

$$\begin{array}{ccc} \mathcal{A}\Sigma & \to & \mathcal{A}M \\ \downarrow & & \downarrow \\ T\Sigma & \to & TM \end{array} \tag{6.2.10}$$

is easily seen to commute. So the adjoint tractor map is a lift of the tangent map.

6.2.4 Relating Tractor Connections on $\mathcal{T}\Sigma$

Using the twisted tractor map and the connection $\nabla^{\mathcal{R}}$ we obtain a connection $\check{\nabla}$ on the standard (co)tractor bundle induced by the ambient tractor connection. Given a standard tractor field u^J and a cotractor field v_J on Σ we define

$$\check{\nabla}_i u^J = \Pi_B^J \nabla_i (\Pi_K^B u^K) \quad \text{and} \quad \check{\nabla}_i v_J = \Pi_J^B \nabla_i (\Pi_B^K v_K) \tag{6.2.11}$$

where by ∇ we mean the ambient standard tractor connection ∇ differentiating in directions tangent to Σ (i.e. pulled back by ι) coupled with the connection $\nabla^{\mathcal{R}}$.

From §§5.3.4 the submanifold intrinsic tractor connection D on a section $v_I \overset{\theta_\Sigma}{=} (\sigma, \tau_\mu, \rho)$ is given by

$$D_\mu v_J \overset{\theta_\Sigma}{=} \begin{pmatrix} D_\mu\sigma - \tau_\mu \\ D_\mu\tau_\nu + iA_{\mu\nu}\sigma \\ D_\mu\rho - p_\mu{}^\nu\tau_\nu + t_\mu\sigma \end{pmatrix}, \quad D_{\bar{\mu}} v_J \overset{\theta_\Sigma}{=} \begin{pmatrix} D_{\bar{\mu}}\sigma \\ D_{\bar{\mu}}\tau_\nu + \boldsymbol{h}_{\mu\bar{\nu}}\rho + p_{\nu\bar{\mu}}\sigma \\ D_{\bar{\mu}}\rho - iA_{\bar{\mu}}{}^\nu\tau_\nu - t_{\bar{\mu}}\sigma \end{pmatrix}, \tag{6.2.12}$$

and

$$D_0 v_J \overset{\theta_\Sigma}{=} \begin{pmatrix} D_0\sigma + \frac{i}{n+1}p\sigma - i\rho \\ D_0\tau_\nu + \frac{i}{n+1}p\tau_\nu - ip_\nu{}^\lambda\tau_\lambda + 2it_\nu\sigma \\ D_0\rho + \frac{i}{n+1}p\rho + 2it^\nu\tau_\nu + is\sigma \end{pmatrix} \tag{6.2.13}$$

where t_μ and s are the submanifold intrinsic versions of T_α and S defined by (5.3.11) and (5.3.12) respectively. By contrast, for $\check{\nabla}$ we have:

Proposition 6.2.9. *The connection $\check{\nabla}$ on a section $v_I \overset{\theta_\Sigma}{=} (\sigma, \tau_\mu, \rho)$ of $\mathcal{E}_I$ is given, in terms of any ambient contact form compatible with θ_Σ, by*

$$\check{\nabla}_\mu v_J \overset{\theta_\Sigma}{=} \begin{pmatrix} D_\mu\sigma - \tau_\mu \\ D_\mu\tau_\nu + iA_{\mu\nu}\sigma \\ D_\mu\rho - P_\mu{}^\nu\tau_\nu + T_\mu\sigma \end{pmatrix}, \tag{6.2.14}$$

$$\check{\nabla}_{\bar{\mu}} v_J \overset{\theta_\Sigma}{=} \begin{pmatrix} D_{\bar{\mu}}\sigma \\ D_{\bar{\mu}}\tau_\nu + \boldsymbol{h}_{\mu\bar{\nu}}\rho + P_{\nu\bar{\mu}}\sigma \\ D_{\bar{\mu}}\rho - iA_{\bar{\mu}}{}^\nu\tau_\nu - T_{\bar{\mu}}\sigma \end{pmatrix}, \tag{6.2.15}$$

and

$$\check{\nabla}_0 v_J \overset{\theta_\Sigma}{=} \begin{pmatrix} D_0\sigma + \frac{i}{n+1}P_\lambda{}^\lambda\sigma - i\rho \\ D_0\tau_\nu + \frac{i}{n+1}P_\lambda{}^\lambda\tau_\nu - iP_\nu{}^\lambda\tau_\lambda + 2iT_\nu\sigma \\ D_0\rho + \frac{i}{n+1}P_\lambda{}^\lambda\rho + 2iT^\nu\tau_\nu + iS\sigma \end{pmatrix}. \tag{6.2.16}$$

Proof. Choose any local isomorphism $\psi : \mathcal{E}_\Sigma(1,0) \to \mathcal{E}(1,0)|_\Sigma$ compatible with the canonical identification of $\mathcal{E}_\Sigma(1,1)$ with $\mathcal{E}(1,1)|_\Sigma$. Replacing σ with $f\sigma$ where $f \in C^\infty(\Sigma, \mathbb{C})$ we may take σ to satisfy $\sigma\bar{\sigma} = \varsigma_\Sigma$ where $\theta_\Sigma = \varsigma_\Sigma \boldsymbol{\theta}_\Sigma$. We can thus factor the components of v_I so that $v_I \overset{\theta_\Sigma}{=} (f\sigma, \xi_\mu\sigma, g\sigma)$, where $\xi_\mu \in \Gamma(\mathcal{E}_\mu)$ and $g \in \Gamma(\mathcal{E}_\Sigma(-1,-1))$. If θ is an ambient contact form compatible with θ_Σ then (splitting the tractor bundles w.r.t. θ, θ_Σ) under the map $\mathcal{T}_\psi\iota\otimes\overline{\psi}$ of Remark 6.2.7

$$(f\sigma,\xi_\mu\sigma,g\sigma)\otimes\overline{\sigma}\mapsto(f\phi,\xi_\alpha\phi,g\phi)\otimes\overline{\phi}$$

where $\phi=\psi(\sigma)$ and $\xi_\alpha=\Pi^\mu_\alpha\xi_\mu$. Thus by definition we have

$$v_B=\Pi^J_B v_J\stackrel{\theta}{=}\begin{pmatrix} f\phi\\ \xi_\beta\phi\\ g\phi\end{pmatrix}\otimes(\overline{\phi}\otimes\overline{\sigma}^{-1})$$

as a section of $\mathcal{E}_B|_\Sigma\otimes\mathcal{R}(1,0)$. Now one simply computes $\nabla_i v_B$ using the formulae (5.3.13), (5.3.14), and (5.3.15) for the tractor connection along with Lemma 6.1.38 which relates $\nabla^{\mathcal{R}}$ to the Tanaka-Webster connections on the ambient and intrinsic density bundles. We have

$$\begin{aligned}\Pi^B_C\nabla_\mu v_B\stackrel{\theta}{=}&\begin{pmatrix}(\nabla_\mu f)\phi+f\nabla_\mu\phi-\xi_\mu\phi\\ \Pi^\beta_\gamma(\nabla_\mu\xi_\beta)\phi+\xi_\gamma\nabla_\mu\phi+i\Pi^\beta_\gamma A_{\mu\beta}f\phi\\ (\nabla_\mu g)\phi+g\nabla_\mu\phi-P_\mu{}^\gamma\xi_\gamma\phi+T_\mu f\phi\end{pmatrix}\otimes(\overline{\phi}\otimes\overline{\sigma}^{-1})\\ &+\begin{pmatrix} f\phi\\ \xi_\gamma\phi\\ g\phi\end{pmatrix}\otimes\nabla^{\mathcal{R}}_\mu(\overline{\phi}\otimes\overline{\sigma}^{-1})\end{aligned}\tag{6.2.17}$$

where $A_{\mu\beta}=\Pi^\alpha_\mu A_{\alpha\beta}$, $P_\mu{}^\gamma=\Pi^\alpha_\mu P_\alpha{}^\gamma$, and $T_\mu=\Pi^\alpha_\mu T_\alpha$. By Corollary 6.1.13 we have $\Pi^\beta_\gamma A_{\mu\beta}=\Pi^\nu_\gamma A_{\mu\nu}$. We also have $\Pi^\beta_\gamma\nabla_\mu\xi_\beta=\Pi^\nu_\gamma D_\mu\xi_\nu$. Now by Lemma 6.1.38 we have

$$\begin{aligned}\nabla^{\mathcal{R}}_\mu(\overline{\phi}\otimes\overline{\sigma}^{-1})&=(\nabla_\mu\overline{\phi})\otimes\overline{\sigma}^{-1}+\overline{\phi}\otimes D_\mu(\overline{\sigma}^{-1})\\ &=(\nabla_\mu\overline{\phi})\otimes\overline{\sigma}^{-1}+(\sigma^{-1}D_\mu\sigma)\overline{\phi}\otimes\overline{\sigma}^{-1}\end{aligned}$$

using that $\overline{\sigma}D_\mu(\overline{\sigma}^{-1})=-\overline{\sigma}^{-1}D_\mu\overline{\sigma}=\sigma^{-1}D_\mu\sigma$ since $D_\mu\varsigma_\Sigma=0$. If $\theta=\varsigma\boldsymbol{\theta}$ then since $\phi=\psi(\sigma)$ we must have $\phi\overline{\phi}=\varsigma|_\Sigma$, and this implies that

$$(\nabla_\mu\phi)\otimes\overline{\phi}+\phi\otimes\nabla_\mu\overline{\phi}=0.$$

Using these to simplify (6.2.17) we have

$$\Pi_C^B \nabla_\mu v_B \overset{\theta}{=} \begin{pmatrix} (D_\mu f)\phi - \xi_\mu \phi \\ \Pi_\gamma^\nu (D_\mu \xi_\nu)\phi + i\Pi_\gamma^\nu A_{\mu\nu} f\phi \\ (D_\mu g)\phi - P_\mu{}^\nu \xi_\nu \phi + T_\mu f\phi \end{pmatrix} \otimes (\overline{\phi} \otimes \overline{\sigma}^{-1})$$
$$+ (\sigma^{-1} D_\mu \sigma) \begin{pmatrix} f\phi \\ \xi_\gamma \phi \\ g\phi \end{pmatrix} \otimes (\overline{\phi} \otimes \overline{\sigma}^{-1}).$$

Applying Π_J^C to the above display gives

$$\Pi_J^B \nabla_\mu v_B \overset{\theta_\Sigma}{=} \begin{pmatrix} (D_\mu f)\sigma - \xi_\mu \sigma \\ (D_\mu \xi_\nu)\sigma + iA_{\mu\nu} f\sigma \\ (D_\mu g)\sigma - P_\mu{}^\nu \xi_\nu \sigma + T_\mu f\sigma \end{pmatrix} + \begin{pmatrix} fD_\mu \sigma \\ \xi_\nu D_\mu \sigma \\ gD_\mu \sigma \end{pmatrix}$$

which proves (6.2.14). Formula (6.2.15) is obtained similarly. In following the same process for (6.2.16) we obtain that

$$\check{\nabla}_0 v_J \overset{\theta_\Sigma}{=} \begin{pmatrix} D_0 \sigma + \frac{i}{n+2} P\sigma - i\rho \\ D_0 \tau_\nu + \frac{i}{n+2} P\tau_\nu - iP_\nu{}^\lambda \tau_\lambda + 2iT_\nu \sigma \\ D_0 \rho + \frac{i}{n+2} P\rho + 2iT^\nu \tau_\nu + iS\sigma \end{pmatrix}$$
$$+ \left(-\frac{i}{(n+1)} P_{N\bar{N}} + \frac{i}{(n+2)(n+1)} P \right) \begin{pmatrix} \sigma \\ \tau_\nu \\ \rho \end{pmatrix},$$

the second term arising from the use of Lemma 6.1.38. Simplifying this gives the result. □

Remark 6.2.10. By construction $\check{\nabla}$ preserves the tractor metric $h_{J\bar{K}}$. One can therefore obtain the formulae for $\check{\nabla}$ acting on sections of $\mathcal{E}^I$ by conjugating the above formulae and using the identification of $\mathcal{E}^I$ with $\mathcal{E}_{\bar{I}}$ via the tractor metric. ▮

One can now easily compare the two connections $\check{\nabla}$ and D on $\mathcal{T}\Sigma$.

Definition 6.2.11. The *difference tractor* S is the tractor endomorphism valued 1-form on Σ given by the difference between $\check{\nabla}$ and D on $\mathcal{T}\Sigma$. Precisely, we have

$$\check{\nabla}_X u = D_X u + \mathsf{S}(X)u \quad \text{and} \quad \check{\nabla}_X v = D_X v - v \circ \mathsf{S}(X) \tag{6.2.18}$$

for $X \in \mathfrak{X}(\Sigma)$, $u \in \Gamma(\mathcal{T}\Sigma)$, and $v \in \Gamma(\mathcal{T}^*\Sigma)$.

Given a contact form θ_Σ on Σ the difference tractor S splits into components $\mathsf{S}_{\mu J}{}^K$, $\mathsf{S}_{\bar{\mu} J}{}^K$ and $\mathsf{S}_{0J}{}^K$ (with only the last of these depending on θ_Σ). From the above formulae for $\check{\nabla}$ and D we have, in terms of a compatible pair of contact forms,

$$\mathsf{S}_{\mu J}{}^K = (P_\mu{}^\lambda - p_\mu{}^\lambda) Z_J W_\lambda^K - (T_\mu - t_\mu) Z_J Z^K, \tag{6.2.19}$$

$$\mathsf{S}_{\bar{\mu} J}{}^K = -(P_{\nu\bar{\mu}} - p_{\nu\bar{\mu}}) W_J^\nu Z^K + (T_{\bar{\mu}} - t_{\bar{\mu}}) Z_J Z^K, \tag{6.2.20}$$

and

$$\begin{aligned} \mathsf{S}_{0J}{}^K = &-\tfrac{i}{m+2}(P_\lambda{}^\lambda - p)\delta_J^K + i(P_\nu{}^\lambda - p_\nu{}^\lambda) W_J^\nu W_\lambda^K \\ &- 2i(T_\nu - t_\nu) W_J^\nu Z^K - 2i(T^\lambda - t^\lambda) Z_J W_\lambda^K - i(S - s) Z_J Z^K, \end{aligned} \tag{6.2.21}$$

where $m + 2 = n + 1$ in this case. Both $\mathsf{S}_{\mu J}{}^K$ and $\mathsf{S}_{\bar{\mu} J}{}^K$ are invariant objects. Both have as projecting part the difference $P_{\mu\bar{\nu}} - p_{\mu\bar{\nu}}$; a manifestly CR invariant expression for this difference was given in Lemma 6.1.42. We can also give matrix formulae for the difference tractor, following the same conventions used in §§5.3.6 we have

$$\mathsf{S}_{\mu J}{}^K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t_\mu - T_\mu & P_\mu{}^\lambda - p_\mu{}^\lambda & 0 \end{pmatrix}, \quad \mathsf{S}_{\bar{\mu} J}{}^K = \begin{pmatrix} 0 & 0 & 0 \\ p_{\nu\bar{\mu}} - P_{\nu\bar{\mu}} & 0 & 0 \\ T_{\bar{\mu}} - t_{\bar{\mu}} & 0 & 0 \end{pmatrix}$$

and

$$\mathsf{S}_{0J}{}^K = \begin{pmatrix} -\frac{i(P_\lambda{}^\lambda - p)}{m+2} & 0 & 0 \\ -2i(T_\nu - t_\nu) & i(P_\nu{}^\lambda - p_\nu{}^\lambda) - \frac{i(P_\lambda{}^\lambda - p)}{m+2} & 0 \\ -i(S - s) & -2i(T^\lambda - t^\lambda) & -\frac{i(P_\lambda{}^\lambda - p)}{m+2} \end{pmatrix}.$$

Remark 6.2.12. Since both tractor connections $\check{\nabla}$ and D preserve the tractor metric on $\mathcal{T}\Sigma$, the difference tractor must take values in skew-Hermitian endomorphisms of the tractor bundle (i.e. S is an $\mathcal{A}\Sigma$-valued 1-form). This can also easily be seen from (6.2.19), (6.2.20) and (6.2.21), from which we see that S is in fact $\mathcal{A}^0\Sigma$-valued. ∎

6.2.5 The Tractor Gauss Formula

In order to write down the Gauss formula in Riemannian geometry one needs the tangent map (more precisely the pushforward) of the embedding, though one typically suppresses this from the notation. In order to give a standard tractor analogue we have sought a canonical 'standard tractor map', but ended up instead with the twisted tractor map $\mathcal{T}^{\mathcal{R}}\iota$. However this poses no problem for constructing a Gauss formula, since the line bundle $\mathcal{R}(1,0)$ we have had to twist with carries an invariant connection $\nabla^{\mathcal{R}}$.

Letting ι_* denote the induced map on sections coming from $\mathcal{T}^{\mathcal{R}}\iota$ we make he following definition:

Definition 6.2.13. We define the *tractor second fundamental form* $\mathbb{L}$ by the *tractor Gauss formula*

$$\nabla_X \iota_* u \;=\; \iota_*(D_X u + \mathsf{S}(X)u) + \mathbb{L}(X)\iota_* u \tag{6.2.22}$$

which holds for any $X \in \mathfrak{X}(\Sigma)$ and $u \in \Gamma(\mathcal{T}\Sigma)$, where ∇ denotes the ambient tractor connection coupled with $\nabla^{\mathcal{R}}$.

This (combined with Theorem 6.2.6) establishes Theorem 4.3.3 for the case $m = n-1$, the result generalises straightforwardly (§§6.3.3).

Remark 6.2.14. By the definition of the difference tensor S, for any $X \in \mathfrak{X}(\Sigma)$ and $u \in \Gamma(\mathcal{T}\Sigma)$ we have that $\mathbb{L}(X)\iota_* u$ is the orthogonal projection of $\nabla_X \iota_* u$ onto $\mathcal{N} \otimes \mathcal{R}(1,0)$. By definition then $\mathbb{L}$ is a 1-form on Σ valued in $\mathrm{Hom}(\mathcal{N}^\perp \otimes \mathcal{R}(1,0), \mathcal{N} \otimes \mathcal{R}(1,0)) = \mathrm{Hom}(\mathcal{N}^\perp, \mathcal{N})$. ∎

Suppressing ι_* we write the tractor Gauss formula as

$$\nabla_X u \;=\; \underbrace{D_X u + \mathsf{S}(X)u}_{\text{`tangential part'}} \;+\; \underbrace{\mathbb{L}(X)u}_{\text{`normal part'}} \tag{6.2.23}$$

for any $X \in \mathfrak{X}(\Sigma)$ and $u \in \Gamma(\mathcal{T}\Sigma)$.

Writing $\Pi^B_J u^J$ as u^B and contracting the Gauss formula on both sides with a unit normal cotractor N_A we get that

$$N_C \mathbb{L}(X)_B{}^C u^B = N_B \nabla_X u^B = -u^B \nabla_X N_B$$

for all sections u^J of $\mathcal{E}^J$ and $X \in \mathfrak{X}(\Sigma)$. Thus $\mathbb{L}$ is given by

$$\mathbb{L}_{iB}{}^C = -N^C \Pi^{B'}_B \nabla_i N_{B'} \tag{6.2.24}$$

for any unit normal cotractor N_C. From this we have:

Proposition 6.2.15. *With respect to a compatible pair of contact forms the components* $\mathbb{L}_{\mu B}{}^{C}$, $\mathbb{L}_{\bar{\mu} B}{}^{C}$, *and* $\mathbb{L}_{0B}{}^{C}$ *of the tractor second fundamental form* $\mathbb{L}$ *are given by*

$$\mathbb{L}_{\mu B}{}^{C} = I\!I_{\mu\nu}{}^{\gamma}\Pi^{\nu}_{\beta}W^{\beta}_{B}W^{C}_{\gamma} + P_{\mu\bar{N}}N^{\gamma}Z_{B}W^{C}_{\gamma}, \tag{6.2.25}$$

$$\mathbb{L}_{\bar{\mu} B}{}^{C} = 0, \tag{6.2.26}$$

and

$$\mathbb{L}_{0B}{}^{C} = -2iT_{\bar{N}}N^{\gamma}Z_{B}W^{C}_{\gamma} \tag{6.2.27}$$

where N^{α} *is some unit holomorphic normal field,* $P_{\mu\bar{N}} = \Pi^{\alpha}_{\mu}P_{\alpha\bar{\beta}}N^{\bar{\beta}}$, *and* $T_{\bar{N}} = T_{\bar{\alpha}}N^{\bar{\alpha}}$.

Proof. One simply chooses a unit holomorphic normal field N^{α} and corresponding normal tractor N^{A}, then calculates $\Pi^{B'}_{B}\nabla_{i}N_{B'}$ using the formulae (5.3.13), (5.3.14), and (5.3.15) for the ambient tractor connection. Using (6.2.24) one immediately obtains (6.2.25); for (6.2.26) one also has to use that $I\!I_{\bar{\mu}\nu}{}^{\gamma} = 0$ (by Proposition 6.1.12) and $\Pi^{\bar{\alpha}}_{\bar{\mu}}A_{\bar{\alpha}\bar{\beta}}N^{\bar{\beta}} = 0$ (by Corollary 6.1.13), and for (6.2.27) one also has to use that $I\!I_{0\nu}{}^{\gamma} = 0$ (again by Proposition 6.1.12). □

The proposition shows that the invariant projecting part of $\mathbb{L}_{\mu B}{}^{C}$ is $I\!I_{\mu\nu}{}^{\gamma}\Pi^{\nu}_{\beta}$, giving a manifestly CR invariant way of defining the CR second fundamental form.

6.3 Higher Codimension Embeddings

It is straightforward to adapt our treatment of CR embeddings in the minimal codimension case to general codimension transversal CR embeddings. Here we consider a CR embedding of $\iota : \Sigma^{2m+1} \to M^{2n+1}$ with $n = m + d$, and $m, d > 0$. We keep our notation for bundles on Σ and M as before. We now have a rank $2d$ real conormal bundle $N^{*}\Sigma$, and the complexification of $N^{*}\Sigma$ splits as

$$\mathbb{C}N^{*}\Sigma = \mathcal{N}_{\alpha} \oplus \mathcal{N}_{\bar{\alpha}} \tag{6.3.1}$$

where $\mathcal{N}_{\alpha}$ is the annihilator of $T^{1,0}\Sigma$ in $(T^{1,0}M)^{*}|_{\Sigma} = \mathcal{E}_{\alpha}|_{\Sigma}$ and $\mathcal{N}_{\bar{\alpha}} = \overline{\mathcal{N}_{\alpha}}$. We denote by $\mathrm{N}^{\alpha}_{\beta}$ the orthogonal projection of $\mathcal{E}^{\beta}|_{\Sigma}$ onto the holomorphic normal bundle $\mathcal{N}^{\alpha}$, and by Π^{α}_{β} the tangential projection, so that $\Pi^{\alpha}_{\beta} + \mathrm{N}^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$. We will also write $\mathrm{N}^{\alpha\bar{\beta}}$ for $\boldsymbol{h}^{\gamma\bar{\delta}}\mathrm{N}^{\alpha}_{\gamma}\mathrm{N}^{\bar{\beta}}_{\bar{\delta}} = \boldsymbol{h}^{\gamma\bar{\beta}}\mathrm{N}^{\alpha}_{\gamma}$.

Remark 6.3.1. Note that in passing to the general codimension there is no restriction on the signatures (or relative signature) of the CR manifolds, provided we have a nondegenerate transversal CR embedding. ∎

6.3.1 Pseudohermitian Calculus

We may continue to work with compatible contact forms in the general codimension case (see Remark 6.1.3). By Remark 6.1.10 the Tanaka-Webster connection ∇ of an admissible ambient contact form θ induces the Tanaka-Webster connection D of θ_Σ via the Webster metric g_θ as in Proposition 6.1.8. We can therefore define the (pseudohermitian) second fundamental form of a pair of compatible contact forms as in Definition 6.1.11 (i.e. via a Gauss formula). By Remark 6.1.14 the only nontrivial components of the pseudohermitian second fundamental form are $II_{\mu\nu}{}^{\gamma}$ and its conjugate. Also by Remark 6.1.14 the pseudohermitian torsion of any admissible ambient contact form satisfies $\Pi^\alpha_\mu A_{\alpha\beta}\mathrm{N}^\beta_\gamma = 0$.

The higher codimension analogue of Lemma 6.1.16 is:

Lemma 6.3.2. *Given compatible contact forms one has*

$$N_\gamma II_{\mu\nu}{}^{\gamma} = -\Pi^\beta_\nu \nabla_\mu N_\beta \tag{6.3.2}$$

for any holomorphic conormal field.

From Lemma 6.3.2 we see again that the component $II_{\mu\nu}{}^{\gamma}$ of the pseudohermitian second fundamental form does not depend on the compatible pair of contact forms used to define it (cf. Corollary 6.1.17).

The Gauss, Codazzi and Ricci equations given in the three propositions of §§6.1.7 hold in the general codimension case with the same proofs (noting that the normal fields used in the proofs of Proposition 6.1.22 and Proposition 6.1.23 were arbitrary).

6.3.2 Relating Densities

As before we define $\Lambda^{1,0}_\perp\Sigma$ to be the bundle of forms in $\Lambda^{1,0}M|_\Sigma$ annihilating $T\Sigma$. Again we may identify $\Lambda^{1,0}_\perp\Sigma$ with $\mathcal{N}_\alpha$ by restriction to $T^{1,0}M|_\Sigma$. We write $\Lambda^{d,0}_\perp\Sigma$ for the line bundle $\Lambda^d(\Lambda^{1,0}_\perp\Sigma)$. The following lemma is easily established (cf. Lemma 6.1.25 and Lemma 6.1.28):

Lemma 6.3.3. *Along Σ the submanifold and ambient canonical bundles are related by the canonical isomorphism which intertwines the Tanaka-Webster connections of any compatible pair of contact forms*

$$\begin{array}{ccccc} \mathscr{K}|_\Sigma & \cong & \mathscr{K}_\Sigma & \otimes & \Lambda_\perp^{d,0}\Sigma \\ \iota^*\nabla & \cong & D & \otimes & \nabla^\perp. \end{array}$$

Identifying $\Lambda_\perp^{1,0}\Sigma$ with $\mathcal{N}_\alpha$ we may write $\Lambda_\perp^{d,0}\Sigma$ as $\mathcal{N}_{[\alpha_1\cdots\alpha_d]}$. Tensoring both sides of the isomorphism of Lemma 6.3.3 with $\mathcal{E}(d,0)|_\Sigma$ we obtain (cf. Corollary 6.1.26 and Corollary 6.1.29):

Corollary 6.3.4. *Along Σ the submanifold and ambient density bundles are related by the canonical isomorphism which intertwines the Tanaka-Webster connections of any compatible pair of contact forms*

$$\begin{array}{ccccc} \mathcal{E}(-m-2,0)|_\Sigma & \cong & \mathcal{E}_\Sigma(-m-2,0) & \otimes & \mathcal{N}_{[\alpha_1\cdots\alpha_d]}(d,0) \\ \iota^*\nabla & \cong & D & \otimes & \nabla^\perp. \end{array}$$

Note that the line bundle $\mathcal{N}_{[\alpha_1\cdots\alpha_d]}(d,0)$ is the d^{th} exterior power of $\mathcal{N}_\alpha(1,0)$. Once again this bundle will turn out to be canonically isomorphic to a subbundle $\mathcal{N}^* = \mathcal{N}_A$ of the ambient cotractor bundle $\mathcal{E}_A|_\Sigma$ (see §§6.3.3), and hence once again $\mathcal{N}_\alpha(1,0)$ carries a canonical invariant connection. As before this connection turns out to be explicitly realised as the normal Weyl connection on $\mathcal{N}_\alpha(1,0)$ of any admissible ambient contact form. The normal Weyl connection on $\mathcal{N}_\alpha(1,0)$ agrees with the normal Tanaka-Webster connection when differentiating in contact directions; when differentiating in Reeb directions the two are related by

$$\nabla_0^{W,\perp}\tau_\alpha = \nabla_0^\perp\tau_\alpha - i\mathrm{N}_\alpha^{\alpha'}P_{\alpha'}{}^{\beta}\tau_\beta + \frac{i}{n+2}P\tau_\alpha \tag{6.3.3}$$

for any section τ_α of $\mathcal{N}_\alpha(1,0)$. The curvature $R^{\Lambda^d\mathcal{N}^*}$ of this connection on $\mathcal{N}_{[\alpha_1\cdots\alpha_d]}(d,0)$ is again generically non zero, and we have

$$R_{\mu\bar{\nu}}^{\Lambda^d\mathcal{N}^*} = (m+2)(P_{\mu\bar{\nu}} - p_{\mu\bar{\nu}}) + (P_\lambda{}^\lambda - p)\boldsymbol{h}_{\mu\bar{\nu}} \tag{6.3.4}$$

(cf. Lemma 6.1.41), $R_{\mu\nu}^{\Lambda^d\mathcal{N}^*} = 0$, $R_{\bar{\mu}\bar{\nu}}^{\Lambda^d\mathcal{N}^*} = 0$, and (cf. (6.1.41))

$$R_{\mu 0}^{\Lambda^d\mathcal{N}^*} = -V_{\mu\bar{\beta}\gamma}\mathrm{N}^{\gamma\bar{\beta}} - iT_{\mu}. \tag{6.3.5}$$

We thus define the ratio bundle of densities $\mathcal{R}(w, w')$ as before (Definition 6.1.32) and see from Corollary 6.3.4 that these bundles carry a canonical connection $\nabla^{\mathcal{R}}$ coming from the connection $\nabla^{\mathcal{N}}$ on $\Lambda^d\mathcal{N}^* = \mathcal{N}_{[\alpha_1\cdots\alpha_d]}(d, 0)$. We have therefore established Proposition 4.3.2. Using Corollary 6.3.4 and (6.3.3) we may relate the connection $\nabla^{\mathcal{R}}$ to the coupled submanifold-ambient Tanaka-Webster connection (cf. Lemma 6.1.38):

Lemma 6.3.5. *In terms of a compatible pair of contact forms, θ, θ_Σ, the connection $\nabla^{\mathcal{R}}$ on a section $\phi\otimes\sigma$ of $\mathcal{E}(w, w')|_\Sigma \otimes \mathcal{E}_\Sigma(-w, -w')$ is given by*

$$\nabla_\mu^{\mathcal{R}}(\phi\otimes\sigma) = (\nabla_\mu\phi)\otimes\sigma + \phi\otimes(D_\mu\sigma), \tag{6.3.6}$$

$$\nabla_{\bar{\mu}}^{\mathcal{R}}(\phi\otimes\sigma) = (\nabla_{\bar{\mu}}\phi)\otimes\sigma + \phi\otimes(D_{\bar{\mu}}\sigma), \tag{6.3.7}$$

and

$$\nabla_0^{\mathcal{R}}(\phi\otimes\sigma) = (\nabla_0\phi)\otimes\sigma + \phi\otimes(D_0\sigma) + \tfrac{w-w'}{m+2}(iP_{\alpha\bar{\beta}}\mathrm{N}^{\alpha\bar{\beta}} - \tfrac{i}{n+2}P)\phi\otimes\sigma. \tag{6.3.8}$$

6.3.3 Relating Tractors

As before we have a canonical isomorphism from $\mathcal{N}_\alpha(1, 0)$ to a subbundle $\mathcal{N}_A$ of $\mathcal{E}_A|_\Sigma$, given with respect to any admissible ambient contact form θ by

$$\tau_\alpha \mapsto \tau_A \overset{\theta}{=} \begin{pmatrix} 0 \\ \tau_\alpha \\ 0 \end{pmatrix}. \tag{6.3.9}$$

There is a corresponding isomorphism of $\mathcal{N}^\alpha(-1, 0)$ with a subbundle $\mathcal{N}^A$ of $\mathcal{E}^A|_\Sigma$, and we alternatively denote the dual pair $\mathcal{N}^A$ and $\mathcal{N}_A$ by $\mathcal{N}$ and $\mathcal{N}^*$ respectively. The normal tractor connection $\nabla^{\mathcal{N}}$ on $\mathcal{N}_A$ agrees with the normal Weyl connection of any admissible ambient contact form on $\mathcal{N}_\alpha(1, 0)$ (cf. Proposition 6.2.3).

Sections §§6.2.2 and §§6.2.3 are valid without change in the general codimension case. In particular, Lemma 6.2.5 and Theorem 6.2.6 hold. Thus we may talk about the twisted standard tractor map

$$\mathcal{T}^{\mathcal{R}}\iota : \mathcal{T}\Sigma \to \mathcal{T}M|_{\Sigma} \otimes \mathcal{R}(1,0)$$

and the corresponding sections Π_I^A of $\mathcal{E}_I \otimes \mathcal{E}^A|_{\Sigma} \otimes \mathcal{R}(1,0)$ and Π_A^I of $\mathcal{E}^I \otimes \mathcal{E}_A|_{\Sigma} \otimes \mathcal{R}(0,1)$. This allows us to define the connection $\check{\nabla}$ on $\mathcal{T}\Sigma$ as in (6.2.11); one can then easily establish the expressions for $\check{\nabla}$ given in Proposition 6.2.9 in the general codimension setting (the proof is essentially the same, with Lemma 6.3.5 generalising Lemma 6.1.38). The difference tractor S, defined as in Definition 6.2.11, is then still given in component form by (6.2.19), (6.2.20), and (6.2.21).

We define the tractor second fundamental form $\mathbb{L}$ by a tractor Gauss formula as in Definition 6.2.13. This establishes Theorem 4.3.3. One then also has that

$$\mathbb{L}_{iB}{}^{C} N_C = -\Pi_B^{B'} \nabla_i N_{B'} \tag{6.3.10}$$

for any section N_A of $\mathcal{N}_A$. From this we get (cf. Proposition 6.2.15):

Proposition 6.3.6. *With respect to a compatible pair of contact forms the components* $\mathbb{L}_{\mu B}{}^{C}$, $\mathbb{L}_{\bar{\mu} B}{}^{C}$, *and* $\mathbb{L}_{0B}{}^{C}$ *of the tractor second fundamental form* $\mathbb{L}$ *are given by*

$$\mathbb{L}_{\mu B}{}^{C} = I\!I_{\mu\nu}{}^{\gamma} \Pi_\beta^\nu W_B^\beta W_\gamma^C + P_{\mu\bar{\delta}} N^{\gamma\bar{\delta}} Z_B W_\gamma^C, \tag{6.3.11}$$

$$\mathbb{L}_{\bar{\mu} B}{}^{C} = 0, \tag{6.3.12}$$

and

$$\mathbb{L}_{0B}{}^{C} = -2i T_{\bar{\delta}} N^{\gamma\bar{\delta}} Z_B W_\gamma^C. \tag{6.3.13}$$

6.4 Invariants of CR Embedded Submanifolds

For many problems in geometric analysis it is important to construct the invariants that are, in a suitable sense, polynomial in the jets of the structure. Riemannian theory along these lines was developed by Atiyah-Bott-Patodi for their approach to the heat equation asymptotics [5], and in [63] Fefferman initiated a corresponding programme for conformal geometry and hypersurface type CR geometry. As explained in [8] there are two steps to such problems. The first is to capture the jets (preferably to all orders) of the geometry invariantly and in an algebraically manageable manner. The second is to use this algebraic structure to construct all invariants. The latter boils down to Lie representation theory, for the case of parabolic geometries this is difficult, and despite the progress in [8] and [75] for conformal geometry and CR geometry many open problems remain. For the conformal and CR cases the first part is treated by the Fefferman and Fefferman-Graham ambient metric constructions [63, 64, 66] and alternatively by the tractor calculus [7, 27, 75]. It is beyond the scope of the current work to fully set up and treat the corresponding invariant theory for CR submanifolds. However we wish to indicate here that the first geometric step, of capturing the jets effectively, is solved via the tools developed above. In particular we will show that it is straightforward to proliferate invariants of a (transversally embedded) CR submanifold. It seems reasonable to hope that these methods will form the basis of a construction of all invariants of CR embeddings (in an appropriate sense).

6.4.1 Jets of the Structure

We now show that the jets of the structure of a CR embedding are captured effectively by the basic invariants we have introduced in our 'tractorial' treatment of CR embeddings.

Observe that the tractor Gauss formula (6.2.13) may be rewritten in the form

$$\nabla_X \mathcal{T}^{\mathcal{R}}\iota = \mathcal{T}^{\mathcal{R}}\iota \circ \mathsf{S}(X) + \mathbb{L}(X) \circ \mathcal{T}^{\mathcal{R}}\iota \tag{6.4.1}$$

for any $X \in T\Sigma$, where $\mathcal{T}^{\mathcal{R}}\iota$ is interpreted as a section of $\mathcal{T}M|_\Sigma \otimes \mathcal{T}^*\Sigma \otimes \mathcal{R}(1,0)$ and ∇ here denotes the (pulled back) ambient tractor connection coupled with the submanifold tractor connection and the canonical connection $\nabla^{\mathcal{R}}$. Using this we have the following proposition:

Proposition 6.4.1. *Given a transversal CR embedding $\iota : \Sigma \to M$, the 2-jet of the map ι at a point $x \in \Sigma$ is encoded by $\iota(x)$, $\mathcal{T}^{\mathcal{R}}_x\iota$, S_x and $\mathbb{L}_x$.*

Proof. Recalling §§§6.2.3.1 we note that the twisted tractor map $\mathcal{T}^{\mathcal{R}}\iota$ determines the adjoint tractor map $\mathcal{A}\iota$ (by restricting $\mathcal{T}^{\mathcal{R}}\iota \otimes \overline{\mathcal{T}^{\mathcal{R}}\iota}$). Since the adjoint tractor map lifts the tangent map, the 1-jet $(\iota(x), T_x\iota)$ of ι at a point $x \in \Sigma$ is also determined by the pair $(\iota(x), \mathcal{T}^{\mathcal{R}}_x\iota)$. The proposition then follows from (6.4.1). □

In the *jets of the structure* of a CR embedding $\iota : \Sigma \to M$ we include the jets of the ambient and submanifold CR structures, along with the jets of the map ι. A CR invariant of the embedding should depend only on these jets evaluated along the submanifold.

The jets of the ambient and submanifold CR structures are determined by the respective tractor curvatures. Thus from Proposition 6.4.1 we have:

Proposition 6.4.2. *The jets of the structure of a transversal CR embedding are determined algebraically by the embedding $\iota : \Sigma \to M$, the submanifold and ambient CR structures (as parabolic geometries), the twisted tractor map $\mathcal{T}^{\mathcal{R}}\iota$, as well as the jets of the difference tensor* S, *the tractor second fundamental form* $\mathbb{L}$, *the submanifold tractor curvature* κ^{Σ}, *and the full (i.e. ambient) jets of the ambient tractor curvature* κ.

In order to complete the first step of the invariant theory programme we need to package the jets of S, $\mathbb{L}$, κ^{Σ} and κ in an algebraically manageable way. Note that the standard tractor bundle, tractor metric, and canonical tractor Z are all determined algebraically from structure of a CR geometry (as a parabolic geometry). Thus if we package the jets of S, $\mathbb{L}$, κ^{Σ} and κ into sequences of CR invariant tractors then one may combine these tractors by tensoring them and using the submanifold and ambient metrics to contract indices. One can also use the twisted tractor map to change submanifold tractor indices to ambient ones before making contractions (the ratio bundle of densities is also determined algebraically from the submanifold and ambient CR structures). This would not only complete the first step of the invariant theory programme, but would also suggest an obvious approach to the second of the two steps.

6.4.2 Packaging the Jets

One way to define iterated derivatives of the difference tractor S and submanifold curvature κ^{Σ} would be to repeatedly apply the submanifold fundamental derivative (or D-operator) of [27]. Denoting the submanifold fundamental derivative by $\boldsymbol{D}$, if f is S or κ_{Σ} then by Theorem 3.3 of [27] the k-jet of f is determined by the section

$$(f, \boldsymbol{D}f, \boldsymbol{D}^2 f, \ldots, \boldsymbol{D}^k f)$$

of $\bigoplus_{l=0}^{k}\left(\bigotimes^{l}\mathcal{A}^*\Sigma\otimes\mathcal{W}\right)$ where $\mathcal{W}$ equals $\Lambda^1\Sigma\otimes\mathcal{A}\Sigma$ or $\Lambda^2\Sigma\otimes\mathcal{A}\Sigma$ respectively. The ambient jets of κ can be similarly captured by iterating the ambient fundamental derivative, and one can also capture the jets of $\mathbb{L}$ by using the submanifold fundamental derivative twisted with the ambient tractor connection. Here instead we parallel the approach taken in [75] to conformal invariant theory by first putting the tractor valued forms S, $\mathbb{L}$,

κ^Σ and κ into tractors (invariantly and algebraically) using the natural inclusion of the cotangent bundle into the adjoint tractor bundle, and then using double-D-operators.

Let $B^a_{A\bar{B}}$ denote the map $T^*M \hookrightarrow \mathcal{A}M$ given explicitly by (5.3.29) and $B^i_{I\bar{J}}$ denote the map $T^*\Sigma \hookrightarrow \mathcal{A}\Sigma$.

Definition 6.4.3. We define the respective *lifted (tractor) expressions* of the tractor valued forms S, $\mathbb{L}$, κ^Σ and κ to be

$$\mathsf{S}_{I\bar{I}'J}{}^K = B^i_{I\bar{I}'}\mathsf{S}_{iJ}{}^K, \quad \mathbb{L}_{I\bar{I}'B}{}^C = B^i_{I\bar{I}'}\mathbb{L}_{iB}{}^C, \quad \kappa^\Sigma_{I\bar{I}'J\bar{J}'K\bar{L}} = B^i_{I\bar{I}'}B^j_{J\bar{J}'}\kappa^\Sigma_{ijK\bar{L}},$$

and

$$\kappa_{A\bar{A}'B\bar{B}'C\bar{D}} = B^a_{A\bar{A}'}B^b_{B\bar{B}'}\kappa_{abC\bar{D}}.$$

Explicitly this means, for example, that

$$\mathsf{S}_{I\bar{J}K}{}^L = \mathsf{S}_{\mu K}{}^L W^\mu_I Z_{\bar{J}} - \mathsf{S}_{\bar{\nu}K}{}^L Z_I W^{\bar{\nu}}_{\bar{J}} - i\mathsf{S}_{0K}{}^L Z_I Z_{\bar{J}}.$$

By (5.3.42) the double-D-operator $\mathbb{D}_{A\bar{B}}$ acting on unweighted ambient tractors can be written as

$$\mathbb{D}_{A\bar{B}} = B^a_{A\bar{B}}\nabla_a \tag{6.4.2}$$

where ∇ is the ambient tractor connection. Similarly the double-D-operator $D_{I\bar{J}}$ acting on unweighted submanifold tractors can be written as

$$\mathbb{D}_{I\bar{J}} = B^i_{I\bar{J}}D_i \tag{6.4.3}$$

where D_i denotes the submanifold tractor connection. By coupling D_i in (6.4.3) with ∇_i we enable the double-D-operator operator $\mathbb{D}_{I\bar{J}}$ to act iteratively on the unweighted (mixed) tractor $\mathbb{L}_{K\bar{L}C}{}^D$. Noting that each of the lifted tractor expressions given in Definition 6.4.3 is unweighted we therefore have:

Proposition 6.4.4. *Let $\iota : \Sigma \to M$ be a transversal CR embedding, and let $\mathbb{D}$ denote the submanifold double-D-operator $\mathbb{D}_{I\bar{J}}$. If f equals S, $\mathbb{L}$, or κ^Σ then the k-jet of f is determined by the section*

$$(\tilde{f}, \mathbb{D}\tilde{f}, \mathbb{D}^2\tilde{f}, \ldots, \mathbb{D}^k\tilde{f}) \tag{6.4.4}$$

of $\bigoplus_{l=0}^k \left(\bigotimes^l \mathcal{A}\Sigma \otimes \mathcal{W}\right)$, where $\tilde{f}$ is the lifted tractor expression for f and $\mathcal{W}$ equals $\otimes^2\mathcal{A}\Sigma$, $\mathcal{A}\Sigma \otimes \mathcal{A}M|_\Sigma$, or $\otimes^3\mathcal{A}\Sigma$ respectively.

Along with the corresponding proposition for the ambient curvature:

Proposition 6.4.5. *Let $\iota : \Sigma \to M$ be a transversal CR embedding, and let $\mathbb{D}$ denote the ambient double-D-operator $D_{A\bar{B}}$. The k-jet of the ambient curvature κ is determined by the section*

$$(\tilde{\kappa}, \mathbb{D}\tilde{\kappa}, \mathbb{D}^2\tilde{\kappa}, \ldots, \mathbb{D}^k\tilde{\kappa}) \tag{6.4.5}$$

of $\bigoplus_{l=0}^{k} \left(\bigotimes^{l+3} \mathcal{A}M\right)$.

By packaging the jets of the basic invariants S, $\mathbb{L}$, κ^Σ and κ into sequences of tractors (i.e. sections of associated bundles corresponding to representations of the appropriate pseudo-special unitary groups) we have solved the first step of the invariant theory.

6.4.3 Making All Invariants

By tensoring together tractors of the form appearing in (6.4.4) and (6.4.5) along Σ, making partial contractions, and taking projecting parts one may proliferate local CR invariant (weighted) scalars and tensors. It is an algebraic problem to show that all such invariants, which are suitably polynomial in the jets of the structure, can be obtained by such a procedure. This is a subtle and difficult problem, which extends Fefferman's parabolic invariant theory programme to the submanifold-relative case (where there are two parabolics around, P and P_Σ). Even in the original case of invariant theory for CR manifolds, despite much progress, important questions remain unresolved [8, 92]. We do not attempt to resolve these issues here.

We do wish to indicate, however, that there is scope for development of the invariant theory for CR manifolds, and now CR embeddings, along the lines of the treatment of invariant theory for conformal and projective structures in [71, 72, 73, 75]. The tractor calculus we have developed for CR embeddings provides all the machinery needed to emulate the constructions of conformal Weyl and quasi-Weyl invariants in [75]. We anticipate that further insight from the projective case [73] will be needed, and our machinery is sufficient for this also. With all the tools in hand this article therefore puts us in good stead in terms of our ability to construct (potentially all) invariants of CR embeddings.

6.4.4 Practical Constructions

Although in principle one may need only the invariant tractors appearing in Propositions 6.4.4 and 6.4.5 for construction of general invariants, in practice it is much more efficient to use the richer calculus which is available. First of all, there are many alternative ways to construct tractor expressions from the basic invariants (recall for instance the curvature tractor of §§5.3.8). Secondly, there are several invariant operators besides the double-D-operators $\mathbb{D}_{I\bar{J}}$ and $\mathbb{D}_{A\bar{B}}$ that can be used to act on these tractor expressions.

6.4.4.1 Alternative tractor expressions

Along with the lifted tractor expressions for the submanifold and ambient tractor curvatures one may of course construct invariants using the curvature tractor of §§5.3.8 or using the tractor defined in equation (5.3.49) of that section. Correspondingly we may also use the middle operators of §§§5.3.7.2 to construct tractors from our basic invariants S and $\mathbb{L}$

$$\mathsf{S}_{IJ}{}^{K} = \mathsf{M}_{I}^{\mu}\mathsf{S}_{\mu J}{}^{K}, \quad \mathsf{S}_{\bar{I}J}{}^{K} = \mathsf{M}_{\bar{I}}^{\bar{\mu}}\mathsf{S}_{\bar{\mu}J}{}^{K}, \quad \text{and} \quad \mathbb{L}_{IB}{}^{C} = \mathsf{M}_{I}^{\mu}\mathbb{L}_{\mu B}{}^{C}$$

using indices to distinguish them from the difference tractor S and the tractor second fundamental form $\mathbb{L}$ (and from their lifted tractor expressions in §§6.4.2). Recall that $\mathbb{L}_{\bar{\mu}B}{}^{C} = 0$.

From (5.3.4) it follows immediately that

$$Z_{[A}W_{B]}^{\beta} = \frac{1}{2}(Z_A W_B^{\beta} - Z_B W_A^{\beta})$$

does not depend on the choice of contact form, so is CR invariant. Using $Z_{[A}W_{B]}^{\beta}$ and $Z_{[I}W_{J]}^{\nu}$ we construct the tractors

$$\mathsf{S}_{II'J}{}^{K} = Z_{[I}W_{I']}^{\mu}\mathsf{S}_{\mu J}{}^{K}, \qquad \mathsf{S}_{\bar{I}\bar{I}'J}{}^{K} = Z_{[\bar{I}}W_{\bar{I}']}^{\bar{\mu}}\mathsf{S}_{\bar{\mu}J}{}^{K},$$

$$\mathbb{L}_{II'B}{}^{C} = Z_{[I}W_{I']}^{\mu}\mathbb{L}_{\mu B}{}^{C},$$

$$\kappa^{\Sigma}_{II'\bar{J}\bar{J}'K\bar{L}} = Z_{[I}W_{I']}^{\mu}Z_{[\bar{J}}W_{\bar{J}']}^{\bar{\nu}}\kappa^{\Sigma}_{\mu\bar{\nu}K\bar{L}},$$

and

$$\kappa_{AA'\bar{B}\bar{B}'C\bar{D}} = Z_{[A}W_{A']}^{\alpha}Z_{[\bar{B}}W_{\bar{B}']}^{\bar{\beta}}\kappa_{\alpha\bar{\beta}C\bar{D}}.$$

Of course one can also make invariant tractors from the invariant components $\mathsf{S}_{\mu J}{}^{K}$, $\mathsf{S}_{\bar{\mu} J}{}^{K}$, $\mathbb{L}_{\mu B}{}^{C}$, and so on, by making contractions with the submanifold (or ambient) CR Levi form. For example, we have the following invariant tractors on Σ

$$\boldsymbol{h}^{\mu\bar{\nu}}\mathsf{S}_{\mu I}{}^{J}\mathsf{S}_{\bar{\nu}K}{}^{L}, \quad \boldsymbol{h}^{\mu\bar{\nu}}\mathbb{L}_{\mu A}{}^{B}\mathsf{S}_{\bar{\nu}\bar{K}}{}^{\bar{L}}, \quad \boldsymbol{h}^{\mu\bar{\rho}}\boldsymbol{h}^{\lambda\bar{\nu}}\kappa^{\Sigma}_{\mu\bar{\nu}K\bar{L}}\mathbb{L}_{\lambda A}{}^{B}\mathbb{L}_{\bar{\rho}\bar{C}}{}^{\bar{D}},$$

where $\mathsf{S}_{\bar{\nu}\bar{K}}{}^{\bar{L}} = \overline{\mathsf{S}_{\nu K}{}^{L}}$ and $\mathbb{L}_{\bar{\rho}\bar{C}}{}^{\bar{D}} = \overline{\mathbb{L}_{\rho C}{}^{D}}$. One can also contract some, or all, of the tractor indices. Note that

$$\mathbb{L}_{\mu B\bar{C}}\mathbb{L}^{\mu B\bar{C}} = I\!I_{\mu\nu\bar{\gamma}}I\!I^{\mu\nu\bar{\gamma}} \tag{6.4.6}$$

whereas

$$\mathsf{S}_{\mu J\bar{K}}\mathsf{S}^{\mu J\bar{K}} = 0 \quad \text{and} \quad \Pi^{J}_{B}\Pi^{\bar{K}}_{\bar{C}}\mathsf{S}_{\mu J\bar{K}}\mathbb{L}^{\mu B\bar{C}} = 0 \tag{6.4.7}$$

from the explicit formulae for S and $\mathbb{L}$ in terms of compatible contact forms and the orthogonality relations (5.3.10) between the splitting tractors.

Remark 6.4.6. Although $\mathsf{S}_{\mu J\bar{K}}\mathsf{S}^{\mu J\bar{K}} = 0$ one can extract a scalar invariant from the partial contraction $\mathsf{S}_{\mu J\bar{K}}\mathsf{S}^{\mu J'\bar{K}}$ by observing that this tractor is of the form $fZ_J Z^{J'}$, so that the $(-1,-1)$ density f must be CR invariant. In fact f is simply the invariant

$$(P_{\mu\bar{\nu}} - p_{\mu\bar{\nu}})(P^{\mu\bar{\nu}} - p^{\mu\bar{\nu}}).$$

One of the difficulties inherent in constructing all invariants is predicting when this type of phenomenon will happen when dealing with various contractions of higher order invariant tractors (such as those appearing in Proposition 6.4.4). ∎

6.4.4.2 Invariant operators

Along with the double-D-operators (6.4.2) and (6.4.3) used in §§6.4.2 one may of course use the submanifold and ambient tractor D-operators of §§§5.3.7.1 and the other double-D-operators $\mathbb{D}_{IJ}$ and $\mathbb{D}_{AB}$. In order to act on tractors of mixed (submanifold-ambient) type, with potentially submanifold and ambient weights, we will need to appropriately couple the submanifold intrinsic invariant D-operators with the ambient tractor connection and with the canonical connection on the density ratio bundles. Note that these operators also form the building blocks for constructing invariant differential operators on CR embedded submanifolds.

We first need to use the ratio bundles to eliminate ambient weights. Let $\mathcal{E}^{\Phi}_{\Sigma}$ denote any submanifold intrinsic tractor bundle and let $\mathcal{E}^{\Phi}_{\Sigma}(w,w')$ denote $\mathcal{E}^{\Phi}_{\Sigma} \otimes \mathcal{E}_{\Sigma}(w,w')$. Let

$\mathcal{E}^{\tilde{\Phi}}(\tilde{w}, \tilde{w}')$ denote any ambient tractor bundle, weighted by ambient densities. We make the identification

$$\mathcal{E}^{\Phi}_{\Sigma}(w, w') \otimes \mathcal{E}^{\tilde{\Phi}}(\tilde{w}, \tilde{w}')|_{\Sigma} = \mathcal{E}^{\Phi}_{\Sigma}(w - \tilde{w}, w' - \tilde{w}') \otimes \mathcal{E}^{\tilde{\Phi}}|_{\Sigma} \otimes \mathcal{R}(\tilde{w}, \tilde{w}') \tag{6.4.8}$$

which motivates the following definition:

Definition 6.4.7. We define the *reduced weight* of a section $f^{\Phi\tilde{\Phi}}$ of the bundle (6.4.8) to be $(\breve{w}, \breve{w}') = (w - \tilde{w}, w' - \tilde{w}')$.

One can extend any of the submanifold D-operators to act on sections of the bundle (6.4.8) by taking the relevant D-operator acting on submanifold tractors with the reduced weight, expressed in terms of a choice of contact form θ_{Σ}, and coupling the Tanaka-Webster connection of θ_{Σ} with the (pulled back) ambient tractor connection and the ratio bundle connection $\nabla^{\mathcal{R}}$.

We illustrate how this works for the submanifold tractor D-operator D_I. We define the CR invariant operator

$$\mathsf{D}_I : \mathcal{E}^{\Phi}_{\Sigma}(\breve{w}, \breve{w}') \otimes \mathcal{E}^{\tilde{\Phi}}|_{\Sigma} \otimes \mathcal{R}(\tilde{w}, \tilde{w}') \to \mathcal{E}_I \otimes \mathcal{E}^{\Phi}_{\Sigma}(\breve{w} - 1, \breve{w}') \otimes \mathcal{E}^{\tilde{\Phi}}|_{\Sigma} \otimes \mathcal{R}(\tilde{w}, \tilde{w}')$$

by

$$\mathsf{D}_I f^{\Phi\tilde{\Phi}} \overset{\theta_{\Sigma}}{=} \begin{pmatrix} \breve{w}(m + \breve{w} + \breve{w}') f^{\Phi\tilde{\Phi}} \\ (m + \breve{w} + \breve{w}') D_{\mu} f^{\Phi\tilde{\Phi}} \\ -\left(D^{\nu} D_{\nu} f^{\Phi\tilde{\Phi}} + i\breve{w} D_0 f^{\Phi\tilde{\Phi}} + \breve{w}(1 + \frac{\breve{w}' - \breve{w}}{m+2}) p f^{\Phi\tilde{\Phi}} \right) \end{pmatrix} \tag{6.4.9}$$

where D denotes the Tanaka-Webster connection of θ_{Σ} coupled with the submanifold tractor connection, the (pulled back) ambient tractor connection, and the ratio bundle connection $\nabla^{\mathcal{R}}$.

6.4.4.3 Computing higher order invariants

Using the tractor calculus we have developed it is now straightforward to construct further local (weighted scalar, or other) invariants of a CR embedding. One can differentiate the various tractors constructed from the basic invariants in §§6.4.2 and §§§6.4.4.1 using the invariant operators of §§§5.3.7.1 and §§§6.4.4.2, tensor these together, and make contractions using the tractor metrics (and the twisted tractor map). One can also make partial contractions and take projecting parts.

To illustrate our construction we give an example invariant and compute the form of the invariant in terms of the Tanaka-Webster calculus of a pair of compatible contact forms: Consider the nontrivial reduced weight $(-2,-2)$ density

$$\mathcal{I} = \mathsf{D}^I \mathsf{D}^{\bar{J}} (\Pi_I^B \Pi_{\bar{J}}^{\bar{D}} \boldsymbol{h}^{\mu\bar{\nu}} h_{C\bar{E}} \mathbb{L}_{\mu B}{}^C \mathbb{L}_{\bar{\nu}\bar{D}}{}^{\bar{E}}). \tag{6.4.10}$$

Since $\Pi_I^B \Pi_{\bar{J}}^{\bar{D}}$ is by definition a section of $\mathcal{E}_{I\bar{J}} \otimes \mathcal{E}^{B\bar{D}}|_\Sigma \otimes \mathcal{R}(1,1)$, and $\mathcal{R}(1,1)$ is canonically trivial and flat, we see that

$$f_{I\bar{J}} = \Pi_I^B \Pi_{\bar{J}}^{\bar{D}} \boldsymbol{h}^{\mu\bar{\nu}} h_{C\bar{E}} \mathbb{L}_{\mu B}{}^C \mathbb{L}_{\bar{\nu}\bar{D}}{}^{\bar{E}} \tag{6.4.11}$$

has reduced weight $(-1,-1)$ and no ratio bundle weight (diagonal ratio bundle weights can be ignored). Therefore in this case we do not need to couple the submanifold tractor D-operator with any ambient connection in order to define $\mathsf{D}^I \mathsf{D}^{\bar{J}} f_{I\bar{J}}$. From the definition of D_J we have

$$\begin{aligned} \mathsf{D}^{\bar{J}} f_{I\bar{J}} = &-(m-2) Y^{\bar{J}} f_{I\bar{J}} + (m-2) W^{\nu\bar{J}} D_\nu f_{I\bar{J}} \\ &- Z^{\bar{J}} \left(D^\nu D_\nu f_{I\bar{J}} - i D_0 f_{I\bar{J}} - p f_{I\bar{J}} \right) \end{aligned} \tag{6.4.12}$$

where D denotes the submanifold tractor connection coupled with the Tanaka-Webster connection of some submanifold contact form θ_Σ and Z, W, Y are the splitting tractors corresponding to the choice of θ_Σ. The tractor $\mathsf{D}^{\bar{J}} f_{I\bar{J}}$ has weight $(-2,-1)$ and so, applying $\mathsf{D}_{\bar{I}}$ and contracting, we have

$$\begin{aligned} \mathsf{D}^I \mathsf{D}^{\bar{J}} f_{I\bar{J}} = &-2(m-3) Y^I \mathsf{D}^{\bar{J}} f_{I\bar{J}} + (m-3) W^{\bar{\mu} I} D_{\bar{\mu}} \mathsf{D}^{\bar{J}} f_{I\bar{J}} \\ &- Z^I \left(D^{\bar{\mu}} D_{\bar{\mu}} \mathsf{D}^{\bar{J}} f_{I\bar{J}} - 2i D_0 \mathsf{D}^{\bar{J}} f_{I\bar{J}} - 2\frac{m+3}{m+2} p \mathsf{D}^{\bar{J}} f_{I\bar{J}} \right). \end{aligned} \tag{6.4.13}$$

If we choose θ admissible and compatible with θ_Σ then (6.3.11) implies

$$f_{I\bar{J}} = I\!I_{\mu\lambda}{}^\gamma I\!I_{\bar{\nu}\bar{\rho}}{}^{\bar{\epsilon}} \boldsymbol{h}^{\mu\bar{\nu}} \boldsymbol{h}_{\gamma\bar{\epsilon}} W_I^\lambda W_{\bar{J}}^{\bar{\rho}} + P_{\mu\bar{\alpha}} \mathrm{N}^{\beta\bar{\alpha}} P_{\beta\bar{\nu}} \boldsymbol{h}^{\mu\bar{\nu}} Z_I Z_{\bar{J}}. \tag{6.4.14}$$

In each term on the right hand side of (6.4.12) and (6.4.13) there is a contraction with a tractor, using the orthogonality relations between the tractor projectors simplifies the calculation significantly since one can ignore terms that will vanish after these contractions. So, for example, one easily computes that

$$W^{\nu\bar{J}} D_\nu f_{I\bar{J}} = D^{\bar{\rho}}(II_{\mu\lambda}{}^{\gamma} II^{\mu}{}_{\bar{\rho}\gamma} W_I^{\lambda}) + m P_{\mu\bar{\alpha}} \mathrm{N}^{\beta\bar{\alpha}} P_\beta{}^\mu Z_I.$$

Another efficient way to compute terms is to commute the splitting tractors forward past each appearance of the connection D using the submanifold versions of (5.3.16)-(5.3.24). Since $Z^{\bar{J}} f_{I\bar{J}} = 0$ and $[D_\nu, Z^{\bar{J}}] = D_\nu Z^{\bar{J}} = 0$ we have $Z^{\bar{J}} D_\nu f_{I\bar{J}} = 0$, from which we get

$$Z^{\bar{J}} D^\nu D_\nu f_{I\bar{J}} = -W^{\nu\bar{J}} D_\nu f_{I\bar{J}}$$

using $[D^\nu, Z^{\bar{J}}] = D^\nu Z^{\bar{J}} = W^{\nu\bar{J}}$; thus two of the terms in (6.4.12) coincide, up to a factor, simplifying our calculations significantly. Computing similarly

$$\begin{aligned} Z^{\bar{J}} D_0 f_{I\bar{J}} &= D_0(Z^{\bar{J}} f_{I\bar{J}}) - (D_0 Z^{\bar{J}}) f_{I\bar{J}} \\ &= i P_{\mu\bar{\alpha}} \mathrm{N}^{\beta\bar{\alpha}} P_\beta{}^\mu Z_I \end{aligned}$$

using that $Z^{\bar{J}} f_{I\bar{J}} = 0$ and $D_0 Z^{\bar{J}} = -iY^{\bar{J}} + \frac{i}{m+2} p Z^{\bar{J}}$. Putting these together yields

$$\mathsf{D}^{\bar{J}} f_{I\bar{J}} = (m-1) D^{\bar{\rho}}(II_{\mu\lambda}{}^{\gamma} II^{\mu}{}_{\bar{\rho}\gamma} W_I^{\lambda}) + (m-1)^2 P_{\mu\bar{\alpha}} \mathrm{N}^{\beta\bar{\alpha}} P_\beta{}^\mu Z_I. \tag{6.4.15}$$

Repeating this procedure for (6.4.13) we eventually obtain

$$\begin{aligned} \mathcal{I} = (m-1) \big[&(m-2) D^\lambda D^{\bar{\rho}} (II_{\mu\lambda}{}^{\gamma} II^{\mu}{}_{\bar{\rho}\gamma}) \\ &+ (D^{\bar{\rho}} D_{\bar{\rho}} - 2i D_0 - \tfrac{(m+4)(m-2)}{m+2} p)(II_{\mu\lambda}{}^{\gamma} II^{\mu\lambda}{}_{\gamma}) \\ &- (m-2)(m-4) p^{\lambda\bar{\rho}} II_{\mu\lambda}{}^{\gamma} II^{\mu}{}_{\bar{\rho}\gamma} \\ &+ (m-1)(m-2)(m-4) P_{\mu\bar{\alpha}} \mathrm{N}^{\beta\bar{\alpha}} P_\beta{}^\mu \big] . \end{aligned} \tag{6.4.16}$$

6.5 A CR Bonnet Theorem

In classical surface theory the Bonnet theorem (or fundamental theorem of surfaces) says that if a covariant 2-tensor II on an abstract Riemannian surface (Σ, g) satisfies the Gauss and Codazzi equations then (locally about any point) there exists an embedding of (Σ, g) into Euclidean 3-space which realises the tensor II as the second fundamental form. A more general version of the Bonnet theorem states that if we specify on a Riemannian manifold (Σ^m, g) a rank d vector bundle $N\Sigma$ with bundle metric and metric preserving connection and an $N\Sigma$-valued symmetric covariant 2-tensor II satisfying the Gauss, Codazzi and Ricci equations then (locally) there exists an embedding of (Σ^m, g) into Euclidean n-space, where $n = m + d$, realising $N\Sigma$ as the normal bundle and II as the second fundamental form. Here we give a CR geometric analogue of this theorem.

6.5.1 Locally Flat CR Structures

The Bonnet theorem given in §§6.5.3 generalises and is motivated by the following well known theorem on locally flat CR structures. The proof we give will be adapted to give a proof of the Bonnet theorem.

Theorem 6.5.1. *A nondegenerate CR manifold (M^{2n+1}, H, J) of signature (p, q) with vanishing tractor curvature is locally equivalent to the signature (p, q) model hyperquadric $\mathcal{H}$.*

Proof. The signature (p, q) model hyperquadric $\mathcal{H}$ can be realised as the space of null (i.e. isotropic) complex lines in the projectivisation of $\mathbb{C}^{p+1,q+1}$. Since the tractor curvature vanishes one may locally identify the standard tractor bundle $\mathcal{T}M$ with the trivial bundle $M \times \mathbb{C}^{p+1,q+1}$ so that the tractor connection becomes the trivial flat connection and the tractor metric becomes the standard inner product on $\mathbb{C}^{p+1,q+1}$. The canonical null line subbundle $L = \mathcal{T}^1 M$ of $\mathcal{T}M$ (spanned by the weighted canonical tractor Z^A) then gives rise to a map from M into the model hyperquadric given by

$$M \ni x \mapsto L_x \subset \mathbb{C}^{p+1,q+1}. \tag{6.5.1}$$

We need to show that the map $f : M \to \mathbb{P}(\mathbb{C}^{p+1,q+1})$ given by (6.5.1) is a local CR diffeomorphism.

The maximal complex subspace in the tangent space to $\mathcal{H}$ at the point ℓ, where $\ell \subset \mathbb{C}^{p+1,q+1}$ is an isotropic line, is the image of $\ell^{\perp}$ under the tangent map of the projection $\mathbb{C}^{p+1,q+1} \to \mathbb{P}(\mathbb{C}^{p+1,q+1})$. Choosing a nowhere zero local section ρ of $L = \mathcal{E}(-1, 0)$ we get a lift of the map f to a map $f_\rho : M \to \mathbb{C}^{p+1,q+1}$. The map $L = \mathcal{E}(-1, 0) \hookrightarrow \mathcal{T}M = \mathcal{E}^A$ is given explicitly by $\rho \mapsto \rho Z^A$. Since the tractor connection is flat the tangent map of f_ρ at $x \in M$ is given by

$$T_x M \ni X \mapsto \nabla_X(\rho Z^A) \in \mathbb{C}^{p+1,q+1}.$$

By the respective conjugates $\nabla_{\bar{\beta}} Z^A = 0$ and $\nabla_\beta Z^A = W^A_\beta$ of (5.3.18) and (5.3.21) (fixing any background contact form and raising indices using the tractor metric) the tangent map $T_x f_\rho$ restricted to contact directions maps onto a complementary subspace to L_x inside $L_x^\perp$ and induces a complex linear isomorphism of H_x with $L_x^\perp / L_x$; combined with (5.3.24) we see that $T_x f_\rho$ is injective and its image is transverse to L_x. Now f is the composition of f_ρ with the projectivisation map $\mathbb{C}^{p+1,q+1} \setminus \{0\} \to \mathbb{P}(\mathbb{C}^{p+1,q+1})$; thus we have that $T_x f$ is injective, and further that f is a local CR diffeomorphism from M to the signature (p, q) model hyperquadric in $\mathbb{P}(\mathbb{C}^{p+1,q+1})$. □

Remark 6.5.2. Throughout this article we have implicitly identified the CR tractor bundle $\mathcal{T}M$ with the holomorphic part of its complexification in the standard way. In the above proof we have therefore also implicitly identified the tangent space to $\mathbb{C}^{p+1,q+1}$ at any point with the holomorphic tangent space; the section ρZ^A should be understood as a section of the holomorphic tractor bundle, the map f_ρ being determined by the corresponding section of the real tractor bundle. ▮

The map constructed in the proof is the usual Cartan developing map for a flat Cartan connection, though constructed using tractors and the projective realisation of the model hyperquadric. The fact that the map constructed is a local CR diffeomorphism relies on the soldering property of the canonical Cartan/tractor connection on M, which is captured in the formulae (5.3.18), (5.3.21), and (5.3.24).

6.5.2 CR Tractor Gauss-Codazzi-Ricci Equations

Our tractor based treatment of transversal CR embeddings in §6.2 and §6.3 has shown us exactly what data should be prescribed on a CR manifold in a CR version of the Bonnet theorem: Consider a transversal CR embedding $\Sigma^{2m+1} \hookrightarrow M^{2n+1}$ between non-degenerate CR manifolds. Then along Σ the ambient standard tractor bundle splits as an orthogonal direct sum $(\mathcal{T}\Sigma \otimes \mathcal{R}(-1,0)) \oplus \mathcal{N}$ with the ratio bundle $\mathcal{R}(-1,0)$ being the dual of an $(m+2)^{th}$ root of the top exterior power $\Lambda^d \mathcal{N}$ of the normal tractor bundle $\mathcal{N}$. It is also easy to see that the (pulled back) ambient tractor connection decomposes along Σ as

$$\nabla = \begin{pmatrix} D \otimes \nabla^{\mathcal{R}} + \mathsf{S} & -\mathbb{L}^\dagger \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \quad \text{on} \quad \begin{matrix} \mathcal{T}\Sigma \otimes \mathcal{R}(-1,0) \\ \oplus \\ \mathcal{N} \end{matrix} \tag{6.5.2}$$

where $D\otimes\nabla^{\mathcal{R}}$ denotes the coupled connection on $\mathcal{T}\Sigma\otimes\mathcal{R}(-1,0)$, with D the submanifold tractor connection and $\nabla^{\mathcal{R}}$ the connection induced on $\mathcal{R}(-1,0)$ by the normal tractor connection $\nabla^{\mathcal{N}}$. The objects S and $\mathbb{L}$ are as defined in §§6.2.4 and §§6.2.5, and $\mathbb{L}^{\dagger}(X)$ is the Hermitian adjoint of $\mathbb{L}(X)$ with respect to the ambient tractor metric for any $X\in\mathfrak{X}(\Sigma)$. The bundle $\mathcal{N}$ carries a Hermitian metric $h^{\mathcal{N}}$ induced by the ambient tractor metric. We refer to the triple $(\mathcal{N},\nabla^{\mathcal{N}},h^{\mathcal{N}})$ along with $(\mathcal{R}(-1,0),\nabla^{\mathcal{R}})$ and the invariants S and $\mathbb{L}$ as the (extrinsic) *induced data* coming from the CR embedding.

The above observations also establish Proposition 4.3.1.

Remark 6.5.3. The Hermitian adjoint $\mathbb{L}^{\dagger}$ of $\mathbb{L}$ appears because of (6.3.10). Note that $\mathbb{L}^{\dagger}_{iB}{}^{C}=\overline{\mathbb{L}_{\bar{i}}{}^{\bar{C}}{}_{\bar{B}}}$ so that in particular $\mathbb{L}^{\dagger}_{\bar{\mu}B}{}^{C}=\overline{\mathbb{L}_{\mu}{}^{\bar{C}}{}_{\bar{B}}}$ and $\mathbb{L}^{\dagger}_{\mu B}{}^{C}=0$. Note also that for any $X\in\mathfrak{X}(\Sigma)$

$$\begin{pmatrix}\mathsf{S}(X) & -\mathbb{L}^{\dagger}(X)\\ \mathbb{L}(X) & 0\end{pmatrix}\tag{6.5.3}$$

is a skew-Hermitian endomorphism of $\mathcal{T}M|_{\Sigma}$ since each of the connections appearing in (6.5.2) preserves the appropriate Hermitian bundle metric. ▮

We can also easily see what the integrability conditions should be on this abstract data: Observe that the curvature of the connection (6.5.2) acting on sections of $\mathcal{T}M|_{\Sigma}$ is given by

$$\begin{pmatrix}D\otimes\nabla^{\mathcal{R}}+\mathsf{S} & -\mathbb{L}^{\dagger}\\ \mathbb{L} & \nabla^{\mathcal{N}}\end{pmatrix}\wedge\begin{pmatrix}D\otimes\nabla^{\mathcal{R}}+\mathsf{S} & -\mathbb{L}^{\dagger}\\ \mathbb{L} & \nabla^{\mathcal{N}}\end{pmatrix}$$
$$=\begin{pmatrix}\kappa^{D\otimes\nabla^{\mathcal{R}}+\mathsf{S}}-\mathbb{L}^{\dagger}\wedge\mathbb{L} & -\mathrm{d}\mathbb{L}^{\dagger}-\mathsf{S}\wedge\mathbb{L}^{\dagger}\\ \mathrm{d}\mathbb{L}+\mathbb{L}\wedge\mathsf{S} & \kappa^{\mathcal{N}}-\mathbb{L}\wedge\mathbb{L}^{\dagger}\end{pmatrix}$$

where $\kappa^{D\otimes\nabla^{\mathcal{R}}+\mathsf{S}}$ is the curvature of $D\otimes\nabla^{\mathcal{R}}+\mathsf{S}$, $\mathrm{d}\mathbb{L}$ and $\mathrm{d}\mathbb{L}^{\dagger}$ are the respective covariant exterior derivatives of $\mathbb{L}$ and $\mathbb{L}^{\dagger}$ with respect to $D\otimes\nabla^{\mathcal{R}}\otimes\nabla^{\mathcal{N}}$, and $\kappa^{\mathcal{N}}$ is the curvature of $\nabla^{\mathcal{N}}$. The above display expresses the pullback of the ambient curvature by the embedding in terms of the induced data of the CR embedding. Writing these relations component-wise leads to the CR tractor Gauss, Codazzi, and Ricci equations; denoting the pullback of the ambient curvature simply by κ these are, respectively,

$$\Pi\circ\kappa\circ\Pi=\kappa^{D\otimes\nabla^{\mathcal{R}}+\mathsf{S}}-\mathbb{L}^{\dagger}\wedge\mathbb{L},\tag{6.5.4}$$

$$\mathrm{N}\circ\kappa\circ\Pi=\mathrm{d}\mathbb{L}+\mathbb{L}\wedge\mathsf{S},\tag{6.5.5}$$

and

$$\mathrm{N} \circ \kappa \circ \mathrm{N} = \kappa^{\mathcal{N}} - \mathbb{L} \wedge \mathbb{L}^{\dagger} \tag{6.5.6}$$

where Π and N denote the complementary 'tangential' and 'normal' projections acting on a section $v = (v^{\top}, v^{\perp})$ of $\mathcal{T}M|_{\Sigma}$. Of course

$$\kappa^{D\otimes\nabla^{\mathcal{R}}+\mathsf{S}} = \kappa^{\Sigma} - \kappa^{\mathcal{R}(1,0)} + \mathrm{d}\mathsf{S} + \mathsf{S} \wedge \mathsf{S}$$

where $\kappa^{\mathcal{R}(1,0)}$ denotes the curvature of $\mathcal{R}(1,0)$ acting as a bundle endomorphism via multiplication, κ^{Σ} is the submanifold tractor curvature, and $\mathrm{d}\mathsf{S}$ is the covariant exterior derivative of S with respect to the submanifold tractor connection. Note that the equation $\Pi \circ \kappa \circ \mathrm{N} = -\mathrm{d}\mathbb{L}^{\dagger} - \mathsf{S} \wedge \mathbb{L}^{\dagger}$ is determined by (6.5.5). Note also that the tractor Ricci equation (6.5.6) determines the normal tractor curvature $\kappa^{\mathcal{N}}$ in terms of the ambient curvature and the tractor second fundamental form.

Remark 6.5.4. One can easily write the terms appearing in the tractor Gauss, Codazzi, and Ricci equations more explicitly using abstract indices. For instance we have

$$(\mathbb{L}^{\dagger} \wedge \mathbb{L})_{ijK}{}^{L} = 2\mathbb{L}^{\dagger}_{[i|E}{}^{L}\mathbb{L}_{|j]K}{}^{E} = 2\mathbb{L}_{[i|}{}^{L}{}_{E}\mathbb{L}_{|j]K}{}^{E}$$

where we use $\mathbb{L}_{jK}{}^{E} = \mathbb{L}_{jC}{}^{E}\Pi^{C}_{K}$ and $\mathbb{L}^{\dagger}_{iE}{}^{L} = \mathbb{L}^{\dagger}_{iE}{}^{D}\Pi^{L}_{D}$ since we are identifying $\mathcal{N}^{\perp} \subset \mathcal{T}M|_{\Sigma}$ with $\mathcal{T}\Sigma \otimes \mathcal{R}(-1,0)$. ▮

6.5.3 The CR Bonnet Theorem

With the notion of induced data on the submanifold from a CR embedding given in the previous section we can now give the following theorem:

Theorem 6.5.5. *Let (Σ^{2m+1}, H, J) be a signature (p,q) CR manifold and suppose we have a complex rank d vector bundle $\mathcal{N}$ on Σ equipped with a signature (p',q') Hermitian bundle metric $h^{\mathcal{N}}$ and metric connection $\nabla^{\mathcal{N}}$. Fix an $(m+2)^{th}$ root $\mathcal{R}$ of $\Lambda^{d}\mathcal{N}$, and let $\nabla^{\mathcal{R}}$ denote the connection induced by $\nabla^{\mathcal{N}}$. Suppose we have a $\mathcal{N} \otimes \mathcal{T}^{*}\Sigma \otimes \mathcal{R}$ valued 1-form $\mathbb{L}$ which annihilates the canonical tractor of Σ and an $\mathcal{A}^{0}\Sigma$ valued 1-form S on Σ such that the connection*

$$\nabla := \begin{pmatrix} D \otimes \nabla^{\mathcal{R}} + \mathsf{S} & -\mathbb{L}^{\dagger} \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \quad \textit{on} \quad \begin{matrix} \mathcal{T}\Sigma \otimes \mathcal{R}^{*} \\ \oplus \\ \mathcal{N} \end{matrix}$$

is flat (where D is the submanifold tractor connection), then (locally) there exists a transversal CR embedding of Σ into the model $(p+p', q+q')$ hyperquadric $\mathcal{H}$, unique up to automorphisms of the target, realising the specified extrinsic data as the induced data.

Proof. Since the complex line bundle $\Lambda^d\mathcal{N}$ is normed by $h^{\mathcal{N}}$, the bundle $\mathcal{R}$ is also normed. This means that the tractor metric $h^{\mathcal{T}\Sigma}$ induces a Hermitian bundle metric on $\mathcal{T}\Sigma\otimes\mathcal{R}^*$, which we again denote by $h^{\mathcal{T}\Sigma}$. We therefore have a Hermitian bundle metric $h = h^{\mathcal{T}\Sigma} + h^{\mathcal{N}}$ on the bundle $(\mathcal{T}\Sigma\otimes\mathcal{R}^*)\oplus\mathcal{N}$. Since S is adjoint tractor valued (i.e. skew-Hermitian endomorphism of $\mathcal{T}\Sigma$ valued) the connection $D\otimes\nabla^{\mathcal{R}}+\mathsf{S}$ on $\mathcal{T}\Sigma\otimes\mathcal{R}^*$ preserves $h^{\mathcal{T}\Sigma}$. Collectively, the terms involving $\mathbb{L}$ and $\mathbb{L}^\dagger$ in the displayed definition of ∇ constitute a one form valued in skew-Hermitian endomorphisms of $(\mathcal{T}\Sigma\otimes\mathcal{R}^*)\oplus\mathcal{N}$. Combined with the fact that $\nabla^{\mathcal{N}}$ preserves $h^{\mathcal{N}}$ this shows that ∇ preserves the Hermitian bundle metric h.

The signature $(p+p', q+q')$ model hyperquadric $\mathcal{H}$ can be realised as the space of null complex lines in the projectivisation of $\mathbb{T}=\mathbb{C}^{p+p'+1,q+q'+1}$. Since the connection ∇ on $(\mathcal{T}\Sigma\otimes\mathcal{R}^*)\oplus\mathcal{N}$ is flat and preserves h one may locally identify this bundle with the trivial bundle $\Sigma\times\mathbb{T}$ such that ∇ becomes the trivial flat connection and h becomes the standard signature $(p+p'+1, q+q'+1)$ inner product on $\mathbb{T}$; this trivialisation is uniquely determined up to the action of $\mathrm{SU}(p+p'+1, q+q'+1)$ on $\mathbb{T}$. The canonical null line subbundle $\mathcal{E}_\Sigma(-1,0)$ of $\mathcal{T}\Sigma$ gives rise to a null line subbundle $L=\mathcal{E}_\Sigma(-1,0)\otimes\mathcal{R}^*$ of $\Sigma\times\mathbb{T}$. The null line subbundle L then gives rise to a smooth map into the model $(p+p'+1, q+q'+1)$ hyperquadric given by

$$\Sigma\ni x\mapsto L_x\subset\mathbb{T}=\mathbb{C}^{p+p'+1,q+q'+1}. \tag{6.5.7}$$

Since the local trivialisation of $(\mathcal{T}\Sigma\otimes\mathcal{R}^*)\oplus\mathcal{N}$ is uniquely determined up to the action of $\mathrm{SU}(\mathbb{T})$ the above displayed map from Σ to $\mathcal{H}$ is determined up to automorphisms of $\mathcal{H}$. It remains to show that this map is a transversal CR embedding inducing the specified extrinsic data.

Let us denote the map (6.5.7) by $f:\Sigma\to\mathcal{H}\subset\mathbb{P}(\mathbb{T})$. Given a nowhere zero local section ρ of $L=\mathcal{E}_\Sigma(-1,0)\otimes\mathcal{R}^*$ we may think of the section ρZ^I of $\mathcal{T}\Sigma\otimes\mathcal{R}^*$ as a section of $\Sigma\times\mathbb{T}$ via inclusion; this section gives rise to a lifted map $f_\rho:\Sigma\to\mathbb{T}$. The tangent map of f_ρ at $x\in\Sigma$ is given by

$$T_x\Sigma\ni X\mapsto\nabla_X(\rho Z^I)\in\mathbb{T}.$$

From (5.3.18) and (5.3.21) we have that $D_{\bar{\nu}} Z^I = 0$ and $D_\nu Z^I = W^I_\nu$ (fixing some background contact form on Σ); using these, the definition of ∇, and the facts that $\mathsf{S}_{iJ}{}^K Z^J \mod Z^K = 0$ (since S is adjoint valued) and that $\mathbb{L}$ annihilates the canonical tractor Z^I, we see that $T_x f_\rho$ restricted to contact directions is injective and induces a complex linear isomorphism of H_x onto a subspace of $L_x^\perp / L_x$; combined with (5.3.24) we see that $T_x f_\rho$ is injective and its image is transverse to L_x. This implies that the composition f of f_ρ with the projectivisation map $\mathbb{T} \setminus \{0\} \to \mathbb{P}(\mathbb{T})$ is a local CR embedding into the model hyperquadric $\mathcal{H}$. Equation (5.3.24) further shows that $T_x f_\rho(T_x \Sigma) \not\subset L_x^\perp$ so f is transversal.

To see that this embedding induces back the specified extrinsic data we simply need to note that we may identify $(\mathcal{T}\Sigma \otimes \mathcal{R}^*) \oplus \mathcal{N} = \Sigma \times \mathbb{T}$ with $\mathcal{T}\mathcal{H}|_\Sigma$, identifying ∇ with the flat tractor connection on $\mathcal{T}\mathcal{H}|_\Sigma$ and h with the tractor metric $h^{\mathcal{T}\mathcal{H}}$ along Σ. Then

$$\mathcal{T}\mathcal{H}|_\Sigma = \begin{array}{c} \mathcal{T}\Sigma \otimes \mathcal{R}^* \\ \oplus \\ \mathcal{N} \end{array}$$

is the usual decomposition of the ambient tractor bundle along the submanifold, and the definition of ∇ in the statement of the theorem gives the usual decomposition of the ambient tractor connection. □

Our formulation and proof of this CR Bonnet theorem is inspired by the conformal Bonnet theorem formulated and proved in terms of standard conformal tractors by Burstall and Calderbank in [23]. The condition that the connection ∇ we define be flat is alternatively given in terms of the prescribed data on $(\Sigma, H_\Sigma, J_\Sigma)$ by the tractor Gauss, Codazzi, and Ricci equations (6.5.4), (6.5.5), and (6.5.6) with the left hand sides equal to zero.

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