

# Conformal submanifolds, distinguished submanifolds, and integrability

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## Abstract

For conformal geometries of Riemannian signature, we provide a comprehensive and explicit treatment of the core local theory for embedded submanifolds of arbitrary dimension. This is based in the conformal tractor calculus and includes a conformally invariant Gauss formula leading to conformal versions of the Gauss, Codazzi, and Ricci equations. It provides the tools for proliferating submanifold conformal invariants, as well for extending to conformally singular Riemannian manifolds the notions of mean curvature and of minimal and CMC submanifolds.

A notion of distinguished submanifold is defined by asking the tractor second fundamental form to vanish. We show that for the case of curves this exactly characterises conformal geodesics, also called conformal circles, while for hypersurfaces it is the totally umbilic condition. So, for other codimensions, this unifying notion interpolates between these extremes, and we prove that in all dimensions this coincides with the submanifold being weakly conformally circular, meaning that ambient conformal circles remain in the submanifold. We prove that submanifolds are conformally circular, meaning submanifold conformal circles coincide with ambient conformal circles, if and only also a second conformal invariant also vanishes.

Next we provide a very general theory and construction of quantities that are necessarily conserved along distinguished submanifolds. This first integral theory thus vastly generalises the results available for conformal circles in [56]. We prove that any normal solution to an equation from the class of first BGG equations can yield such a conserved quantity, and we show that it is easy to provide explicit formulae for these.

Finally we prove that the property of being distinguished is also captured by a type of moving incidence relation. This second characterisation is used to show that, for suitable solutions of conformal Killing-Yano equations, a certain zero locus of the solution is necessarily a distinguished submanifold.

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# 1 Introduction

Submanifolds are one of the most fundamental structures in differential geometry. They have a crucial role in the geometric analysis associated with PDEs, complex analysis, and functional analysis, as well as in many global questions. In the special case of Riemannian geometry, submanifold theory is a classical area and the basic local theory is well understood and captured in the celebrated equations of Gauss, Codazzi, and Ricci, see, e.g., [78, 87].

Conformal manifolds  $(M, \mathbf{c})$  are structures where a smooth  $n$  manifold  $M$  is equipped not with a metric, but rather with just an equivalence class of smooth metrics  $\mathbf{c}$ , where  $g, \hat{g} \in \mathbf{c}$  means that  $\hat{g} = e^{2\omega}g$  for some smooth function  $\omega$ . There is currently a growing interest in the study of conformal submanifolds, and conformally distinguished curves, including the relationships between these objects [27, 32, 43, 64, 73, 75, 76, 83, 84]. Some of these developments have been inspired and driven by the links to physics [42, 66], the discovery that there are higher dimensional analogues of the classical Willmore energy and invariant [70, 58, 98, 63], and the development of a holographic approach to submanifolds (see [57, 59, 61]) that is an analogue of Fefferman and Graham's holographic approach to intrinsic conformal geometry as in [37, 38].

In the conformal setting there is no distinguished connection on the tangent bundle, so even the local theory of submanifolds provides a challenge. Toward resolving this, a logical step is to use the conformal Cartan/tractor connection of [24, 95, 5, 22], and for the special case of *hypersurfaces*, meaning embedded submanifolds of codimension one, an effective approach was initiated in [5]. With a view to various applications, this hypersurface theory was extended in the works [9, 67, 93, 51, 98] and this approach has proved to be central in a number of further extensions and applications [2, 8, 57, 58, 59, 62, 61, 75]. Rather separately from the general consideration of submanifolds, the *distinguished curves* in conformal manifolds known as *conformal circles* or *conformal geodesics* have been studied classically (see, e.g., [39, 85, 86, 100, 101]) and from various modern perspectives recently [44, 4, 5, 96, 90, 56, 35, 68, 72, 14].

In the first part of our work here we develop a comprehensive basic local theory for conformal submanifolds of all proper codimensions. This is based in the conformal tractor calculus, and by construction is conformally invariant. It builds on the mentioned approach to hypersurfaces from [5, 9, 67, 98, 33] and its extension into higher codimension by the first and third named authors in [34] and [91]. There are also links to the somewhat more abstract theory developed in the preprint [11] (cf. Remark 3.16 below and also the discussion in [34]). The result is a theory and collection of explicit calculational tools that treats curves and higher dimensional submanifolds, embedded in conformal manifolds, by a single uniform approach. These tools may be used in an obvious way to proliferate submanifold invariants (including for curves), in the spirit of Fefferman's parabolic invariant theory

programme [36, 37, 6], and we touch on this in Section 3.8. We also use these tools to effectively capture, via tractors, the mean curvature, and with it the notions of minimal, constant mean curvature, and parallel mean curvature submanifolds. This leads immediately to a generalisation of these notions that is applicable for the study of conformally singular geometries (such as Poincaré-Einstein geometries, and more generally conformally compact structures). See Section 4.

For submanifolds of any codimension we introduce the notion of a *distinguished submanifold*. By definition this means that an object called the tractor second fundamental form vanishes, see Theorem 1.1. In the case of curves this coincides precisely with the unparametrised conformal circle equation, see Theorem 1.2, whereas for hypersurfaces it recovers the usual condition of total umbilicity [5]. Thus the notion interpolates between these. Interestingly, in all codimensions greater than one the condition is stronger than total umbilicity (outside of the conformally flat setting, where these notions coincide). Part of our motivation in defining this notion is to treat certain key applications that form the second main objective of this work.

It has long been known that Killing tensors and Killing-Yano tensors may be used to provide first integrals for geodesics [25, 41, 88, 92, 99], and this is used for a host of applications [1, 26, 28, 45, 46, 77, 74]. Conformal circles are governed by a higher order equation than geodesics, so an analogous theory has been lacking aside from certain specific examples [96]. However, in [56] this was solved and a very general theory of first integrals was developed by understanding a characterisation of conformal circles as a parallel condition on a fundamental tractor 3-form that one can associate to any non-null curve. Using this, it was established that essentially any normal solution of a class of equations known as first BGG equations (see [21, 23], or Section 6.1, for the meaning of these terms) can provide, or contribute to, such conserved quantities; in fact in many cases more general solutions produce first integrals. See [68] for some applications of this perspective. The conformal Killing equations on tensors and the conformal Killing-Yano equations are all first BGG equations. But in fact the class of first BGG equations is vastly wider than this suggests. In Section 3.7 we show that, just as for curves, higher dimensional distinguished submanifolds can be characterised by a parallel condition on a tractor form. Then, as an application, we obtain a theory of first integrals for distinguished submanifolds of all codimensions in a form that includes the case of conformal circles as special case. See Theorem 1.5 and Corollary 1.4.

In another direction, an important question for submanifolds of dimension 2 or greater is characterising the conformal analogue of the notion of being totally geodesic; that is, to capture some sense of “total conformal circularity”. This was touched on in [5] for hypersurfaces, and treated for submanifolds in general by Belgun [7]. We show in Section 5 that our tools give efficient new proofs these results, and show that Belgun’s results have an elegant interpretation in this tractor picture. See Theorem 1.3 and Theorem 5.10 below.

Finally, in Section 7 we show that distinguished submanifolds can also be char-

acterised by a very simple moving incidence relation, see Theorem 7.1, or its paraphrasing in Theorem 1.5. As an application we prove that the zero locus of suitable overdetermined PDE solutions are necessarily distinguished submanifolds; see Theorem 7.2. This shows how distinguished submanifolds fit into the *curved orbit* theory of [20, 21] and, along with Proposition 7.3, is a first step toward understanding how to generalise the holography approach of [2, 8, 58, 59, 60, 62, 61] to higher codimensions.

## 1.1 Main results and a technical overview

Now we give the approach and results with more technical detail. The first step in developing a calculus for hypersurfaces in a Riemannian manifold is the observation that any oriented hypersurface is equipped with a canonical unit conormal field  $n_a$ . Similarly it was established in [5] that each hypersurface  $\Sigma$  in a Riemannian signature conformal manifold determines canonically a basic conformally invariant tractor field  $N_A$  that plays an analogous role at a tractor level. For example it's failure to be parallel along the hypersurface is captured by a tractor second fundamental form  $\mathbb{L}$ . In particular one obtains, in a simple explicit way, conformal analogues of the Gauss-Codazzi-Ricci theory, see [33] and references therein. Moreover the normal tractor has a remarkable link to other objects in, for example, Poincaré-Einstein manifolds and related structures, that has led to some deep results (e.g., that link the so-called conformal volume anomaly to higher Willmore invariants [63, 2, 59]).

Here we establish that one higher codimension analogue of the normal tractor is a conformally invariant alternating tractor form  $N_{A_1 \dots A_d}$ , where  $d$  is the codimension of the submanifold. An equivalent object to  $N_{A_1 \dots A_d}$  is its tractor Hodge-star, that we denote  $\star N^{A_1 A_2 \dots A_{m+2}}$ , see (2.27). For a submanifold  $\Sigma$  (of any non-trivial codimension), the intrinsic tractor bundle  $\mathcal{T}\Sigma$  can be identified with the annihilator in  $\mathcal{T}M$  of  $N_{A_1 A_2 \dots A_d}$ , see Section 3.3. (This is still true when  $\Sigma$  has dimension 1 or 2, though in this case it is less obvious if one starts with the jet bundle construction of  $\mathcal{T}\Sigma$ ; see the discussion in Section 3.5 which treats, from our point of view here, the natural Möbius structures induced on low dimensional submanifolds, cf. [13]). Thus one has an orthogonal decomposition of the ambient tractor bundle  $\mathcal{T}M$ ,

$$\mathcal{T}M|_{\Sigma} = \mathcal{T}\Sigma \oplus \mathcal{N},$$

which also defines the *normal tractor bundle*  $\mathcal{N}$ . Denote by  $N_B^A$  the projection  $\mathcal{T}M|_{\Sigma} \rightarrow \mathcal{N}$  (here we are using abstract index notation, cf. Section 2). There is also a projector  $\mathcal{T}M|_{\Sigma} \rightarrow \mathcal{T}\Sigma$ , and thus (provided  $\dim M \geq 3$  so that the conformal structure on  $M$  determines a canonical Cartan/tractor connection) one has a tractor Gauß formula which defines a tractor second fundamental form,  $\mathbb{L}_{i,j}^C$ ; see (3.22) and its refinement (3.40) (cf. [11, 34]).

Using these tools we develop and present explicitly the fundamental conformal submanifold calculus in Section 3.4, and, in particular, the conformal Gauss-

Codazzi-Ricci equations (3.42), (3.43), (3.44). Also, the normal forms, and their equivalents, may be combined with standard conformal tractor calculus, and ideas for using this to construct conformal manifold invariants (as in, e.g., [48]), to manufacture submanifold invariants. This is the subject of Section 3.8.

Although the tractor approach is conformal, mean curvature (which is far from being conformally invariant) can be nicely described by introducing an object called the *scale tractor*,  $I$ , see Section 2.4. In particular minimal submanifolds are seen to be exactly those submanifolds whose tractor normal form is orthogonal to the scale tractor, see Corollary 4.2, and constant mean curvature notions are similarly captured, see Proposition 4.4. This means that these concepts generalise to Poincaré-Einstein, and more generally conformally compact manifolds, with the submanifold extending to the conformal infinity, as discussed in Section 4.

The definition here of a submanifold being *distinguished* is that  $\mathbb{L}_{iJ}^C = 0$ , i.e. the vanishing of the tractor second fundamental form. A key result is that this may be alternatively captured as in the following theorem.

**Theorem 1.1.** *Let  $(M, c)$  be a conformal manifold and  $\Sigma \hookrightarrow M$  a conformal submanifold of codimension  $d$ . Then the following are equivalent:*

1.  $\mathbb{L}_{iJ}^C = 0$ ;
2.  $\nabla_i N_{A_2}^{A_1} = 0$ ;
3.  $\nabla_i N_{A_1 A_2 \dots A_{d-1} A_d} = 0$ ;
4.  $\nabla_i \star N^{A_1 A_2 \dots A_{m+2}} = 0$ ,

where  $\nabla_i$  indicates the pullback to  $\Sigma$  of the ambient tractor connection.

A hypersurface has  $\mathbb{L}_{iJ}^C = 0$  if and only if it is totally umbilic, meaning the trace-free second fundamental form vanishes, but for higher codimension it means a certain conditional invariant must also vanish. (When the trace-free second fundamental form vanishes this conditional invariant becomes Belgun's  $\mu$  invariant [7], and the relationship between  $\mu$  and  $\mathbb{L}$  is discussed in Section 5.2). Thus for codimensions greater than one, a distinguished submanifold is necessarily totally umbilic, but the converse is not true in general. Moreover, in the case of 1-dimensional submanifolds (where the totally umbilic condition becomes vacuous) the vanishing of  $\mathbb{L}$  precisely characterises conformal circles. That is, in our current terminology, 1-dimensional conformally distinguished submanifolds are exactly unparametrised conformal circles:

**Theorem 1.2.** *Let  $(M, c)$  be a conformal manifold and  $\gamma \hookrightarrow M$  a curve. Then  $\gamma$  is an unparametrised conformal circle if, and only if,  $\mathbb{L}_{iJ}^C = 0$ , or equivalently any one of the conditions in Theorem 1.1 holds.*

*Proof.* Proposition 4.13 from [56] asserts that the unparametrised conformal circle equation is equivalent to a certain 3-tractor being parallel along the curve, and equation (5.13) of Section 5.1 below asserts that this 3-tractor is precisely  $\star N^{A_1 A_2 A_3}$ . The theorem therefore follows from the equivalence of items 1 and 4 in Theorem 1.1.  $\square$

For the convenience of the reader we discuss Proposition 4.13 from [56] in Section 5.1 on conformal circles below.

Next we observe that Theorem 1.2 generalises to give an interpretation of any of the conditions of Theorem 1.1 for higher dimensional submanifolds:

**Theorem 1.3.** *A submanifold  $\Sigma$  is distinguished ( $\mathbb{L}_{iJ}^C = 0$ ) if, and only if, it is weakly conformally circular.*

Here, adapting terminology from [7], *weakly conformally circular* means that an ambient conformal circle with tangential initial conditions remains in the submanifold for some time, cf. [5] for the case of hypersurfaces. This is a repackaging into tractors of Belgun's [7, Theorem 5.4(2)]. The proof here is via tractors.

There is an alternative natural notion of conformal circularity for submanifolds, namely that any submanifold conformal circle is also an ambient conformal circle. This is stronger than the previous notion and we refer to such a submanifold as *conformally circular*. It turns out that for this property, in addition to any of the requirements of Theorem 1.1, one also requires that the trace-free part of the Filkoff tensor vanishes; see Theorem 5.11. Taking into account parametrisations gives rise to yet another notion of conformal circularity, which refer to as being *strongly conformally circular*, and Theorem 5.11 should be contrasted with the analogous result [7, Theorem 5.4(3)] (cf. Theorem 5.10 below) for projectively parametrised conformal circles. A key point here is that the tractor point of view developed in this paper gives a natural conceptual framework for characterising these three different notions of conformal circularity, and this leads to simpler proofs.

Another important feature of Theorem 1.1 is that points 2.–4. characterise distinguished submanifolds in a way that immediately allows the proliferation of conserved quantities. As mentioned above, Killing tensors, Killing-Yano tensors and their conformal analogues are well-established as tools for providing first integrals for geodesics. These are each examples of solutions to first BGG equations, a large class of overdetermined natural equations [23, 12]. For such equations, there is a class of solutions called *normal* solutions that are in one-to-one correspondence with parallel sections of the corresponding tractor bundle [21]. In particular, on conformally flat manifolds, all solutions to first BGG equations are normal. Let us state our result rather informally as follows.

**Corollary 1.4.** *Suppose a conformal manifold admits a BGG normal solution corresponding to a parallel tractor  $S$ , and  $\Sigma$  is a distinguished submanifold. Let*

$$\langle \otimes^\ell S, \otimes^k N \rangle$$

denote a scalar quantity constructed from linear combinations of tensor powers of  $S$  and linear combinations of tensor powers of the normal tractor form  $N$  and with contractions using the conformal tractor metric and possibly the tractor volume form. Then  $\langle \otimes^\ell S, \otimes^k N \rangle$  is a first integral for the distinguished submanifold.

This result generalises the large family of conformal circle first integrals constructed in [56] to the case of distinguished submanifolds of arbitrary dimension. We should say that rather than using the tractor normal form in the Corollary 1.4 above one may equally alternatively (or additionally use)  $\star N$ , or  $N_B^A$ . Precise statements can be found in Section 6.2, where we also show that it is easy to compute explicit examples.

Towards another key application, we show that there is yet another characterisation of distinguished conformal submanifolds that takes the form of a moving incidence relation. For this we need the first elements of conformal tractor calculus. On any smooth manifold, one has the bundle of conformal 1-densities that we call  $\mathcal{E}[1]$ , which is a root of the squared canonical bundle, see Section 2. Its 2-jet bundle  $J^2\mathcal{E}[1]$  admits the exact sequence at 2-jets,

$$0 \rightarrow S^2T^*M[1] \rightarrow J^2\mathcal{E}[1] \rightarrow J^1\mathcal{E}[1] \rightarrow 0, \quad (1.1)$$

where  $\mathcal{V}[w] := \mathcal{V} \otimes \mathcal{E}[w]$  for any vector bundle  $\mathcal{V}$  and any  $w \in \mathbb{R}$ .

The introduction of a conformal structure determines a canonical splitting of  $S^2T^*M[1]$  as  $S_0^2T^*M[1] \oplus \mathbf{g} \cdot \mathcal{E}[-1]$ , where  $\mathbf{g} \in \Gamma(S^2T^*M[2])$  is the conformal metric. The standard conformal cotractor bundle  $\mathcal{T}^*$  (or  $\mathcal{T}^*M$ ) is the quotient of  $J^2\mathcal{E}[1]$  by the image of  $S_0^2T^*M[1]$  and so has a filtration as given by the exact sequence

$$0 \rightarrow \mathcal{E}[-1] \xrightarrow{X} \mathcal{T}^* \rightarrow J^1\mathcal{E}[1] \rightarrow 0. \quad (1.2)$$

(In the case that  $M$  is of dimension one then  $S_0^2T^*M[1]$  is trivial and  $\mathcal{T}^* = J^2\mathcal{E}[1]$ .) There is canonically a conformally invariant metric  $h$  on the bundle  $\mathcal{T}^*$ , and hence  $\mathcal{T}^*$  is identified with its dual  $\mathcal{T}$ , which we call the *tractor bundle*. The bundle injection  $X$  which maps  $\mathcal{E}[-1] \rightarrow \mathcal{T}^*$  is typically viewed as a section  $X \in \Gamma(\mathcal{T}^*[1])$ , and called the *canonical tractor*. This invariantly encodes information about position on the manifold and plays a very important role in our developments here.

A well known feature of  $\mathcal{T}^*$  is that, in the case where  $\dim M \geq 3$ , it is naturally equipped with the canonical conformally invariant tractor connection [5], which is equivalent (see [17]) to the normal Cartan connection as in [24]. This preserves the tractor metric. Using this object and language we have the following result.

**Theorem 1.5.** *In a conformal manifold  $(M, \mathbf{c})$  of dimension at least 3, let  $\Sigma \hookrightarrow M$  be a connected submanifold of codimension  $d$ . Then  $\Sigma$  is distinguished if, and only if, either (equivalently both) of the following holds*

- *there exists a nontrivial  $\Psi \in \Gamma(\Lambda^d \mathcal{T}^*)$  such that  $X \lrcorner \Psi = 0$  and  $\nabla_i \Psi = 0$  along  $\Sigma$ , or*



- *there exists a nontrivial  $\star\Psi \in \Gamma(\Lambda^{n+2-d}\mathcal{T}^*)$  such that  $X \wedge \star\Psi = 0$  and  $\nabla_i \star\Psi = 0$  along  $\Sigma$ .*

Again this generalises a result for non-null conformal circles from [56]. If either of the above conditions hold, then  $\Psi$  is necessarily (up to constant factor) the tractor normal form of the submanifold. This will be proved in Section 7. Note that in the model case of the conformal sphere, viewed as the projectivised null cone of a Minkowski space  $\mathbb{M}$  of two higher dimensions, the distinguished submanifolds all arise from cutting the null cone with a subspace (and projectivising); in this case the tractor  $\star\Psi$  corresponding to a distinguished submanifold  $\Sigma$  is constant and simple (it is the wedge product of vectors that span the corresponding subspace of the Minkowski space) and, identifying the canonical tractor  $X$  with the position vector in  $\mathbb{M}$ , the condition  $X \wedge \star\Psi = 0$  is precisely the incidence relation saying that  $X$  is a point in the subspace corresponding to  $\star\Psi$ .

Theorem 1.5 is a useful result in that it allows us to immediately conclude that certain zero loci of normal solutions of appropriate BGG equations are distinguished submanifolds. Recall that on a Riemannian manifold, an alternating tensor  $k_{a_1 a_2 \dots a_d}$  is a conformal Killing form if the trace-free part of  $\nabla_c k_{a_1 a_2 \dots a_d}$  is completely alternating. For a suitable conformal weight, this condition is conformally invariant, see Section 6.3. We have the following consequence of Theorem 1.5, where the operator  $L$  is explained in (6.8) (and Theorem 6.3).

**Theorem 1.6.** *Suppose  $k_{a_1 \dots a_{d-1}}$  is a normal solution of the conformal Killing form equation on  $(M, \mathfrak{c})$  such that the parallel tractor  $L(k_{a_1 \dots a_{d-1}})$  is simple. Then the zero locus of*

$$\mathcal{K} := (k_{a_1 \dots a_{d-1}}, \nabla^c k_{ca_2 \dots a_{d-1}}), \text{ for any } g \in \mathfrak{c} \text{ with Levi-Civita connection } \nabla,$$

*is either empty, an isolated point, or a distinguished conformal submanifold of codimension  $d$ .*

In fact finer information is available, see Theorem 7.2. This is an analogue for normal Killing solutions of the results for almost Einstein scales found in [49, 51, 33]. Those results for Einstein scales (and their generalisations to so-called ASC scales in [51]) were key in the (earlier mentioned) development of a holographic approach to hypersurfaces via a singular Yamabe problem in [2, 8, 58, 59, 61], as well as a boundary calculus of asymptotically hyperbolic manifolds [57]. We believe the results in Theorem 7.2 should provide one of the key insights for the analogous treatment of submanifolds of higher codimension. Indeed, toward this end we provide a simple direct proof of a similar zero locus result, for fields satisfying weaker (than normal Killing-Yano) conditions, in Proposition 7.3.

## 2 Conventions and conformal geometry

Often we will use the standard abstract index notation of Penrose. For example we may write  $\mathcal{E}^a$  for the tangent bundle  $TM$  of a manifold  $M$  and  $v^a$  for a vector field on  $M$ . Similarly  $\mathcal{E}_a$  denotes the cotangent bundle  $T^*M$ , and  $\omega_a \in \Gamma(\mathcal{E}_a)$  a 1-form field. Then we write  $v^a\omega_a$  for the canonical pairing between vector fields and 1-forms. We denote by the Kronecker delta  $\delta^b_a$  the identity section of the bundle  $\text{End}(TM)$  of endomorphisms of  $TM$ . Indices enclosed by round (respectively by square) brackets indicate symmetrisation (respectively skew-symmetrisation) over the enclosed indices. For example, if  $T_{ab}$  is a rank 2 tensor then

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}) \quad \text{and} \quad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}).$$

We also use this notation for bundles. For example,  $\mathcal{E}_{[a_1 a_2 \dots a_d]}$  denotes the bundle of  $d$ -forms. When tractor bundles are introduced these will also be adorned with abstract indices when convenient, with the same convention for symmetrisation and skew-symmetrisation.

For simplicity of exposition we assume throughout that the basic manifold  $M$  studied is connected.

### 2.1 Conventions for Riemannian geometry

A Riemannian manifold is a pair  $(M^n, g)$ , consisting of a manifold  $M$  and a positive definite metric  $g$ . We assume that the dimension  $n$  (of  $M$ ) is at least 2. All structures are assumed smooth, meaning  $C^\infty$ . This is to simplify the discussion. For all the theory a much lower level of regularity is required, but this varies throughout and at any point is easily calculated by the reader. We will also typically assume for convenience that  $M$  is oriented, with volume form  $\epsilon_{a_1 a_2 \dots a_n}$  normalised by  $\epsilon^{a_1 a_2 \dots a_n} \epsilon_{a_1 a_2 \dots a_n} = n!$ , where indices are raised using the inverse of the metric  $g$ .

Writing  $\nabla$  for the Levi-Civita connection, the *Riemannian curvature tensor*  $R_{ab}{}^c{}_d$  is defined by

$$R_{ab}{}^c{}_d v^d = [\nabla_a, \nabla_b]v^c. \quad v^a \in \Gamma(\mathcal{E}^a), \quad (2.1)$$

in the abstract index notation.

In dimensions  $n \geq 3$ , this decomposes into trace-free and a trace part:

$$R_{abcd} = W_{abcd} + 2g_{c[a}P_{b]d} - 2g_{d[a}P_{b]c},$$

where  $W_{ab}{}^c{}_d$  is the *Weyl tensor* and  $P_{ab}$  is the *Schouten tensor*. Equivalently, the Schouten tensor is characterised by

$$R_{ab} = (n-2)P_{ab} + Jg_{ab}, \quad (2.2)$$

where  $R_{ab} := R_{ca}{}^c{}_b$  is the Ricci tensor, and  $J := g^{ab}P_{ab}$ . The Weyl tensor is totally trace free, and satisfies the algebraic Bianchi identities. In dimension 3 this implies that the Weyl tensor is zero.

The *Cotton* tensor (also for  $n \geq 3$ ) is defined by

$$C_{abc} := 2\nabla_{[a}P_{b]c}. \quad (2.3)$$

In dimension 2, it is easy to show that the Riemannian curvature is pure trace:

$$R_{abcd} = K(g_{ik}g_{jl} - g_{il}g_{jk}),$$

where  $K$  is the Gaußian curvature. Hence the Ricci tensor is also pure trace, and the Weyl curvature is zero.

Later, we will need to consider 1-dimensional submanifolds  $\Sigma$  equipped with a Riemannian metric. On a 1-dimensional manifold  $\Sigma$ , a Riemannian metric  $g_\Sigma$  takes the form  $u \otimes u$ , where  $u$  is a non-vanishing 1-form, and requiring  $u$  to be the volume form corresponding to  $g_\Sigma$  and the orientation fixes the sign of  $u$ . Thus there is a unique connection preserving this metric, namely the connection  $D$  that preserves  $u$ . We will term this the Levi-Civita connection for  $(\Sigma, g_\Sigma)$ . The curvature of any such connection is clearly zero.

## 2.2 Conformal geometry

Two metrics  $g, \hat{g}$  are said to be *conformally related* if

$$\hat{g} = \Omega^2 g, \quad (2.4)$$

where  $\Omega \in C^\infty(M)$  is a positive function. Then  $\mathbf{c}$  denotes an equivalence class of conformally related metrics, i.e. if  $g, \hat{g} \in \mathbf{c}$ , then they are related according to (2.4) for some smooth  $\Omega$ , and we may write  $\mathbf{c} = [g]$ . A conformal manifold is then a pair  $(M, \mathbf{c})$ .

Recall that on a manifold  $M$ , for any  $\alpha \in \mathbb{R}$ , one has the bundle of  $\alpha$ -densities. This is the associated bundle to the linear frame bundle of  $M$  via the 1-dimensional  $\mathrm{GL}(n)$ -representation  $A \mapsto |\det(A)|^{-\alpha}$ . Sections of this bundle are called  $\alpha$ -densities. There is a correspondence between 1-densities and sections of  $\Lambda^n T^*M$  when  $M$  is oriented, or in general between the square of these bundles, which we outline here. Let  $v_{a_1 a_2 \dots a_n} \in \Gamma(\mathcal{E}_{[a_1 a_2 \dots a_n]})$  be an  $n$ -form. Choosing a frame,  $v$  is represented by its components  $v_{a_1 a_2 \dots a_n}$ , where here the change of font indicates that the indices  $a_j$  should be understood as concrete indices. By contraction and summation with two copies of the Levi-Civita symbol, the given choice of frame then induces a map from  $S^2 \Lambda^n T^*M \rightarrow \mathbb{R}$ , and changing frame by some  $A \in \mathrm{GL}(n)$  rescales the result of this map by a factor of  $(\det(A))^2$ . Thus each  $v \in \Gamma(\Lambda^n T^*M)$  determines (via  $\otimes^2 v$  and then taking a square root) a map  $f$  from the frame bundle to  $\mathbb{R}$  which is equivariant with respect to the  $\mathrm{GL}(n)$ -action, explicitly

$$f(A \cdot u) = |\det(A)| f(u).$$

Thus by the usual correspondence between sections of associated bundles and equivariant functions, we have constructed from  $v$  a 1-density.

Separately, we have the bundle of conformal densities of weight  $w$ , which we denote by  $\mathcal{E}[w]$ , and which are defined by

$$\mathcal{E}[w] := \mathcal{Q} \times_{\rho_w} \mathbb{R}, \quad (2.5)$$

where  $\mathcal{Q}$  is the ray bundle of conformally related metrics and  $\rho_w$  is the 1-dimensional  $\mathbb{R}_+$ -representation  $\rho_w(s)(t) := s^{-\frac{w}{2}}t$ . The bundles  $\mathcal{E}[w]$  are evidently oriented and we write  $\mathcal{E}_+[w]$  for the ray suundle of positive elements. As detailed in, e.g., [18], the conformal densities of weight  $w$  are in bijective correspondence with densities of weight  $(-\frac{w}{n})$ . In particular, this means that 1-densities also correspond to conformal densities of weight  $-n$ , and so together with the discussion of the previous paragraph we have an isomorphism  $(\Lambda^n T^*M)^2 \xrightarrow{\cong} \mathcal{E}[-2n]$ , and dually  $(\Lambda^n TM)^2 \xrightarrow{\cong} \mathcal{E}[2n]$ . If  $\mathcal{B}$  is a vector bundle on  $M$  we will write  $\mathcal{B}[w]$  as a shorthand for  $\mathcal{B} \otimes [w]$ .

If  $M$  is oriented, as we henceforth assume, we write  $\epsilon = \epsilon_{a_1 a_2 \dots a_n} \in \Gamma(\mathcal{E}_{[a_1 a_2 \dots a_n]}[n])$  for the canonical map  $\Lambda^n TM \rightarrow \mathcal{E}[n]$ , given by contraction, and call  $\epsilon_{a_1 a_2 \dots a_n}$  the *conformal volume form* or *weighted volume form*. Since  $\mathcal{E}[w]$  is an associated bundle, its sections may be thought of as equivariant functions  $f : \mathcal{Q} \rightarrow \mathbb{R}$  such that  $f(s^2 g_x) = s^w f(g_x)$ . So we may think of a section of  $\mathcal{E}[w]$  as an equivalence class of pairs  $(g, f)$ , where  $(g, f) \sim (\Omega^2 g, \Omega^w f)$ . The conformal volume form can therefore similarly be thought of as the equivalence class of  $(g, \epsilon)$  for any  $g \in \mathfrak{c}$ , where  $\epsilon$  is the Riemannian volume form of  $g$  and  $(g, \epsilon) \sim (\Omega^2 g, \Omega^n \epsilon)$ .

Corresponding to  $g \in \mathfrak{c}$  there is evidently a corresponding section  $\sigma_g \in \Gamma(\mathcal{E}[1])$ , represented by the pair  $(g, 1)$ . It follows that the conformal structure  $\mathfrak{c}$  determines a tautological section  $\mathbf{g} \in \Gamma(S^2 T^*M \otimes \mathcal{E}_+[2])$  that is given by

$$\mathbf{g} = (\sigma_g)^2 g, \quad (2.6)$$

for any metric  $g \in \mathfrak{c}$  (but which is independent of this choice); equivalently, the tautological section  $\mathbf{g} \in \Gamma(S^2 T^*M \otimes \mathcal{E}_+[2])$  may be thought of as the equivalence class of  $(g, g)$  for any metric  $g \in \mathfrak{c}$ , where  $(g, g) \sim (\Omega^2 g, \Omega^2 g)$ . This is called the *conformal metric*. We will henceforth typically use the conformal metric to raise and lower indices, even when a choice of  $g \in \mathfrak{c}$  has been made (thus raising and lowering indices typically introduces a density bundle weight). For example the Riemann curvature with indices all down  $R_{abcd}$  will now be considered to have weight 2 as it is

$$R_{abcd} = \mathbf{g}_{ec} R_{ab}{}^e{}_d,$$

the Weyl-Schouten decomposition of the Riemann curvature becomes

$$R_{abcd} = W_{abcd} + 2\mathbf{g}_{c[a} P_{b]d} - 2\mathbf{g}_{d[a} P_{b]c},$$

for dimensions  $n \geq 3$ , and  $J$  will mean  $\mathbf{g}^{ab} P_{ab}$ . From (2.6) we see that if we use  $\sigma_g$  to trivialise density bundles, then the conformal metric  $\mathbf{g}$  becomes  $g$ . However,

usually we will avoid trivialising density bundles. This becomes significant when we write down conformal rescaling laws, since then there are two different metrics that could be used to trivialize the density bundles (and correspondingly two different flat connections on sections of density bundles) and many formulae are simplified when we work with weighted objects.

Each metric  $g \in \mathbf{c}$  determines a corresponding Levi-Civita connection  $\nabla$ . This naturally acts on sections of density bundles and, tautologically from the construction above, preserves  $\sigma_g$ . Thus as well as preserving  $g$ , the Levi-Civita connection  $\nabla$  preserves  $\mathbf{g}$  and  $\epsilon$  (cf. [33]). Under a change to  $\hat{g} = \Omega^2 g \in \mathbf{c}$  we have

$$\hat{\nabla}_a u^b = \nabla_a u^b + \Upsilon_a u^b - \Upsilon^b u_a + \Upsilon_c u^c \delta_a^b, \quad \text{on } u^b \in \Gamma(\mathcal{E}^b), \quad (2.7)$$

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b - \Upsilon_a \omega_b - \Upsilon_b \omega_a + \Upsilon^c \omega_c g_{ab}, \quad \text{on } \omega_b \in \Gamma(\mathcal{E}_b), \quad (2.8)$$

and

$$\hat{\nabla}_a \tau = \nabla_a \tau + w \Upsilon_a \tau \quad \text{on } \tau \in \Gamma(\mathcal{E}[w]), \quad (2.9)$$

where  $\Upsilon_a := \Omega^{-1} \nabla_a \Omega$ .

The Weyl curvature  $W_{ab}{}^c{}_d$  is conformally invariant, while the Schouten tensor transforms according to,

$$\hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} \Upsilon_c \Upsilon^c g_{ab}. \quad (2.10)$$

Equations (2.7) and (2.8) still hold when  $M$  has dimension 1, although the final two terms of both equations cancel. Equation (2.10) only holds when  $\dim M \geq 3$ , since in lower dimensions the Schouten tensor is not defined.

## 2.3 The tractor connection and calculus

Recall from the introduction, the tractor bundle is recovered from jets of the conformal density bundle  $\mathcal{E}[1]$ . The inverse of the conformal metric maps  $S^2 T^* M[1] \rightarrow \mathcal{E}[-1]$ , with kernel  $S_0^2 T^* M[1]$  and hence we have a decomposition

$$S^2 T^* M[1] = S_0^2 T^* M[1] \oplus \mathbf{g} \cdot \mathcal{E}[-1].$$

Then the bundle  $\mathcal{T}^*$  (that we also denote  $\mathcal{E}_A$  in the abstract index notation) is  $J^2 \mathcal{E}[1]$  modulo the image of  $S_0^2 T^* M[1]$  under the map  $S^2 T^* M[1] \rightarrow J^2 \mathcal{E}[1]$  of the jet exact sequence at 2-jets (1.1). Thus we obtain (1.2). From the jet exact sequence at 1-jets

$$0 \rightarrow T^* M[1] \rightarrow J^1 \mathcal{E}[1] \rightarrow \mathcal{E}[1] \rightarrow 0, \quad (2.11)$$

we then see that  $\mathcal{T}^*$  has the composition series

$$\mathcal{T}^* = \mathcal{E}[1] \oplus T^* M[1] \oplus \mathcal{E}[-1].$$

Here the semidirect sum notation  $\oplus$  simply encodes the information of the exact sequences (cf. [5]). Note that this construction still applies when  $M$  has dimension 1 or 2, but in dimension 1,  $S_0^2 T^* M[1]$  is trivial and hence  $\mathcal{T}^*$  is simply  $J^2 \mathcal{E}[1]$ . Recall that we denote by  $X_A \in \Gamma(\mathcal{E}_A[1])$  the canonical tractor which provides the embedding  $\mathcal{E}[-1] \rightarrow \mathcal{E}_A$ . Let us also note that by the definition of the tractor bundle, there is an invariant differential operator  $\mathbb{D} : \Gamma(\mathcal{E}[1]) \rightarrow \Gamma(\mathcal{T}^*)$ , where  $\frac{1}{n} \mathbb{D}$  is the differential operator corresponding to the linear map  $J^2 \mathcal{E}[1] \rightarrow \mathcal{T}^*$ .

Let us now fix  $n \geq 3$ . Given a choice of metric  $g \in \mathfrak{c}$ , the formula

$$\sigma \mapsto \frac{1}{n} [\mathbb{D}_A \sigma]_g := \begin{pmatrix} \sigma \\ \nabla_a \sigma \\ -\frac{1}{n} (\Delta + J) \sigma \end{pmatrix}, \quad (2.12)$$

where  $\Delta = \nabla^a \nabla_a$ , gives a second-order differential operator on  $\mathcal{E}[1]$  which is a linear map  $J^2 \mathcal{E}[1] \rightarrow \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$  that clearly factors through  $\mathcal{T}^*$  and so determines an isomorphism

$$\mathcal{T}^* \xrightarrow{\sim} [\mathcal{T}^*]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1], \quad (2.13)$$

and hence the sequences (1.2) and (2.11) split, as discussed in, e.g., [16, 33]. When using a choice of metric  $g$  to split the tractor bundle we will typically indicate this by writing  $\frac{g}{\sim}$  rather than applying the bracket notation  $[\cdot]_g$  to the object we are breaking up into slots.

In the subsequent discussions, we will use (2.13) to split the tractor bundles without further comment. Thus, given  $g \in \mathfrak{c}$ , an element  $V_A$  of  $\mathcal{E}_A$  may be represented by a triple  $(\sigma, \mu_a, \rho)$ , or equivalently by

$$V_A = \sigma Y_A + \mu_a Z_A^a + \rho X_A. \quad (2.14)$$

The last display defines the algebraic splitting operators  $Y : \mathcal{E}[1] \rightarrow \mathcal{T}^*$  and  $Z : T^* M[1] \rightarrow \mathcal{T}^*$  (determined by the choice  $g \in \mathfrak{c}$ ) which may be viewed as sections  $Y_A \in \Gamma(\mathcal{E}_A[-1])$  and  $Z_A^a \in \Gamma(\mathcal{E}_A^a[-1])$ . We call these sections  $X_A, Y_A$  and  $Z_A^a$  *tractor projectors*. Note that with this convention, (2.12) is, tautologically, an explicit formula for the invariant operator  $\mathbb{D}$ , in terms of the splitting given by the choice of metric  $g$ .

While  $X_A$  is conformally invariant, a change of tractor splitting given by (2.4) determines the transformations

$$\hat{Z}_A^a = Z_A^a + \Upsilon^a X_A, \quad \hat{Y}_A = Y_A - \Upsilon_a Z_A^a - \frac{1}{2} \Upsilon^a \Upsilon_a X_A \quad (2.15)$$

where, as usual,  $\Upsilon_a = \Omega^{-1} \nabla_a \Omega$ . These transformations mean that the tractor triples transform by

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_a \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^b & \delta_a^b & 0 \\ -\frac{1}{2} \Upsilon^c \Upsilon_c & -\Upsilon_a & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix}. \quad (2.16)$$

One then observes that the symmetric tractor field given by

$$h^{AB} := 2X^{(A}Y^{B)} + g^{ab}Z_a^AZ_b^B \quad (2.17)$$

is invariant under (2.15), and so determines a conformally invariant metric on  $\mathcal{T}^*$ . We will hence use this and its inverse  $h_{AB}$  (called the *tractor metric*) to identify  $\mathcal{T}^*$  and its dual, the *standard tractor bundle*, which we denote by simply  $\mathcal{T}$ . Using this we obtain

$$X^AY_A = 1, \quad Z_A^aZ_b^A = \delta_b^a, \quad (2.18)$$

and all other (tractor-index) pairings of the splitting operators give a zero section. For example  $X^AX_A = 0$ .

The canonical conformally invariant (normal) tractor connection on  $\mathcal{T}$  will also be denoted  $\nabla_a$ , or sometimes  $\nabla_a^{\mathcal{T}}$  for emphasis. It can be coupled to the Levi-Civita connection of any metric  $g \in \mathfrak{c}$ , and its action on the tractor projectors is then given by

$$\nabla_a X^A = Z_a^A, \quad \nabla_a Z_b^A = -P_{ab}X^A - g_{ab}Y^A, \quad \nabla_a Y^A = P_a^b Z_b^A. \quad (2.19)$$

In fact, these formulae determine the tractor connection as the general action on a section of a tractor bundle follows from the Leibniz rule. It is easily verified that the tractor connection is conformally invariant and preserves the tractor metric. The latter means that the tractor connection agrees with its dual. It extends in the obvious way to tensor powers of the tractor bundle and these extensions are all referred to as the tractor connection. The coupled tractor-Levi Civita connection will always be denoted simply  $\nabla$  and will be used, usually without comment, according to context.

As with any linear connection,  $\nabla = \nabla^{\mathcal{T}}$  has a curvature. The *tractor curvature*  $\Omega_{ab}^C{}_D$  of the tractor connection is defined by  $\Omega_{ab}^C{}_D \Phi^D := 2\nabla_{[a}\nabla_{b]}\Phi^C$ , for any  $\Phi^A \in \Gamma(\mathcal{T})$ . In the splitting determined by a choice of metric  $g \in \mathfrak{c}$  it is given explicitly by the formula

$$\Omega_{abCD} = W_{abcd}Z_C^cZ_D^d - 2C_{abc}X_{[C}Z_{D]}^c. \quad (2.20)$$

A conformal structure is said to be (*locally*) *flat* if this tractor curvature vanishes as this happens if and only if, locally, there is a metric in the conformal class that is flat.

The tractor objects developed above form the initial objects of a conformal tractor calculus that can be used, for example, to construct conformal invariants [47, 48]. We will not discuss this in detail, but one particularly important object is the *Thomas operator*  $\mathbb{D}$  that extends (2.12) to a conformally invariant operator between weighted tractor bundles,

$$\mathbb{D}_A : \Gamma(\mathcal{E}^\Phi[w]) \rightarrow \Gamma(\mathcal{E}_A \otimes \mathcal{E}^\Phi[w-1]),$$

where  $\mathcal{E}^\Phi$  indicates any tensor power of  $\mathcal{E}^A$ , or  $SO(h)$ -invariant part thereof. It is given, with respect to  $g \in \mathfrak{c}$ , by the formula

$$\Gamma(\mathcal{E}^\Phi[w]) \in V \mapsto \mathbb{D}_A V \stackrel{g}{=} \begin{pmatrix} (n+2w-2)wV \\ (n+2w-2)\nabla_a V \\ -(\Delta V + wJV) \end{pmatrix}. \quad (2.21)$$

where (as usual)  $\nabla$  is the coupled tractor-Levi-Civita connection and  $\Delta$  the corresponding Laplacian.

All of the above has a clear geometric interpretation in the case of the model, the conformal  $n$ -sphere. This should be thought of as the ray projectivisation of  $\mathcal{C}_+$ , where  $\mathcal{C}_+$  is the future directed part of the null quadric  $\mathcal{C} := \{X \in \mathbb{R}^{n+2} \mid h(X, X) = 0\}$  in  $\mathbb{R}^{n+2}$  equipped with a fixed symmetric non-degenerate bilinear form  $h$  of signature  $(n+1, 1)$  and a time-orientation. The resulting manifold  $M := \mathbb{P}_+(\mathcal{C}_+) \cong S^n$  is acted on transitively by  $G := SO_0(h) \cong SO_0(n+1, 1)$ , where the 0 here denotes taking the connected component of the identity, and the stabiliser of a point is a parabolic subgroup that we denote  $P$  (so  $M \cong G/P$ ). Moreover, it is straightforward to verify that  $h$  induces a Riemannian metric on each section of the map  $\mathcal{C}_+ \rightarrow M$ , and different sections result in conformally related metrics. Thus  $M$  is equipped canonically with a conformal structure, and clearly the group  $G$  acts on  $M$  by conformal isometries, see, e.g., [33, 51] for a more detailed discussion of this model.

From this point of view the standard tractor bundle for the model is  $T\mathbb{R}^{n+2}|_{\mathcal{C}_+}/\sim$ , where the equivalence relation is  $U_p \sim V_q$  if one is mapped to the other by standard  $\mathbb{R}^{n+2}$  parallel transport (i.e., from the affine structure of  $\mathbb{R}^{n+2}$ ) along a null ray. The tractor metric is then induced in an obvious way from the ambient Minkowski signature metric  $h$ , and parallel tractor fields are equivalent to vector fields in  $\Gamma(T\mathbb{R}^{n+2}|_{\mathcal{C}_+})$  that are constant along  $\mathcal{C}_+$ . Moreover the parallel tensor fields on connected regions of  $\mathcal{C}_+$  may all be viewed as arising from the restriction of tensor fields parallel on  $\mathbb{R}^{n+1,1}$ , and these give the parallel sections of the corresponding tensor powers of the tractor bundle. Finally, in this picture, the canonical tractor  $X^A$  is identified with the Euler vector field of  $\mathbb{R}^{n+2}$  along  $\mathcal{C}_+$ .

Although we will not make use of this below, for completeness we note that the Thomas operator  $\mathbb{D}_A$  also has a concrete geometric interpretation in the model. Sections of the weight  $w$  conformal density bundle on the model can be identified with functions on  $\mathcal{C}_+$  that are homogeneous of degree  $w$  with respect to the  $\mathbb{R}_+$ -action. Weighted tractors on the model can therefore be identified with tensor fields along  $\mathcal{C}_+$  of the appropriate homogeneity. The Thomas operator  $\mathbb{D}_A$  on such sections is then given (up to an overall factor) by formally extending such tensor fields off  $\mathcal{C}_+$  to be ‘‘harmonic’’ with respect to the ambient Minkowski metric  $h$  and then taking the directional derivative (at points along  $\mathcal{C}_+$ ) in the flat ambient space  $\mathbb{R}^{n+2}$ ; see, [18].



## 2.4 The scale tractor

Recall that, from Section 2.2, a metric  $g \in \mathbf{c}$  is equivalent to a section  $\sigma_g \in \Gamma(\mathcal{E}_+[1])$  by the relation

$$g = \sigma_g^{-2} \mathbf{g}.$$

Given any section  $\sigma \in \Gamma(\mathcal{E}[1])$  we can form

$$I_A := \frac{1}{n} \mathbb{D}_A \sigma,$$

and we will term this a *scale tractor* if  $I_A$  is nowhere zero. In this case  $\sigma$  is clearly non-vanishing on an open dense subset of  $M$ , on which it determines a metric  $g := \sigma^{-2} \mathbf{g}$  from the conformal class. So for a scale tractor  $I_A$  we will term  $\sigma = X^A I_A$  a *generalised scale* – or sometimes simply a *scale*. Following [51], a conformal manifold  $(M, \mathbf{c})$  equipped with a *scale tractor* will said to be an *almost-Riemannian* manifold (since it has a metric almost everywhere). Given a Riemannian metric  $g = \sigma_g^{-2} \mathbf{g}$ , we term  $I_A := \frac{1}{n} \mathbb{D}_A \sigma_g$  the scale tractor of  $g$ .

It follows easily from (2.19) that if a tractor  $I_A \neq 0$  is parallel then it is a scale tractor and  $g := \sigma^{-2} \mathbf{g}$  is Einstein; see [53, 51]. In this case we say  $(M, \mathbf{c}, I)$  is *almost Einstein*.

An important example of almost-Riemannian manifolds arise in connection with conformally compact manifolds. A complete Riemannian manifold  $(M, g)$  is conformally compact if  $M$  is the interior of a manifold with boundary  $\overline{M}$ , and on  $\overline{M}$  there is a metric  $\overline{g}$  (so a metric that is smooth up to the boundary) such that on  $M$

$$g_{ab} = r^{-2} \overline{g}_{ab}$$

for some smooth defining function  $r$  for the boundary  $\partial M$  (meaning that  $r > 0$  on  $M$ ,  $\partial M$  is the zero locus of  $r$ , and  $dr$  is nowhere zero on  $\partial M$ ). A conformally compact manifold is said to *asymptotically hyperbolic* if  $|dr|_{\overline{g}} = 1$  along  $\partial M$  (which is equivalent to requiring that the sectional curvatures of  $g$  all tend to  $-1$  as one approaches  $\partial M$ ) and *Poincaré-Einstein* if  $g$  is Einstein. It is easily verified that in the latter case the scalar curvature is negative. If the Poincaré-Einstein metric is normalised so that

$$\text{Sc}^g = -n(n-1)$$

(as is usually assumed) then the manifold is necessarily asymptotically hyperbolic. These structures have been the subject of sustained interest; see, e.g., [29, 30, 31, 65, 69, 79, 97] and the many references therein.

It is easily verified that a conformal compactification of a manifold  $(M, g)$  is the same as a conformal manifold with boundary  $(\overline{M}, \mathbf{c})$ , with interior  $M$ , and equipped with a scale tractor  $I_A$  with the following properties: the zero locus  $Z(\sigma) = \sigma^{-1}(0)$  of  $\sigma := X^A I_A$  is  $\partial M$ , and along  $\partial M$  the 1-jet  $j^1 \sigma$  (of  $\sigma$ ), is nowhere zero (we say that  $\sigma$  is a *defining density* for  $\partial M$ ). Thus the conformal compactification is

almost-Riemannian; in the following we will therefore think of a conformally compact manifold as an almost-Riemannian manifold for which  $\sigma$  is a defining density for  $\partial M$ . Such a manifold is asymptotically hyperbolic if  $I^A I_A = 1$  along  $\partial M$ , and Poincaré-Einstein if  $I_A$  is parallel. If  $I^A I_A = 1$  on  $\overline{M}$  then  $\text{Sc}^g = -n(n-1)$ . See, e.g., [51, 61] for more details.

There are many structures such as certain notions of asymptotically flat manifolds that can be similarly be understood in terms of almost-Riemannian structures. So this notion provides a uniform framework for approaching a range of singular geometries [33].

## 2.5 Form tractors

We will use the term *form tractor* to describe sections of the exterior powers of the tractor bundle [10, 54]. It is useful to introduce some notation for form tractors. From the composition series for the standard tractor bundle, one sees that for the  $k$ -th exterior power of the standard tractor bundle, one has the composition series

$$\mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]} = \mathcal{E}_{[a_2 \cdots a_k]}[k] \oplus \begin{array}{c} \mathcal{E}_{[a_1 a_2 \cdots a_{k-1} a_k]}[k] \\ \oplus \\ \mathcal{E}_{[a_3 \cdots a_k]}[k-2] \end{array} \oplus \mathcal{E}_{[a_2 \cdots a_k]}[k-2]. \quad (2.22)$$

The tractor projectors for the standard tractor bundle induce tractor projectors on the bundles of tractor forms. Since these will be very important for us, we introduce dedicated notation for these:

$$\begin{aligned} \mathbb{Y}_{A_1 A_2 \cdots A_{k-1} A_k}^{a_2 \cdots a_{k-1} a_k} &:= Y_{[A_1} Z_{A_2}^{a_2} \cdots Z_{A_{k-1}}^{a_{k-1}} Z_{A_k}^{a_k} && \in \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]}^{a_2 \cdots a_{k-1} a_k}[-k] \\ \mathbb{Z}_{A_1 A_2 \cdots A_{k-1} A_k}^{a_1 a_2 \cdots a_{k-1} a_k} &:= Z_{[A_1}^{a_1} Z_{A_2}^{a_2} \cdots Z_{A_{k-1}}^{a_{k-1}} Z_{A_k}^{a_k} && \in \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]}^{a_1 a_2 \cdots a_{k-1} a_k}[-k] \\ \mathbb{W}_{A_1 A_2 A_3 \cdots A_{k-1} A_k}^{a_3 \cdots a_{k-1} a_k} &:= X_{[A_1} Y_{A_2} Z_{A_3}^{a_3} \cdots Z_{A_{k-1}}^{a_{k-1}} Z_{A_k}^{a_k} && \in \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]}^{a_3 \cdots a_{k-1} a_k}[-k+2] \\ \mathbb{X}_{A_1 A_2 \cdots A_{k-1} A_k}^{a_2 \cdots a_{k-1} a_k} &:= X_{[A_1} Z_{A_2}^{a_2} \cdots Z_{A_{k-1}}^{a_{k-1}} Z_{A_k}^{a_k} && \in \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]}^{a_2 \cdots a_{k-1} a_k}[-k+2] \end{aligned} \quad (2.23)$$

For example,  $\mathbb{Y}_{A_1 A_2 \cdots A_{k-1} A_k}^{a_2 \cdots a_{k-1} a_k}$  gives the injection

$$\mathbb{Y}_{A_1 A_2 \cdots A_{k-1} A_k}^{a_2 \cdots a_{k-1} a_k} : \mathcal{E}_{[a_2 \cdots a_k]}[k] \rightarrow \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]},$$

determined by a choice of metric  $g \in \mathfrak{c}$ . Similarly  $\mathbb{X}_{A_1 A_2 \cdots A_{k-1} A_k}^{a_2 \cdots a_{k-1} a_k}$

$$\mathbb{X}_{A_1 A_2 \cdots A_{k-1} A_k}^{a_2 \cdots a_{k-1} a_k} : \mathcal{E}_{[a_2 \cdots a_k]}[k-2] \rightarrow \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]}, \quad (2.24)$$

but in this case the map is not dependent on any choice of metric in  $\mathfrak{c}$ . For  $1 \leq k \leq$

$n + 2$  and a choice of scale, one has

$$\begin{aligned}
\nabla_b \mathbb{Y}_{A_1 A_2 A_3 \dots A_k}^{a_2 a_3 \dots a_k} &= P_{ba_1} \mathbb{Z}_{A_1 A_2 A_3 \dots A_k}^{a_1 a_2 a_3 \dots a_k} + (k - 1) P_b^{a_2} \mathbb{W}_{A_1 A_2 A_3 \dots A_k}^{a_3 \dots a_k} \\
\nabla_b \mathbb{Z}_{A_1 A_2 \dots A_k}^{a_1 a_2 \dots a_k} &= -k \cdot P_b^{a_1} \mathbb{X}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k} - k \cdot \delta_b^{a_1} \mathbb{Y}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k} \\
\nabla_b \mathbb{W}_{A_1 A_2 A_3 \dots A_k}^{a_3 \dots a_k} &= -g_{ba_2} \mathbb{Y}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k} + P_{ba_2} \mathbb{X}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k} \\
\nabla_b \mathbb{X}_{A_1 A_2 A_3 \dots A_k}^{a_2 a_3 \dots a_k} &= g_{ba_1} \mathbb{Z}_{A_1 A_2 A_3 \dots A_k}^{a_1 a_2 a_3 \dots a_k} - (k - 1) \delta_b^{a_2} \mathbb{W}_{A_1 A_2 A_3 \dots A_k}^{a_3 \dots a_k},
\end{aligned} \tag{2.25}$$

where sequentially labeled indices are alternating, and any term involving the alternation of  $n + 1$  or more tensor (i.e. lower case) indices should be interpreted as zero.

In particular, we observe that  $\mathbb{W}_{A_1 A_2 A_3 \dots A_{n+2}}^{a_3 \dots a_{n+2}}$  is parallel in any scale, and hence there is a distinguished parallel section of the top exterior power of the standard tractor bundle, which we term the *tractor volume form*

$$\epsilon_{A_1 A_2 A_3 \dots A_{n+2}} := (n + 2)(n + 1) \epsilon_{a_3 \dots a_{n+2}} \mathbb{W}_{A_1 A_2 A_3 \dots A_{n+2}}^{a_3 \dots a_{n+2}}, \tag{2.26}$$

where  $\epsilon_{a_3 \dots a_{n+2}} \in \mathcal{E}_{[a_3 \dots a_{n+2}]}[n]$  is the weighted volume form of Section 2.2 (note that our normalisation is such that  $\epsilon^{A_1 A_2 A_3 \dots A_{n+2}} \epsilon_{A_1 A_2 A_3 \dots A_{n+2}} = -(n + 2)!$ ). That this is parallel now follows from the fact that  $\epsilon_{a_3 \dots a_{n+2}}$  is parallel for any Levi-Civita connection in the conformal class. Of course, the existence of the tractor volume form also reflects the fact that the conformal tractor connection is equivalent to an  $\text{SO}(n + 1, 1)$ -Cartan connection.

Finally in this section we need the tractor Hodge-star. For a tractor  $k$ -form  $\Psi_{A_1 \dots A_k}$  this is

$$\star \Psi_{B_1 \dots B_{n+2-k}} = \frac{1}{k!} \epsilon^{A_1 \dots A_k}{}_{B_1 \dots B_{n+2-k}} \Psi_{A_1 \dots A_k}. \tag{2.27}$$

This satisfies  $\star \star = -(-1)^{k(n-k)}$ , since the tractor metric has Lorentzian signature. Note also that this tractor Hodge-star operation commutes with the tractor covariant derivative:

$$\nabla_a \star \Psi_{B_1 \dots B_{n+2-k}} := \frac{1}{k!} \epsilon^{A_1 \dots A_k}{}_{B_1 \dots B_{n+2-k}} \nabla_a \Psi_{A_1 \dots A_k}.$$

### 3 Submanifold geometry and submanifold tractors

Given a smooth  $n$ -manifold  $M$ , a *submanifold* will mean a smooth embedding  $\iota : \Sigma \rightarrow M$  of a smooth  $m$ -dimensional manifold  $\Sigma$ , where  $1 \leq m \leq n - 1$ , and the image has *codimension*  $d := n - m$ . Typically we will suppress explicit mention of the embedding map and identify  $\Sigma$  with its image  $\iota(\Sigma) \subset M$ . We refer to  $M$  as the *ambient* manifold.

Regarding abstract indices, we adopt the convention that Latin letters from the start of the alphabet ( $a, b, c, \dots$ ) will denote ambient tensor indices, while indices from later in the alphabet ( $i, j, k, \dots$ ) will denote submanifold tensor indices. So, for example,  $\mathcal{E}^a$  is the usual tangent bundle  $TM$ ,  $\mathcal{E}^i$  is the tangent bundle of the submanifold  $T\Sigma$ , and  $\mathcal{E}_i^a$  denotes the bundle  $T^*\Sigma \otimes TM|_\Sigma$ . Note that indices alone will not distinguish sections of  $TM$  and  $TM|_\Sigma$ , so  $v^a$  could be a section of either  $\mathcal{E}^a$  or a section of  $\mathcal{E}^a|_\Sigma$ , where  $\mathcal{E}^a|_\Sigma \rightarrow \Sigma$  is the pullback bundle  $\iota^*TM$ .

Given a submanifold  $\iota : \Sigma \rightarrow M$ , its derivative  $T\iota : T\Sigma \rightarrow TM$  will be written  $\Pi_i^a$  and viewed as a section of  $T^*\Sigma \otimes TM|_\Sigma$ . We frequently identify  $T\Sigma$  with its image in  $TM|_\Sigma$  under this map. Note that  $\Pi_i^a$  also gives the canonical map  $\Pi_a^i : \mathcal{E}_a|_\Sigma \rightarrow \mathcal{E}_i$ , which is dual to  $T\iota$ . We will temporarily use the notation  $T_{M/\Sigma}$  for the normal bundle  $TM|_\Sigma/T\iota(T\Sigma)$ , and  $(T_{M/\Sigma})^* \subset T^*M|_\Sigma$  for the conormal bundle.

#### 3.1 Basic Riemannian submanifold theory

We now move to the setting of a submanifold  $\Sigma$  in a Riemannian manifold  $(M, g)$  (cf. discussions in, e.g., [78, 87]). In the Riemannian setting, we only require that  $\dim M \geq 2$ , and  $\Sigma$  satisfies  $1 \leq m = \dim \Sigma \leq n - 1$ . The exact sequence defining the normal bundle  $T_{M/\Sigma}$  then splits

$$0 \longrightarrow \mathcal{E}^i \xrightarrow{\Pi_i^a} \mathcal{E}^a|_\Sigma \longrightarrow T_{M/\Sigma} \longrightarrow 0, \quad (3.1)$$

$\longleftarrow \underbrace{\hspace{1.5cm}}_{\Pi_a^i} \longrightarrow$

where  $\Pi_a^i$  is the orthogonal projection map  $TM|_\Sigma \rightarrow T\Sigma$ . We may then identify  $T_{M/\Sigma}$  with the kernel  $N\Sigma$  of  $\Pi_a^i$ , via the splitting, and we denote the orthogonal projection onto this by  $N_a^b : \mathcal{E}^a|_\Sigma \rightarrow N\Sigma^b$ . The complementary projection is  $\Pi_b^a = \delta_b^a - N_a^b = \Pi_i^a \Pi_b^i$  which is the orthogonal projection onto  $T\Sigma$  viewed as submanifold of  $TM|_\Sigma$ .

The Riemannian metric  $g$  on  $M$  induces a Riemannian metric  $g_\Sigma$  on  $\Sigma$  by restriction, which we call the *induced metric*. We usually omit the explicit reference to  $\Sigma$  when abstract indices are used. So the induced metric will be denoted by  $g_{ij}$ . Note that

$$g_{ij} = \Pi_i^a \Pi_j^b g_{ab}. \quad (3.2)$$

Next we observe that (3.1) can be used to decompose the ambient Levi-Civita connection. First and most simply, we have the *normal connection*  $\nabla^\perp$  which is a connection on the bundle  $N\Sigma \rightarrow \Sigma$  defined by

$$\nabla_i^\perp \nu^a := N_b^a \nabla_i \nu^b, \quad (3.3)$$

where  $\nabla_i$  denotes the pullback connection of the ambient Levi-Civita connection (meaning, in this context, its restriction to differentiating along vectors tangent to  $\Sigma$ ). Complementary to this, we also have induced a connection  $D_i$  on  $T\Sigma \rightarrow \Sigma$  defined by

$$D_i V^j := \Pi_b^j \nabla_i \left( \Pi_k^b V^k \right). \quad (3.4)$$

It is elementary to verify that both (3.3) and (3.4) define connections. Indeed, it is also straightforward to verify that  $D$  is torsion-free and preserves the induced metric, and so is in fact the Levi-Civita connection of  $(\Sigma, g_\Sigma)$ . The fundamental ingredient of submanifold calculus, in this setting, is the *Gauß formula* which, for a section  $V \in \Gamma(T\Sigma|_\Sigma)$ , provides the decomposition of  $\nabla_i V^c = \nabla_i(\Pi_j^c V^j)$  into its tangential and normal parts:

$$\nabla_i V^c = \Pi_j^c D_i V^j + II_{ij}{}^c V^j, \quad (3.5)$$

and this defines  $II_{ij}{}^c \in \Gamma(S^2 T^* \Sigma \otimes N\Sigma)$ , which is the *second fundamental form* of  $\Sigma$  in  $(M, g)$ . We also define the *mean curvature*

$$H^c := \frac{1}{m} g^{ij} II_{ij}{}^c \quad (3.6)$$

and set  $\mathring{II}_{ij}{}^c := II_{(ij)_0}{}^c$ , the trace-free part of the second fundamental form. Thus one has

$$II_{ij}{}^c = \mathring{II}_{ij}{}^c + g_{ij} H^c. \quad (3.7)$$

Using (3.5), one can derive

$$R_{ijkl} = R_{ijkl}^\Sigma + 2g_{cd} II_{l[i}{}^c II_{j]k}{}^d, \quad (3.8)$$

$$R_{ij}{}^c{}_k N_c^d = 2D_{[i} II_{j]k}{}^d, \quad (3.9)$$

and

$$R_{ij}{}^a{}_b N_a^c N_c^d = R^\perp{}_{ij}{}^c{}_d + 2g^{kl} II_{l[i}{}^c II_{j]kd}, \quad (3.10)$$

where  $R_{ijkl} := \Pi_i^a \Pi_j^b \Pi_k^c \Pi_l^d R_{abcd}$  is the curvature of the ambient Levi-Civita connection restricted to  $\Sigma$ ,  $R_{ijkl}^\Sigma$  is the intrinsic Riemann curvature tensor (i.e. the curvature of the connection  $D$ ),  $D$  is the intrinsic Levi-Civita connection coupled to the normal connection and  $R^\perp{}_{ij}{}^c{}_d$  is the curvature of the normal connection  $\nabla_i^\perp$ . All these formulae are derived by substituting the Gauß formula (3.5) into equation (2.1) which defines the curvature of the pullback connection  $\nabla_i$ , as follows.

Using the decomposition  $TM|_\Sigma = T\Sigma \oplus N\Sigma$ , we may write a section  $V^c \in \Gamma(\mathcal{E}^c|_\Sigma)$  as a tuple  $(\Pi_d^c V^d, N_d^c V^d)$ . Since  $\Pi_d^c V^d$  is a tangent vector to  $\Sigma$ , we abuse notation slightly and typically write the tuple as  $(V^k, N_d^c V^d)$  where  $V^k = \Pi_d^k V^d$ . Computing the action of the Riemann curvature  $R_{ij}{}^c{}_d$  on such a tuple, we see that

$$\begin{aligned} & (\nabla_i \nabla_j - \nabla_j \nabla_i) \begin{pmatrix} V^k \\ N_d^c V^d \end{pmatrix} \\ &= \begin{pmatrix} R_{ij}{}^\Sigma{}^k{}_l V^l - 2D_{[i}(\Pi_{j]}{}^k{}_d V^d) - 2g_{ef} \Pi_{[i}{}^k{}_f \Pi_{j]}{}^e{}_l V^l - 2\Pi_{[i}{}^k{}_{|e|} \nabla_{j]}^\perp(N_d^e V^d) \\ 2\Pi_{k[i}{}^c(D_{j]}V^k) - 2g^{kl} \Pi_{k[i}{}^c \Pi_{j]}{}^l{}_d V^d + 2\nabla_{[i}^\perp(\Pi_{j]}{}^k{}_c V^k) + R_{ij}{}^\perp{}^c{}_d V^d \end{pmatrix} \\ &= \begin{pmatrix} R_{ij}{}^\Sigma{}^k{}_l + 2g_{cd} \Pi_{l[i}{}^c \Pi_{j]}{}^k{}_d & -2D_{[i} \Pi_{j]}{}^k{}_d \\ 2D_{[i} \Pi_{j]}{}^c{}_l & R_{ij}{}^\perp{}^c{}_d + 2g^{kl} \Pi_{l[i}{}^c \Pi_{j]}{}^k{}_d \end{pmatrix} \begin{pmatrix} V^l \\ N_e^d V^e \end{pmatrix}, \end{aligned}$$

and the equations (3.8), (3.9) and (3.10) all follow from this by simply projecting the appropriate entry of the matrix.

Note that in dimension  $m = 1$ , the trace-free part of the second fundamental form is zero. Also, in dimension  $m = 1$  equations (3.8)-(3.10) are valid, but trivial in that in each case both sides are identically zero.

### 3.2 Conformal Submanifolds

Consider now a submanifold  $\Sigma$  satisfying  $1 \leq \dim \Sigma \leq n - 1$  in a conformal manifold  $(M, \mathbf{c})$ . Observe that the conformal structure is sufficient to determine an orthogonal complement of  $T\Sigma \subset TM$  and so the splitting of (3.1) is in fact conformally invariant. We carry over to this setting the same notation for the normal projection  $N_b^a$  and the orthogonal projection  $\Pi_a^i$ . Since each  $g \in \mathbf{c}$  induces a Riemannian metric on  $\Sigma$  by restriction, it follows immediately that  $\mathbf{c}$  induces a conformal structure on  $\Sigma$  that we denote  $\mathbf{c}_\Sigma$ . We therefore have intrinsic to  $(\Sigma, \mathbf{c}_\Sigma)$  density bundles  $\mathcal{E}_\Sigma[w]$ . In fact, for any  $w \in \mathbb{R}$ ,

$$\mathcal{E}_\Sigma[w] = \mathcal{E}[w]|_\Sigma,$$

which can be seen immediately from the interpretation of densities as equivalence classes of a metric and a function, see the discussion of Section 2.2. From these observations, it follows immediately that the conformal metric  $\mathbf{g}_\Sigma$  is simply the restriction to  $\otimes^2 T\Sigma$  of the ambient conformal metric  $\mathbf{g}$ . Using these, it is straightforward to see that the orthogonal projection may be thought of as a composition  $\mathbf{g}_\Sigma^{-1} \circ \Pi \circ \mathbf{g}$ , meaning

$$\Pi_a^i = \mathbf{g}^{ij} \Pi_j^b \mathbf{g}_{ab}, \quad (3.11)$$

where we are raising and lowering indices using the conformal metric (and we omit the subscript  $\Sigma$  when the indices imply that we are using  $\mathbf{g}_\Sigma$  or its inverse). Since this involves  $\mathbf{g}$  and  $\mathbf{g}^{-1}$ , the resulting section still has conformal weight zero. When a metric  $g \in \mathbf{c}$  is chosen we have as usual the Gauß, Codazzi and Ricci equations ((3.8), (3.9) and (3.10)), but it will be convenient to work with weighted versions

of these, with the ambient or intrinsic conformal metrics replacing any instances of their scale-dependent counterparts. For example the weighted version of the Gauß equation is

$$R_{ijkl} = R_{ijkl}^\Sigma + 2g_{cd}II_{[i}{}^c II_{j]k}{}^d,$$

where now  $R_{ijkl}$  and  $R_{ijkl}^\Sigma$  have weight 2 (and the second fundamental form naturally has conformal weight 0).

Coupling the normal connection (3.3) to the Levi-Civita connection on the bundle  $\mathcal{E}_\Sigma[-1]$  yields a connection on  $N\Sigma[-1]$ , which we shall also denote  $\nabla^\perp$ . It is easily verified using (2.7) and (2.9) that this is conformally invariant. In fact, more generally, on sections of  $N\Sigma[w]$  the transformation law is

$$\widehat{\nabla}_i^\perp \nu^a = \nabla_i^\perp \nu^a + (w+1)\Upsilon_i \nu^a \quad (3.12)$$

when  $\widehat{g} = \Omega^2 g$  and  $\Upsilon_i = \Omega^{-1} \nabla_i \Omega$ . The conformal metric  $g$  induces a bundle metric on  $N\Sigma[-1]$ , and this is preserved by  $\nabla^\perp$ .

Since the Levi-Civita connection changes under a conformal rescaling, the Gauß formula is not conformally invariant. Using (2.7), we conclude that under a conformal transformation,

$$\widehat{H}_{ij}{}^c = H_{ij}{}^c - g_{ij} N_d^c \Upsilon^d. \quad (3.13)$$

Since this transformation is by pure trace, it follows immediately that  $\widehat{H}_{ij}{}^c$  is conformally invariant:

$$\widehat{\widehat{H}}_{ij}{}^c = \widehat{H}_{ij}{}^c.$$

Thus the transformation (3.13) is entirely due to the transformed mean curvature, whence

$$\widehat{H}^c = H^c - N_d^c \Upsilon^d. \quad (3.14)$$

Note that the mean curvature is now defined using the conformal metric:  $H^c := \frac{1}{m} g^{ij} H_{ij}{}^c$ . As a consequence of (3.14) we have the following very useful proposition [9, 51, 34]:

**Proposition 3.1.** *Let  $\Sigma$  be a submanifold of a conformal manifold  $(M, \mathbf{c})$ . Then any metric  $g_\Sigma$  in the induced conformal class of metrics on  $\Sigma$  can be extended to a metric  $g \in \mathbf{c}$  such that the mean curvature of  $\Sigma$  with respect to  $g$  vanishes.*

*Proof.* Let  $g_\Sigma$  be as in the proposition, and let  $g \in \mathbf{c}$  be any extension of  $g_\Sigma$ . We look for a rescaled metric  $\widehat{g}$  satisfying the requirements of the proposition. Set  $\widehat{g} = e^{2\omega} g$  with  $\omega$  to be determined. Since we require  $\widehat{g}|_\Sigma = g_\Sigma$  we set  $\omega = 0$  along  $\Sigma$ . Now, by (3.14),  $\widehat{H}^c = 0$  if, and only if,  $H^c = N_d^c \nabla^d \omega$  along  $\Sigma$ . Since the latter merely amounts to specifying the normal derivatives of  $\omega$  along  $\Sigma$ , such an  $\omega$  clearly exists (unique modulo functions that vanish and have vanishing differential along  $\Sigma$ ). This proves the proposition.  $\square$

We refer to a metric  $g \in \mathbf{c}$  such that  $H^c = 0$  as a *minimal scale* for  $\Sigma$ . The freedom to work in a minimal scale when computing conformally invariant quantities helps to simplify many calculations.

**Remark 3.2.** *In the case of a 1-dimensional submanifold a minimal scale  $g$  is easily seen to be one for which the curve is an unparametrised geodesic, since in this case, after parametrising the curve by arc length, the mean curvature vector can be identified with the acceleration of the curve. Moreover, in the case where the curve  $\gamma$  in  $(M, \mathbf{c})$  is already parametrised one can take  $g_\Sigma$  to be the “metric” on  $\gamma$  corresponding to the parametrisation and extend  $g_\Sigma$  to  $g \in \mathbf{c}$  as in Proposition 3.2 to obtain a metric  $g$  for which  $\gamma$ , with its original parametrisation, is a parametrised geodesic.*

### 3.3 Submanifold tractors

As a conformal manifold in its own right,  $(\Sigma, \mathbf{c}_\Sigma)$  possesses its own standard tractor bundle, which we will call the *intrinsic tractor bundle*, and denote by  $\mathcal{T}\Sigma$ , or  $\mathcal{E}^I$  in abstract indices, carrying over our convention that later Latin letters will be used for sections of submanifold bundles, with upper case indices for tractor bundles. We now wish to relate this to the corresponding ambient tractor bundle,  $\mathcal{T}M$ , which will continue to denote by  $\mathcal{T}$ .

As we have already seen in Section 3.1, the Gauß formula plays a central role in the setting of Riemannian submanifold geometry. Crucially, the Gauß formula uses that  $T\Sigma$  may be identified with a subbundle of  $TM$ . In fact, there is an analogous notion for the intrinsic and ambient tractor bundles, and this explains our abstract index notation for submanifolds being similar to that for ambient tractors.

First, there is a mapping  $N\Sigma[-1] \rightarrow \mathcal{T}$  defined by

$$n^a \mapsto N^A = N_a^A n^a \stackrel{g}{=} \begin{pmatrix} 0 \\ n^a \\ n_c H^c \end{pmatrix}. \quad (3.15)$$

This is easily seen to be conformally invariant using the transformation laws for the tractor projectors (2.15) and the mean curvature (3.14):

$$\begin{aligned} \widehat{N}^A &= \widehat{n}^a \widehat{Z}_a^A + \widehat{n}_a \widehat{H}^a \widehat{X}^A \\ &= n^a (Z_a^A + \Upsilon_a X^A) + n_a (H^a - \Upsilon^b N_b^a) X^A \\ &= n^a Z_a^A + n^a \Upsilon_a X^A + n_a H^a X^A - n_b \Upsilon^b X^A \\ &= n^a Z_a^A + n_a H^a X^A \\ &= N^A. \end{aligned}$$

Thus the image of the injective map (3.15) defines, along  $\Sigma$ , a subbundle of  $\mathcal{T}$  which is canonically isomorphic to  $N\Sigma[-1]$ . We call this the *normal tractor bundle* and denote this  $\mathcal{N}$ , or  $\mathcal{N}^A$  with indices. We summarise, as follows.



**Lemma 3.3.** *The map (3.15) defines a conformally invariant isomorphism*

$$N_a^A : N\Sigma[-1] \xrightarrow{\cong} \mathcal{N} \subset \mathcal{T}|_\Sigma. \quad (3.16)$$

The bundle  $\mathcal{N}$  admits an orthogonal complement in  $\mathcal{T}$ ,  $\mathcal{N}^\perp$ , and so  $\mathcal{T}|_\Sigma$  decomposes as

$$\mathcal{T}|_\Sigma = \mathcal{N}^\perp \oplus \mathcal{N}.$$

Write  $\Pi_B^A : \mathcal{E}^B \rightarrow \mathcal{N}^{\perp A}$  and  $N_B^A : \mathcal{E}^B \rightarrow \mathcal{N}^A$  for the orthogonal projections onto the respective factors of this decomposition. So  $\delta_B^A = \Pi_B^A + N_B^A$ . Note that for any  $N^B \in \Gamma(\mathcal{N}^B)$ , one has  $h_{AC}N^C\Pi_B^A = 0$  since  $\Pi_B^A$  is valued in  $\mathcal{N}^{\perp A}$ . Thus,  $h_{AC}N_B^A\Pi_D^C = 0$ . Substituting  $\Pi_D^C = \delta_D^C - N_D^C$ , it follows that

$$N_{BC} = N_B^AN_D^Ch_{AC}.$$

So  $N_{AB}$  and  $\Pi_{AB}$  are symmetric, where in each case an index has been lowered with the tractor metric, and  $\Pi_B^A$  and  $N_B^A$  give the orthogonal decomposition of the cotractor bundle  $\mathcal{E}_A$ .

Note that

$$N_a^AN_B^a = N_B^A \quad \text{and} \quad N_B^aN_b^B = N_b^a, \quad (3.17)$$

where  $N_A^a$  is the inverse to (3.16). From the symmetry of  $N_{AB}$ , and corresponding observation of symmetry for  $N_{ab}$ , it follows that  $N_A^a$  is obtained from  $N_b^B$  by raising and lowering indices using the tractor and conformal metrics.

A straightforward direct calculation shows that the isomorphism (3.16) intertwines the tractor and normal Levi-Civita connections in the sense of the following lemma.

**Lemma 3.4.** *For any section  $n^a \in \Gamma(N\Sigma[-1])$  we have*

$$N_b^C\nabla_i^\perp n^b = N_B^C\nabla_i(N_b^B n^b).$$

*Proof.* This follows immediately from the definitions if we work in a minimal scale.  $\square$

We note here that  $\mathcal{N}$  is a rank  $d = n - m$  vector bundle, and hence  $\mathcal{N}^\perp$  has rank  $(n + 2) - d = m + 2$ , which coincides with the rank of  $\mathcal{T}\Sigma$ . This is not a coincidence, and it turns out that there is an isomorphism of vector bundles  $\mathcal{T}\Sigma \rightarrow \mathcal{N}^\perp$ . Let us initially understand this in submanifold dimensions  $m \geq 3$ .

**Theorem 3.5.** *Let  $\Sigma$  be a submanifold of dimension  $m \geq 3$  in a conformal manifold  $(M, \mathbf{c})$ . The intrinsic tractor bundle  $\mathcal{T}\Sigma$  is canonically isomorphic to the orthogonal complement  $\mathcal{N}^\perp$  of the normal tractor bundle via a bundle isomorphism which preserves both the metric and the filtration. We denote this isomorphism  $\Pi_I^A$ . Explicitly, in a general ambient scale  $g \in \mathbf{c}$ , it is given by*

$$\mathcal{T}\Sigma \ni V^I \stackrel{g_\Sigma}{\cong} \begin{pmatrix} \sigma \\ \mu^i \\ \rho \end{pmatrix} \xrightarrow{\Pi_I^A} \begin{pmatrix} \sigma \\ \mu^a - H^a\sigma \\ \rho - \frac{1}{2}H^a H_a\sigma \end{pmatrix} \stackrel{g}{\cong} V^A \in \mathcal{N}^\perp, \quad (3.18)$$

where  $\mu^a = \Pi_i^a \mu^i$ .

The map  $\Pi : \mathcal{T}\Sigma \rightarrow \mathcal{N}^\perp$  is a filtration and metric preserving isomorphism.

*Proof.* Fix a scale  $g_\Sigma \in \mathfrak{c}_\Sigma$ , and let  $g \in \mathfrak{c}$  be a scale that satisfies  $\iota^* g = g_\Sigma$ . The map is clearly injective, and the image is also clearly annihilated by any section of  $\mathcal{N}$ . We need to show that the map (3.18) is unchanged if we replace  $g$  by some conformally related  $\widehat{g} = \Omega^2 g$ , and  $g_\Sigma$  by  $\widehat{g}_\Sigma = \Omega^2 g_\Sigma$ . (In the latter  $\Omega$  is restricted to  $\Sigma$  – this is clear by context and so we do introduce additional notation.) Equivalently, we need to show that the following diagram commutes

$$\begin{array}{ccc} [\mathcal{E}^I]_{g_\Sigma} & \xrightarrow{\Pi_I^A} & [\mathcal{E}^A]_g|_\Sigma \\ \downarrow & & \downarrow \\ [\mathcal{E}^J]_{\widehat{g}_\Sigma} & \xrightarrow{\Pi_J^B} & [\mathcal{E}^B]_{\widehat{g}}|_\Sigma \end{array} \quad (3.19)$$

where the vertical maps are conformal change of tractor splitting, as given in (2.16), and the horizontal maps are (3.18) in the appropriate scale.

Write  $\Upsilon_a = \Omega^{-1} \nabla_a \Omega$ , and  $\Upsilon_i = \Omega^{-1} D_i \Omega$ . Note that  $\Upsilon_i = \Pi_i^a \Upsilon_a$ . Applying  $\Pi_I^A$  and then rescaling is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^b & \delta_a^b & 0 \\ -\frac{1}{2} \Upsilon^c \Upsilon_c & -\Upsilon_a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -H^a & \Pi_i^a & 0 \\ -\frac{1}{2} H^c H_c & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^b - H^b & \Pi_i^b & 0 \\ -\frac{1}{2} \Upsilon^c \Upsilon_c + H^a \Upsilon_a - \frac{1}{2} H^c H_c & -\Upsilon_i & 1 \end{pmatrix},$$

while first rescaling and then applying  $\Pi_J^B$  corresponds to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -\widehat{H}^b & \Pi_j^b & 0 \\ -\frac{1}{2} \widehat{H}^c \widehat{H}_c & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^j & \delta_i^j & 0 \\ -\frac{1}{2} \Upsilon^k \Upsilon_k & -\Upsilon_i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\widehat{H}^b + \Pi_j^b \Upsilon^j & \Pi_i^b & 0 \\ -\frac{1}{2} \widehat{H}^c \widehat{H}_c - \frac{1}{2} \Upsilon^k \Upsilon_k & -\Upsilon_i & 1 \end{pmatrix}.$$

Using equation (3.14), we see that

$$-\widehat{H}^b + \Pi_j^b \Upsilon^j = -H^b + N_c^b \Upsilon^c + \Pi_j^b \Upsilon^j = -H^b + \Upsilon^b$$

and

$$\begin{aligned} -\frac{1}{2} \widehat{H}^c \widehat{H}_c - \frac{1}{2} \Upsilon^k \Upsilon_k &= -\frac{1}{2} H^c H_c + H^e \Upsilon_e - \frac{1}{2} (\Upsilon^k \Upsilon_k + N^{cd} \Upsilon_c \Upsilon_d) \\ &= -\frac{1}{2} H^c H_c + H^e \Upsilon_e - \frac{1}{2} \Upsilon^c \Upsilon_c, \end{aligned}$$

whence the above two matrix products are equal. Hence the map is conformally invariant.

It is easily verified that the map  $\Pi_I^A$  is metric preserving and sends  $X^I$  to  $X^A$ , and so is filtration preserving. An easy calculation shows that it is metric preserving, cf. Remark 3.6 below. □

**Remark 3.6.** *Note that the calculations in the above proof and hence the existence of the canonical metric and filtration preserving map  $\Pi : \mathcal{T}\Sigma \rightarrow \mathcal{N}^\perp$  are greatly simplified if we choose to work only with minimal scales ( $g \in \mathfrak{c}$  with  $H^c = 0$ , cf. Proposition 3.2); in a minimal scale, the map  $\Pi_I^A$  simply maps  $(\sigma, \mu_i, \rho) \mapsto (\sigma, \mu_a, \rho)$  and it is clear that this map preserves the metric and the filtration.*

Motivated by the result in Theorem 3.5 above, for the cases of dimensions  $m = 1, 2$  we (for now) define  $\mathcal{T}\Sigma$  to be  $\mathcal{N}^\perp$ . (Then Theorem 3.5 again applies, and is effectively just changing the splitting to give triples that transform in the usual way.) In Section 3.5 below we will then show that in dimension  $m = 1$  and  $m = 2$  it is still that case that  $\mathcal{T}\Sigma$  is canonically  $J^2\mathcal{E}[1]/S_0^2T^*\Sigma[1]$  (where in the  $m = 1$  case  $S_0^2T^*\Sigma[1]$  is the “zero vector bundle”), consistent with the discussion given in the introduction.

For convenience, we will say that sections of  $\mathcal{N}^\perp$  are tractors *tangent* to the submanifold, and similarly, sections of  $\mathcal{N}$  are tractors *normal* to the submanifold.

It will also be convenient to record the relationship between the submanifold and ambient splitting tractors corresponding to the isomorphism in Theorem 3.5, namely:

$$X^I = \Pi_A^I X^A, \quad Z_i^I = \Pi_A^I \Pi_i^a Z_a^A, \quad \text{and} \quad Y^I = \Pi_A^I (Y^A - H^a Z_a^A - \frac{1}{2} H^a H_a X^A) \quad (3.20)$$

along  $\Sigma$ , where  $\Pi_A^I$  can be interpreted as the inverse of the map  $\Pi_I^A$  given by Theorem 3.5 or, better, as the orthogonal projection from  $\mathcal{T}$  to  $\mathcal{N}^\perp$  followed by the isomorphism  $\mathcal{N}^\perp \rightarrow \mathcal{T}\Sigma$  (this is completely analogous to our use of the notation  $\Pi_a^i$  applied to tangent vectors, see (3.1)).

We have already mentioned that there is a tractor Gauß formula, namely a decomposition of the ambient tractor connection which is compatible with the decomposition  $\mathcal{T} = \mathcal{N}^\perp \oplus \mathcal{N}$ . Define the “checked” connection  $\check{\nabla}$  on  $\mathcal{T}\Sigma$  by

$$\check{\nabla}_i V^J := \Pi_B^J \nabla_i (\Pi_K^B V^K), \quad (3.21)$$

where  $\nabla_i$  on the right-hand side is the (pullback of the) ambient tractor connection. This is essentially the tangential part of the ambient connection. We may then define the *tractor second fundamental form*  $\mathbb{L}_{iJ}^C$  analogously to the Riemannian case, namely as the 1-form with values in maps  $\mathcal{T}\Sigma \rightarrow \mathcal{N}$  which characterises the normal part of the ambient connection:

$$\nabla_i V^B = \Pi_J^B \check{\nabla}_i V^J + \mathbb{L}_{iJ}^B V^J, \quad (3.22)$$

where  $V^B = \Pi_J^B V^J$  is a section of the ambient tractor bundle which is tangent to the submanifold. The linear operator  $\mathbb{L}_{iJ}^B$  is well defined by this since both  $\iota^* \nabla$  and  $\tilde{\nabla}$  satisfy the Leibniz rule. We call (3.22) the *tractor Gauß formula*.

The ambient tractor connection also induces a connection on the normal tractor bundle in the obvious way:

$$\nabla_i^{\mathcal{N}} N^A := N_B^A \nabla_i N^B, \quad (3.23)$$

where  $N^A$  is a section of  $\mathcal{N}$ . Such  $N^A$  are of the form  $N^A \stackrel{g}{=} (0, n^a, n_c H^c)$ , where  $n^a \in \Gamma(N\Sigma^a[-1])$ . As a tractor with zero in the top slot, it follows from (2.15) that the middle slot of  $\nabla_i^{\mathcal{N}} N^A$  is necessarily conformally invariant. But this exactly recovers the invariant connection on  $N\Sigma[-1]$  discussed in Section 3.2 (cf. Lemma 3.4). In summary, we have the following.

**Proposition 3.7.** *The canonical isomorphism  $N\Sigma[-1] \xrightarrow{\cong} \mathcal{N}$  preserves the invariant parallel transports defined on each bundle.*

Essential to our direction in this article is that the tractor fundamental form may be captured in several equivalent ways, the first of which we give here.

**Proposition 3.8.** *The tractor second fundamental form is given by*

$$\mathbb{L}_{iK}^B = \Pi_K^C N_A^B \nabla_i \Pi_C^A, \quad (3.24)$$

or equivalently,

$$\mathbb{L}_{iK}^B = -\Pi_K^C N_A^B \nabla_i N_C^A. \quad (3.25)$$

*Proof.* Let  $N^A$  be a section of the normal tractor bundle  $\mathcal{N}$ . Note that  $\Pi_C^A N_A = 0$ , and hence

$$0 = \nabla_i (\Pi_C^A N_A) = (\nabla_i \Pi_C^A) N_A + \Pi_B^A \nabla_i N_A,$$

whence

$$\Pi_C^A \nabla_i N_A = -N_A \nabla_i \Pi_C^A. \quad (3.26)$$

As a consequence of the tractor Gauß formula (3.22),

$$N_B \mathbb{L}_{iK}^B V^K = N_B \nabla_i V^B = -V^B \nabla_i N_B = -V^K \Pi_K^B \nabla_i N_B$$

for all  $V^K \in \Gamma(\mathcal{E}^K)$ , and therefore

$$N_B \mathbb{L}_{iK}^B = -\Pi_K^B \nabla_i N_B = -\Pi_K^C \Pi_C^A \nabla_i N_A.$$

Combining this with (3.26), we have that

$$N_B \mathbb{L}_{iK}^B = -\Pi_K^C (-N_A \nabla_i \Pi_C^A) = N_B \Pi_K^C N_A^B \nabla_i \Pi_C^A,$$

and this must hold for any section  $N^B$  of the normal tractor bundle, whence the result follows. Substituting  $\Pi_C^A = \delta_C^A - N_C^A$  into (3.24) and using that  $\delta_C^A$  is parallel for the tractor connection then gives the second equality of the proposition.  $\square$

It will be convenient to have a second (equivalent) object that we also term the *tractor second fundamental form*, which we denote by  $\bar{\mathbb{L}}$  and which is the section of  $T^*\Sigma \otimes \mathcal{N}^{\perp*} \otimes \mathcal{N}$  defined by

$$\bar{\mathbb{L}}_{iA}{}^B = \Pi_A^J \mathbb{L}_{iJ}{}^B. \quad (3.27)$$

Since  $\Pi_A^J$  is an isomorphism  $\mathcal{T}^*\Sigma \rightarrow \mathcal{N}^{\perp*}$ , this is clearly equivalent to the original tractor second fundamental form (and eventually when there is no possibility of confusion we will simply denote both objects by  $\mathbb{L}$ ). The gain of using  $\bar{\mathbb{L}}$  is that both its tractor indices are ambient tractor indices, as we shall see shortly.

It is useful to observe that  $\bar{\mathbb{L}}$  arises naturally in several different ways.

**Lemma 3.9.** *The B index of  $N_A^C \nabla_i N_B^A$  is tangential, i.e., for any  $N^B \in \Gamma(\mathcal{N}^B)$ , one has*

$$N^B N_A^C \nabla_i N_B^A = 0.$$

*Proof.* Let  $N^B \in \Gamma(\mathcal{N}^B)$ . We calculate  $\nabla_i(N_B^A N^B)$  in two different ways. On the one hand, one has

$$\nabla_i(N_B^A N^B) = \nabla_i N^A,$$

while on the other

$$\nabla_i(N_B^A N^B) = N_B^A \nabla_i N^B + N^B \nabla_i N_B^A.$$

Hence

$$N^B \nabla_i N_B^A = \nabla_i N^A - N_B^A \nabla_i N^B.$$

Thus

$$N^B N_A^C \nabla_i N_B^A = N_A^C (\nabla_i N^A - N_B^A \nabla_i N^B) = N_A^C \nabla_i N^A - N_B^C \nabla_i N^B = 0.$$

□

Thus we see that the  $\Pi_K^C$  of equation (3.25) is merely identifying the already tangential  $C$  index with a submanifold tractor index. Thus we see the following.

**Proposition 3.10.**

$$\bar{\mathbb{L}}_{iB}{}^C = -N_A^C \nabla_i N_B^A. \quad (3.28)$$

*Proof.* By equation (3.25) we have

$$\bar{\mathbb{L}}_{iB}{}^C = -\Pi_B^K \Pi_K^D N_A^C \nabla_i N_D^A = -(\delta_B^D - N_B^D) N_A^C \nabla_i N_D^A,$$

and so the result follows from the previous lemma. □

**Remark 3.11.** *Note that (3.28) is equivalent to  $\mathbb{L}_{iJ}{}^C = -\Pi_J^B N_A^C \nabla_i N_B^A$ . A similar argument shows that*

$$H_{ij}{}^c = -\Pi_j^b N_a^c \nabla_i N_b^a.$$

**Lemma 3.12.** *Let  $N_B^C$  be the normal tractor projector. Then*

$$\nabla_i N_B^C = -\bar{\mathbb{L}}_i^C{}_B - \bar{\mathbb{L}}_i B^C, \quad (3.29)$$

where

$$\bar{\mathbb{L}}_i^C{}_B = h^{CD} h_{AB} \bar{\mathbb{L}}_i D^A.$$

*Proof.* Noting that  $N_B^C = N_A^C N_B^A$ , we have

$$\nabla_i N_B^C = N_B^A \nabla_i N_A^C + N_A^C \nabla_i N_B^A.$$

The second term is exactly the negative of equation (3.28). Using that the normal projector is symmetric, the first term is clearly a transpose of this (so for example on this term, the  $C$  index is tangential).  $\square$

We now use this result to compute an explicit formula for the tractor second fundamental form. We first prove a lemma about the tractor normal projector.

**Lemma 3.13.** *For a choice of scale, the tractor normal projector is given by*

$$N_B^A = N_b^a Z_a^A Z_B^b + H^a Z_a^A X_B + H_b X^A Z_B^b + (H^d H_d) X^A X_B \quad (3.30)$$

where the  $H^c$  is the mean curvature vector in the chosen scale.

*Proof.* The right-hand side of (3.30) defines a conformally invariant bundle map  $\mathcal{E}^A \rightarrow \mathcal{N}^B$  which moreover acts as the identity on sections of  $\mathcal{N}^A$  as defined in following (3.15). The latter is easily verified by working in a minimal scale.

Alternatively, (3.30) follows immediately from (3.17), as in any scale

$$N_B^a = N_b^a Z_B^b + H^a X_B,$$

and  $N_B^A = N_a^A N_B^a$ .  $\square$

**Theorem 3.14.** *The tractor second fundamental form is given by*

$$\begin{aligned} \mathbb{L}_{iJ}^C &= \mathring{\mathbb{I}}_{ij}{}^c Z_j^C Z_c^C + N_a^c (P_i^a - \nabla_i H^a) X_J Z_c^C \\ &\quad + H_c \mathring{\mathbb{I}}_{ij}{}^c Z_j^C X^C + H_a (P_i^a - \nabla_i H^a) X_J X^C. \end{aligned} \quad (3.31)$$

*Proof.* We compute  $N_A^C \nabla_i N_B^A$  using the formula from Lemma 3.13. We then apply  $\Pi_J^B$ , the formula for which is given in Theorem 3.5 to complete the proof.

First, differentiating (3.30) gives

$$\begin{aligned}
\nabla_i N_B^A &= (\nabla_i N_b^a) Z_a^A Z_B^b + N_b^a (-P_{ia} X^A - \mathbf{g}_{ia} Y^A) Z_B^b + N_b^a Z_a^A (-P_i^b X_B - \Pi_i^b Y_B) \\
&\quad + (\nabla_i H^a) Z_a^A X_B + H^a (-P_{ia} X^A - \mathbf{g}_{ia} Y^A) X_B + H^a Z_a^A Z_{Bi} \\
&\quad + (\nabla_i H_b) X^A Z_B^b + H_b Z_i^A Z_B^b + H_b X^A (-P_i^b X_B - \Pi_i^b Y_B) \\
&\quad + 2(H^d \nabla_i H_d) X^A X_B + H^d H_d Z_i^A X_B + H^d H_d X^A Z_{Bi} \\
&= (\nabla_i N_b^a + H^a \mathbf{g}_{ib} + H_b \Pi_i^a) Z_a^A Z_B^b \\
&\quad + \left( -N_b^a P_{ia} + \nabla_i H_b + H_d H^d \mathbf{g}_{ib} \right) X^A Z_B^b \\
&\quad + \left( -N_b^a P_i^b + \nabla_i H^a + H_d H^d \Pi_i^a \right) Z_a^A X_B \\
&\quad + \left( -H^a P_{ia} - H_b P_i^b + 2H^d \nabla_i H_d \right) X^A X_B,
\end{aligned}$$

where, recall,  $\mathbf{g}_{ia}$  means  $\Pi_i^b \mathbf{g}_{ba}$ . From (3.30), it follows that

$$N_A^C Z_a^A = N_a^c Z_c^C + H_a X^C \quad \text{and} \quad N_A^C X^A = 0.$$

Hence

$$\begin{aligned}
N_A^C \nabla_i N_B^A &= (\nabla_i N_b^a + H^a \mathbf{g}_{ib} + H_b \Pi_i^a) (N_a^c Z_c^C + H_a X^C) Z_B^b \\
&\quad + \left( -N_b^a P_i^b + \nabla_i H^a + H_d H^d \Pi_i^a \right) (N_a^c Z_c^C + H_a X^C) X_B \\
&= (N_a^c \nabla_i N_b^a + H^c \mathbf{g}_{ib}) Z_B^b Z_c^C + N_a^c (\nabla_i H^a - P_i^a) X_B Z_c^C \\
&\quad + H_a (\nabla_i N_b^a + H^a \mathbf{g}_{ib}) Z_B^b X^C + H_a (\nabla_i H^a - P_i^a) X_B X^C.
\end{aligned}$$

All that remains is to apply the tangential tractor projector  $\Pi_J^B$ . According to (3.18),

$$\Pi_J^B Z_B^b = \Pi_j^b Z_J^j \quad \text{and} \quad \Pi_J^B X_B = X_J. \quad (3.32)$$

Therefore

$$\begin{aligned}
\Pi_J^B N_A^C \nabla_i N_B^A &= \Pi_j^b (N_a^c \nabla_i N_b^a + H^c \mathbf{g}_{ib}) Z_J^j Z_c^C + N_a^c (\nabla_i H^a - P_i^a) X_J Z_c^C \\
&\quad + H_a \Pi_j^b (\nabla_i N_b^a + H^a \mathbf{g}_{ib}) Z_J^j X^C + H_a (\nabla_i H^a - P_i^a) X_J X^C \\
&= (-\mathbb{I}_{ij}{}^c + H^c \mathbf{g}_{ij}) Z_J^j Z_c^C + N_a^c (\nabla_i H^a - P_i^a) X_J Z_c^C \\
&\quad + H_c (-\mathbb{I}_{ij}{}^c + H^c \mathbf{g}_{ib}) Z_J^j X^C + H_a (\nabla_i H^a - P_i^a) X_J X^C \\
&= -\mathring{\mathbb{I}}_{ij}{}^c Z_J^j Z_c^C + N_a^c (\nabla_i H^a - P_i^a) X_J Z_c^C \\
&\quad - H_c \mathring{\mathbb{I}}_{ij}{}^c Z_J^j X^C + H_a (\nabla_i H^a - P_i^a) X_J X^C.
\end{aligned}$$

where we note that  $H_a \nabla_i N_b^a = H_c N_a^c \nabla_i N_b^a$ , and we have used the observation from Remark 3.11 to replace  $\Pi_j^b N_a^c \nabla_i N_b^a$  with  $-\mathbb{I}_{ij}{}^c$ .

Finally, applying  $\Pi_J^B$  to equation (3.29) shows that  $\mathbb{L}_{iJ}{}^C$  is equal to negative of the above, from which the claim in the theorem follows.  $\square$

### 3.4 Conformal submanifolds of dimension $m \geq 3$

For submanifold dimensions  $m \geq 3$  the conformal structure of  $\Sigma$  determines a tractor connection  $D$  and compatible tractor metric  $h_\Sigma$  on  $\mathcal{T}\Sigma$ . We refer to these as the *intrinsic* tractor connection and metric for  $\mathcal{T}\Sigma$ . We continue to use abstract indices from the later part of the alphabet to distinguish submanifold objects from their ambient analogues, so, e.g., we write  $h_\Sigma$  as  $h_{IJ} \in \Gamma(\mathcal{E}_{(IJ)})$ .

Unlike the Riemannian case, the checked connection is not exactly the intrinsic tractor connection.

**Proposition 3.15.** *Along a submanifold  $\Sigma$  of dimension  $m \geq 3$  the checked and intrinsic tractor connections are related by*

$$\check{\nabla}_i V^J = D_i V^J + S_i^J{}_K V^K, \quad (3.33)$$

with  $D_i$  the intrinsic submanifold tractor connection and

$$S_{iJK} := 2 \left( P_{ij} - p_{ij} + H_c \mathring{\Pi}_{ij}{}^c + \frac{1}{2} H_c H^c g_{ij} \right) Z_{[J}^j X_{K]}, \quad (3.34)$$

where  $\Pi_{ij} := \Pi_i^a \Pi_j^b P_{ab}$  is the restriction of the ambient Schouten tensor to the submanifold and  $p_{ij}$  is the intrinsic Schouten tensor.

*Proof.* Fix metrics  $g \in \mathbf{c}$  and  $g_\Sigma \in \mathbf{c}_\Sigma$  such that  $\iota^* g = g_\Sigma$  to facilitate calculation. The inverse isomorphism of (3.18) is the map  $\mathcal{N}^\perp \rightarrow \mathcal{T}\Sigma$  given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -H^a & \Pi_i^a & 0 \\ -\frac{1}{2} H^c H_c & 0 & 1 \end{pmatrix}. \quad (3.35)$$



Using this we have

$$\begin{aligned}
\check{\nabla}_i V^J &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi_b^j & 0 \\ -\frac{1}{2}H^c H_c & -H_b & 1 \end{pmatrix} \nabla_i \left[ \begin{pmatrix} 1 & 0 & 0 \\ -H^b & \Pi_k^b & 0 \\ -\frac{1}{2}H^c H_c & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^k \\ \rho \end{pmatrix} \right] \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi_b^j & 0 \\ -\frac{1}{2}H^c H_c & -H_b & 1 \end{pmatrix} \nabla_i \begin{pmatrix} \sigma \\ \mu^b - H^b \sigma \\ \rho - \frac{1}{2}H^c H_c \sigma \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi_b^j & 0 \\ -\frac{1}{2}H^c H_c & -H_b & 1 \end{pmatrix} \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i (\mu^b - H^b \sigma) + P_i^b \sigma + \Pi_i^b (\rho - \frac{1}{2}H^c H_c \sigma) \\ \nabla_i (\rho - \frac{1}{2}H^c H_c \sigma) - P_{ic} (\mu^c - H^c \sigma) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi_b^j & 0 \\ -\frac{1}{2}H^c H_c & -H_b & 1 \end{pmatrix} \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu^b - (\nabla_i H^b) \sigma - H^b \nabla_i \sigma + P_i^b \sigma + \Pi_i^b \rho - \frac{1}{2} \Pi_i^b H^c H_c \sigma \\ \nabla_i \rho - (H^c \nabla_i H_c) \sigma - \frac{1}{2} H^c H_c \nabla_i \sigma - P_{ic} \mu^c + P_{ic} H^c \sigma \end{pmatrix} \\
&= \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \Pi_b^j (\nabla_i \mu^b - (\nabla_i H^b) \sigma - H^b \nabla_i \sigma + P_i^b \sigma + \Pi_i^b \rho - \frac{1}{2} \Pi_i^b H^c H_c \sigma) \\ -\frac{1}{2} H^c H_c (\nabla_i \sigma - \mu_i) - H_b (\nabla_i \mu^b - (\nabla_i H^b) \sigma - H^b \nabla_i \sigma + P_i^b \sigma + \Pi_i^b \rho - \frac{1}{2} \Pi_i^b H^c H_c \sigma) \\ + \nabla_i \rho - (H^c \nabla_i H_c) \sigma - \frac{1}{2} H^c H_c \nabla_i \sigma - P_{ic} \mu^c + P_{ic} H^c \sigma \end{pmatrix}
\end{aligned}$$

Using the agreement of the intrinsic Levi-Civita connection with the pullback of the ambient Levi-Civita connection then gives

$$\begin{aligned}
\check{\nabla}_i V^J &= \begin{pmatrix} D_i \sigma - \mu_i \\ D_i \mu^j + P_i^j \sigma + \delta_i^j \rho - (\Pi_b^j \nabla_i H^b) \sigma - \frac{1}{2} \delta_i^j H^c H_c \sigma \\ D_i \rho - P_{ic} \mu^c + \frac{1}{2} H^c H_c \mu_i - H_b \nabla_i \mu^b \end{pmatrix} \\
&= \begin{pmatrix} D_i \sigma - \mu_i \\ D_i \mu^j + p_i^j \sigma + \delta_i^j \rho - (-\mathring{\Pi}_i^j b - \delta_i^j H_b) H^b \sigma - \frac{1}{2} \delta_i^j H^c H_c \sigma + (P_i^j - p_i^j) \sigma \\ D_i \rho - p_{ic} \mu^c + \frac{1}{2} H^c H_c \mu_i - H_b (\mathring{\Pi}_{ij}^b + \mathbf{g}_{ij} H^b) \mu^j + (P_i^j - p_i^j) \mu^j \end{pmatrix} \\
&= \begin{pmatrix} D_i \sigma - \mu_i \\ D_i \mu^j + p_i^j \sigma + \delta_i^j \rho \\ D_i \rho - p_{ic} \mu^c \end{pmatrix} + \begin{pmatrix} 0 \\ (P_i^j - p_i^j + H_b \mathring{\Pi}_i^{jb} + \frac{1}{2} H^c H_c \delta_i^j) \sigma \\ - (P_{ij} - p_{ij} + H_b \mathring{\Pi}_{ij}^b + \frac{1}{2} H_b H^b \mathbf{g}_{ij}) \mu^j \end{pmatrix} \\
&= D_i^{\mathcal{T}\Sigma} \begin{pmatrix} \sigma \\ \mu^j \\ \rho \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{F}_i^j & 0 & 0 \\ 0 & -\mathcal{F}_{ij} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^j \\ \rho \end{pmatrix}.
\end{aligned}$$

where

$$\mathcal{F}_{ij} := P_{ij} - p_{ij} + H_b \mathring{\Pi}_{ij}^b + \frac{1}{2} H_b H^b \mathbf{g}_{ij}. \quad (3.36)$$

□

We call the tensor  $\mathcal{F}_{ij}$  given in (3.36) the *Fialkow tensor* (since this quantity seems to have appeared first in the work of Fialkow [40], cf. [75, 98]). Since the checked connection, the intrinsic tractor connection and  $Z_{[J}^j X_{K]}$  are all conformally invariant, it follows that the Fialkow tensor is also conformally invariant. Proposition 3.15 tells us that the Fialkow tensor measures the failure of the induced tractor connection  $\check{\nabla}$  to be *normal* (in the sense of corresponding to a normal Cartan connection [17]).

There is an alternative formula for the Fialkow tensor which is manifestly conformally invariant. To derive it, one substitutes the Weyl-Schouten decomposition of the ambient and intrinsic Riemann tensors into the Gauß formula, and then applies the map

$$T_{ijkl} \mapsto \frac{1}{m-2} \left( T_{ikj}{}^k - \frac{T_{kl}{}^{kl}}{2(m-1)} \mathbf{g}_{ij} \right)$$

to both sides (to take the “submanifold Schouten” part). After writing the second fundamental form as  $\mathring{H}_{ij}{}^c + \mathbf{g}_{ij} H^c$  and rearranging, one finds that the Fialkow tensor (3.36) is equal to

$$\mathcal{F}_{ij} = \frac{1}{m-2} \left( W_{icjd} N^{cd} + \frac{W_{abcd} N^{ac} N^{bd}}{2(m-1)} \mathbf{g}_{ij} + \mathring{H}_i{}^{kc} \mathring{H}_{jkc} - \frac{\mathring{H}_{klc} \mathring{H}{}^{klc}}{2(m-1)} \mathbf{g}_{ij} \right). \quad (3.37)$$

All objects on the right-hand side are conformally invariant. Since the Fialkow tensor has already been observed to be conformally invariant, it is sufficient to establish the formula (3.37) in a minimal scale. Let  $g \in \mathfrak{c}$  be a minimal scale and  $g_\Sigma = \iota^* g$ . Thus  $II = \mathring{H}$ . Applying the submanifold and ambient Weyl-Schouten decompositions in the Gauss equation we obtain

$$\begin{aligned} W_{ijkl} + P_{ik} \mathbf{g}_{jl} - P_{jk} \mathbf{g}_{il} - P_{il} \mathbf{g}_{jk} + P_{jl} \mathbf{g}_{ik} &= w_{ijkl} + p_{ik} \mathbf{g}_{jl} - p_{jk} \mathbf{g}_{il} - p_{il} \mathbf{g}_{jk} + p_{jl} \mathbf{g}_{ik} \\ &\quad + \mathbf{g}_{cd} \mathring{H}_{li}{}^c \mathring{H}_{jk}{}^d - \mathbf{g}_{cd} \mathring{H}_{lj}{}^c \mathring{H}_{ik}{}^d \end{aligned}$$

where  $W_{ijkl}$  denotes the full projection  $\Pi_i^a \Pi_j^b \Pi_k^c \Pi_l^d W_{abcd}$  of the ambient Weyl curvature,  $P_{ij} = \Pi_i^a \Pi_j^b P_{ab}$ , and  $w_{ijkl}$  denotes the submanifold intrinsic Weyl tensor. Applying the map  $T_{ijkl} \mapsto \frac{1}{m-2} \left( T_{ikj}{}^k - \frac{T_{kl}{}^{kl}}{2(m-1)} \mathbf{g}_{ij} \right)$  on both sides of the above display we get

$$-\frac{1}{m-2} \left( W_{icjd} N^{cd} + \frac{W_{abcd} N^{ab} N^{cd}}{2(m-1)} \mathbf{g}_{ij} \right) + P_{ij} = p_{ij} + \frac{1}{m-2} \left( \mathring{H}_i{}^{kc} \mathring{H}_{jkc} - \frac{\mathring{H}{}^{klc} \mathring{H}_{klc}}{2(m-1)} \mathbf{g}_{ij} \right),$$

noting that  $\mathbf{g}^{kl} W_{ikjl} = \mathbf{g}^{kl} \Pi_k^c \Pi_l^d W_{icjd} = -N^{cd} W_{icjd}$  since  $W_{abcd}$  is trace free, and similarly  $W_{kl}{}^{kl} = W_{abcd} N^{ab} N^{cd}$ . The result then follows from (3.36).

Recall that along  $\Sigma$  we may decompose the ambient standard tractor bundle  $\mathcal{T} = \mathcal{T}M$  as  $\mathcal{T}|_\Sigma = \mathcal{T}\Sigma \oplus \mathcal{N}$ . If  $V \in \Gamma(\mathcal{T}|_\Sigma)$  is given by  $(V^\top, V^\perp)$  with respect to

this decomposition, then by (3.22), (3.23), (3.33) and the fact that  $\nabla$  preserves the ambient tractor metric we have

$$\nabla_X V = \begin{pmatrix} D_X + \mathbf{S}(X) & -\mathbb{L}(X)^T \\ \mathbb{L}(X) & \nabla_X^{\mathcal{N}} \end{pmatrix} \begin{pmatrix} V^\top \\ V^\perp \end{pmatrix} \quad (3.38)$$

for any  $X \in \mathfrak{X}(\Sigma)$ , where  $\mathbb{L}(X)^T$  is the transpose of  $\mathbb{L}(X)$  with respect to the ambient tractor metric (cf. Proposition 3.8). We therefore write

$$\iota^* \nabla = \begin{pmatrix} D + \mathbf{S} & -\mathbb{L}^T \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \quad (3.39)$$

on  $\mathcal{T}|_\Sigma = \mathcal{T}\Sigma \oplus \mathcal{N}$ . We will most often make use of the above display in the case where  $V$  is a section of  $\mathcal{T}\Sigma$  ( $V^\perp = 0$ ). In this case we obtain the following form of the *tractor Gauß formula*:

$$\nabla_i V^B = \Pi_J^B (D_i V^J + \mathbf{S}_i^J{}_K V^K) + \mathbb{L}_{iK}{}^B V^K, \quad (3.40)$$

for any section  $V$  of  $\mathcal{T}\Sigma$ .

By writing the curvature  $\iota^* \Omega$  of  $\iota^* \nabla$ , somewhat informally, as

$$\iota^* \Omega = \begin{pmatrix} D + \mathbf{S} & -\mathbb{L}^T \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \wedge \begin{pmatrix} D + \mathbf{S} & -\mathbb{L}^T \\ \mathbb{L} & \nabla^{\mathcal{N}} \end{pmatrix} \quad (3.41)$$

one may easily obtain conformal tractor analogues of the Riemannian Gauss, Codazzi, and Ricci equations (cf. the derivation of the Riemannian Gauß-Codazzi-Ricci equations in Section 3.1):

$$\Omega_{ijKL} = \Omega_{ijKL}^\Sigma + 2D_{[i} \mathbf{S}_{j]KL} + 2\mathbf{S}_{[i|KM} \mathbf{S}_{j]ML} - 2\mathbb{L}_{[i|KC} \mathbb{L}_{j]LC} \quad (3.42)$$

$$\Omega_{ijKD} N_E^D = -2D_{[i} \mathbb{L}_{j]KE} - 2\mathbf{S}_{[i|KL} \mathbb{L}_{j]L}{}^E \quad (3.43)$$

$$\Omega_{ij}{}^A{}_B N_A^C N_D^B = \Omega^{\mathcal{N}}{}_{ij}{}^C{}_D + 2\mathbb{L}_{[i}{}^{KC} \mathbb{L}_{j]KD}, \quad (3.44)$$

where indices between bars are exempt from antisymmetrisation,  $\Omega_{ijKL} = \Omega_{ijCD} \Pi_K^C \Pi_L^D$  for  $\Omega_{ijCD}$  the curvature of the pullback connection, and  $\Omega^{\mathcal{N}}$  is the curvature of the normal tractor connection, characterised by

$$\Omega^{\mathcal{N}}{}_{ij}{}^C{}_D N^D = (\nabla_i^{\mathcal{N}} \nabla_j^{\mathcal{N}} - \nabla_j^{\mathcal{N}} \nabla_i^{\mathcal{N}}) N^C, \quad (3.45)$$

for any section  $N$  of the normal tractor bundle  $\mathcal{N}$ .

**Remark 3.16.** In [11] Burstall and Calderbank define a ‘Möbius reduction’ to be a rank  $(m+2)$  subbundle  $\mathcal{V}$  of  $\mathcal{T}|_\Sigma$  containing the rank  $m+1$  subbundle spanned by the canonical tractor  $X^A$  and its covariant derivatives in submanifold tangential

directions (with respect to the ambient tractor connection coupled with the Levi-Civita connection of some, equivalently any, metric  $g \in \mathfrak{c}$ ). One then decomposes the ambient tractor connection along  $\Sigma$  as (using notation similar to the above)

$$\iota^* \nabla = \begin{pmatrix} \nabla^\mathcal{V} & -(\mathbb{L}^\mathcal{V})^T \\ \mathbb{L}^\mathcal{V} & \nabla^{\mathcal{V}^\perp} \end{pmatrix}$$

on  $\mathcal{T}|_\Sigma = \mathcal{V} \oplus \mathcal{V}^\perp$ . The definition of ‘Möbius reduction’ implies that  $\mathbb{L}^\mathcal{V}_{iJ}{}^C X^J = 0$  and  $\mathbb{L}^\mathcal{V}_{iJ}{}^C X_C = 0$ , so that there is a well-defined projection  $\Pi^\mathcal{V}_{ij}{}^c := \mathbb{L}^\mathcal{V}_{iJ}{}^C Z_j^J Z_c^C$  of  $\mathbb{L}^\mathcal{V}_{iJ}{}^C$ . Burstall and Calderbank then define the unique ‘canonical Möbius reduction’  $\mathcal{V}_\Sigma$  by imposing an algebraic normalisation condition on

$$\begin{pmatrix} 0 & -(\mathbb{L}^\mathcal{V})^T \\ \mathbb{L}^\mathcal{V} & 0 \end{pmatrix}$$

similar to the algebraic normalisation condition imposed on the curvature of the normal Cartan/tractor connection [22, 17], see Section 9.3 of [11]. This algebraic normalisation condition amounts to the requirement that  $\mathbf{g}^{ij} \Pi^\mathcal{V}_{ij}{}^c = 0$ . Since by Theorem 3.14 the tractor second fundamental form has invariant projection  $\check{\Pi}_{ij}{}^c = \mathbb{L}_{iJ}{}^C Z_j^J Z_c^C$  the ‘canonical Möbius reduction’  $\mathcal{V}_\Sigma$  is the same as the orthogonal complement  $\mathcal{N}^\perp$  of the normal tractor bundle and hence gives an abstract characterization of this bundle (equivalently of the normal tractor bundle  $\mathcal{N}$ ). Our approach differs in that we explicitly construct  $\mathcal{N}$ , and then further explicitly identify  $\mathcal{N}^\perp$  with the intrinsic tractor bundle  $\mathcal{T}\Sigma$ .

### 3.5 Low-dimensional conformal submanifolds

In this section we treat submanifolds  $\Sigma$  such that  $\dim(\Sigma)$  is  $m = 1$  or  $m = 2$ . Note that Section 3.3 has no restriction on the submanifold dimension  $m$ . However in Section 3.4, just above, we make the restriction to  $m \geq 3$  to discuss the intrinsic tractor connection then available. When  $m = 1, 2$  the conformal structure on  $\Sigma$  is not sufficient to determine a canonical connection on  $\mathcal{T}\Sigma$ . The purpose of this section is to observe that in these dimensions the conformal embedding does determine distinguished tractor connections on  $\mathcal{T}\Sigma$ , and then using this we get analogues of the results from Section 3.4.

First recall that Equation (3.21) defines a connection  $\check{\nabla}$  on the bundle  $\mathcal{T}\Sigma$  also when  $\dim \Sigma$  is 1 or 2.

Riemannian manifolds of dimension 1 or 2 are not naturally equipped with an (intrinsically determined) Schouten tensor. However conformal submanifolds of these dimensions inherit a natural replacement, as follows. First recall that for submanifolds of dimension at least 3, the difference tractor of is equivalent to the Fialkow tensor of the submanifold according to expressions (3.34) and (3.36). In dimensions 1 and 2, we will, in essence, turn this around and use the formula (3.36) to determine a submanifold Schouten tensor. In these dimensions all terms in (3.36) are

well defined, as usual, except the submanifold Schouten  $p_{ij}$  and the Fialkow tensor  $\mathcal{F}_{ij}$ . Note that fixing one of these two determines the other via (3.36). Moreover, from the conformal transformation formulae of the terms in (3.36), it follows that *any* natural conformally invariant choice of  $\mathcal{F}_{ij}$  determines a submanifold tensor  $p_{ij} \in S^2T^*\Sigma$  that transforms conformally according to (2.10). Such a Schouten tensor  $p_{ij}$  then yields a conformally invariant tractor connection  $D$  on  $\mathcal{T}\Sigma$  (in a choice of scale) according to the usual formula, namely

$$D_i \begin{pmatrix} \sigma \\ \mu_j \\ \rho \end{pmatrix} = \begin{pmatrix} D_i\sigma - \mu_i \\ D_i\mu_j + p_{ij}\sigma + g_{ij}\rho \\ D_i\rho - p_{ij}\mu^j \end{pmatrix}, \quad (3.46)$$

where  $D$  on the right-hand side is the intrinsic Levi-Civita connection. Equivalently the tractor connection is determined by formula (3.33) with

$$S_{iJK} := 2\mathcal{F}_{ij}Z_{[J}^jX_{K]}.$$

Thus it remains to specify an invariant  $\mathcal{F}_{ij}$ , or equivalently a  $p_{ij}$  that transforms according to (2.10). Recall that in all dimensions we have the conformally invariant tractor connection  $\check{\nabla}$  on  $\mathcal{T}\Sigma$ . It is given by

$$\check{\nabla}_i V^J = \begin{pmatrix} D_i\sigma - \mu_i \\ D_i\mu^j + P_i^j\sigma + \delta_i^j\rho - (\Pi_b^j\nabla_i H^b)\sigma - \frac{1}{2}\delta_i^j H^c H_c\sigma \\ D_i\rho - P_{ic}\mu^c + \frac{1}{2}H^c H_c\mu_i - H_b\nabla_i\mu^b \end{pmatrix}$$

where we have computed using a choice of ambient scale  $g \in \mathfrak{c}$ , and  $V^J \stackrel{g\Sigma}{=} (\sigma, \mu^j, \rho)$ . But by the conformal invariance of  $\check{\nabla}$ , the right hand side depends only on  $g_\Sigma$ , as this is sufficient to determine the tractor bundle splitting. Now, arguing similarly to the previous section we have

$$\begin{aligned} \check{\nabla}_i V^J &= \begin{pmatrix} D_i\sigma - \mu_i \\ D_i\mu^j + P_i^j\sigma + \delta_i^j\rho - (-\check{\Pi}_i^j{}_b - \delta_i^j H_b)H^b\sigma - \frac{1}{2}\delta_i^j H^c H_c\sigma \\ D_i\rho - P_{ij}\mu^j + \frac{1}{2}H^c H_c\mu_i - H_b(\check{\Pi}_{ij}^b + g_{ij}H^b)\mu^j \end{pmatrix} \\ &= \begin{pmatrix} D_i\sigma - \mu_i \\ D_i\mu^j + \left(P_i^j + H_b\check{\Pi}_i^j{}_b + \frac{1}{2}H^c H_c\delta_i^j\right)\sigma + \delta_i^j\rho \\ D_i\rho - \left(P_{ij} + H_b\check{\Pi}_{ij}^b + \frac{1}{2}H_b H^b g_{ij}\right)\mu^j \end{pmatrix}. \end{aligned}$$

In particular

$$\check{\nabla}_i Y^J = \left(P_i^j + H_b\check{\Pi}_i^j{}_b + \frac{1}{2}H^c H_c\delta_i^j\right) Z_j^J,$$

cf. (2.19). From this and the conformal invariance of  $\check{\nabla}$  it follows that  $P_i^j + H_b\check{\Pi}_i^j{}_b + \frac{1}{2}H^c H_c\delta_i^j$  transforms in the same way as a Schouten tensor and depends only on

$g_\Sigma \in \mathbf{c}_\Sigma$  (rather than  $g \in \mathbf{c}$ ) and the conformal embedding of  $\Sigma$  in  $(M, \mathbf{c})$ . That is, if we define

$$p_{ij} := \left( P_{ij} + H_b \mathring{\Pi}_{ij}{}^b + \frac{1}{2} H_b H^b g_{ij} \right) - \mathcal{F}_{ij},$$

for any (decreed to be) conformally invariant  $\mathcal{F}_{ij} \in \Gamma(\mathcal{E}_{(ij)})$ , then  $p_{ij}$  depends only on  $g_\Sigma$  (and the conformal embedding of  $\Sigma$  in  $(M, \mathbf{c})$ ) and has conformal transformation

$$\hat{p}_{ij} = p_{ij} - D_i \Upsilon_j + \Upsilon_i \Upsilon_j - \frac{1}{2} g_{ij} \Upsilon^k \Upsilon_k.$$

This formula is also easily checked directly. When such a choice of submanifold Schouten tensor has been made we will denote its trace by

$$j := g^{ij} p_{ij}. \quad (3.47)$$

Then under conformal change

$$\hat{j} = j - D_i \Upsilon^i + \left(1 - \frac{m}{2}\right) \Upsilon^i \Upsilon_i. \quad (3.48)$$

In dimension 1 we will simply set  $\mathcal{F}_{ij} := 0$ . Thus given any submanifold scale  $g_\Sigma$  and any extension  $g \in \mathbf{c}$  one has

$$p_{ij} := P_{ij} + \frac{1}{2} H_b H^b g_{ij},$$

since, for a curve, the trace-free second fundamental form is trivially zero. In this case, of course,  $p_{ij} = jg_{ij}$  where  $j := g^{ij} p_{ij}$ . Note that the extension  $g \in \mathbf{c}$  can be chosen such that  $H^b = 0$  and then one simply has  $p_{ij} := P_{ij}$ .

In dimension 2 we shall set  $\mathcal{F}_{(ij)_0} = 0$ . By (3.36) (which determines the relation between  $p_{ij}$  and  $\mathcal{F}_{ij}$ ) this is equivalent to taking the trace-free part of  $p_{ij}$  in any scale to be

$$p_{(ij)_0} := P_{(ij)_0} + H_b \mathring{\Pi}_{ij}{}^b,$$

for any metric  $g \in \mathbf{c}$  extending the given submanifold scale  $g_\Sigma$  (of course,  $g$  can be taken to be a minimal scale in which case we simply have  $p_{(ij)_0} := P_{(ij)_0}$ ). From the point of view of Cartan geometry, or of the tractor connection  $D$  (which is to be determined from (3.33) by the choice of  $p_{ij}$ ), it is then natural to determine the trace part of  $p_{ij}$  in any scale  $g_\Sigma$  by setting

$$j = g^{ij} p_{ij} := \frac{1}{2} g^{ij} \text{Ric}_{ij}^{g_\Sigma},$$

so that  $j$  agrees with the Gauss curvature of  $g_\Sigma$  (which we are viewing as section of  $\mathcal{E}[-2]|_\Sigma$ ). This transforms conformally by the formula (3.48), and so the trace part of the right-hand side of (3.36) is then conformally invariant with this definition. We have thus given a conformally invariant definition of  $\mathcal{F}_{ij}$  as a pure trace tensor

(indeed, one can show that this gives  $\mathcal{F}_{ij} = \frac{1}{2}|\mathring{II}|^2 \mathbf{g}_{ij}$  [91]). Besides this, our choice of  $j = \mathbf{g}^{ij} p_{ij}$  is significant in that it is the unique choice that enables us to write

$$R_{ijkl}^{g\Sigma} = 2\mathbf{g}_{k[i} p_{j]l} - 2\mathbf{g}_{l[i} p_{j]k}.$$

Requiring the above display to hold is a normalisation condition on the tractor connection  $D$  determined by  $p_{ij}$  as in (3.46), equivalent to requiring that the “middle slot” (the  $Z_K^k Z_L^l$  component) of the tractor curvature vanishes. Indeed, with the conventions we have just established, the curvature of the tractor connection  $D$  is given by

$$\Omega_{ijKL}^\Sigma = -2c_{ijk} X_{[K} Z_{L]}^k,$$

where  $c_{ijk} := 2D_{[i} p_{j]k}$ . This should be compared with (2.20) in the three dimensional case where the Weyl tensor term vanishes. Note that, while in three and higher dimensions the Cotton tensor is trace free, since we are in two dimensions the tensor  $c_{ijk}$  can be written as

$$c_{ijk} = \frac{1}{2} \epsilon_{ij} c_k$$

where  $c_k = \epsilon^{ij} c_{ijk}$  and hence  $c_{ijk}$  is determined by its trace

$$c_{ijk} \mathbf{g}^{jk} = \frac{1}{2} \epsilon_{ij} c^j = D_{ij} - D^j p_{ij}.$$

In the 2-dimensional case, the choice of  $p_{ij}$  for each scale (with the trace part normalized by setting  $g^{ij} p_{ij}$  to be the Gauß curvature) is equivalent to a choice of Möbius structure on  $\Sigma$  in the sense of [13]. The invariant  $c_{ijk}$ , which we refer to as the *Cotton tensor* in this setting, is precisely the curvature of this Möbius structure and vanishes if and only if  $\Sigma$  (with the Möbius structure just defined) is locally equivalent to the conformal Möbius sphere (i.e. comes from a system of local coordinates on  $\Sigma$  related by Möbius transformations); see [13] for more details. Note that the Möbius structure we have just defined on 2-dimensional submanifolds  $\Sigma$  agrees with the notion of *induced conformal Möbius structure* in [11], cf. [34].

In both dimensions 1 and 2 we then have that, in any scale  $g \in \mathfrak{c}$ , and with  $g_\Sigma = g|_{T\Sigma}$ , the tractor connection on  $[T\Sigma]_{g_\Sigma}$  is given by (3.46). We note that this formula exactly agrees with the formula for the usual tractor connection, as defined by (2.19) (but has  $p_{ij}$  as defined here). This connection in turn defines a Thomas-D operator  $\mathbb{D} : \mathcal{E}[1] \rightarrow \mathcal{T}\Sigma$  via the usual BGG splitting operator characterisation (see e.g. [21]), namely that, for any  $\sigma \in \Gamma(\mathcal{E}[1])$ ,  $D_i \mathbb{D}_I \sigma$  must be 0 in the top slot and be trace-free in the middle slot. Thus, in a scale  $g_\Sigma \in \mathfrak{c}_\Sigma$ , this takes the form

$$\frac{1}{m} \mathbb{D}_K \sigma \stackrel{g_\Sigma}{=} \begin{pmatrix} \sigma \\ D_k \sigma \\ -\frac{1}{m} (\mathbf{g}^{i\ell} D_i D_\ell + j) \sigma \end{pmatrix}, \quad (3.49)$$

(cf. (2.12)). It follows at once that (as in higher dimensions  $m$ ) any parallel standard tractor  $I$  is necessarily in the image of  $\frac{1}{m}\mathbb{D}$ . Moreover we have that, for any  $\sigma \in \Gamma(\mathcal{E}[1])$ , and on the set where  $\sigma$  is non-vanishing, the scale tractor

$$I := \frac{1}{m}\mathbb{D}\sigma$$

satisfies that

$$h^{KL}I_K I_L = -\frac{2}{m}j\sigma^2 = -\frac{2}{m}j^{g_\Sigma}$$

where  $j^{g_\Sigma} = j\sigma^2$  is the (weight zero true)  $J$ -curvature for the scale  $g_\Sigma$ , and this is thus constant if  $I$  is parallel. (E.g., for surfaces the scale tractor parallel being implies the corresponding metric has constant Gauß curvature.) So this fits with the situation in higher dimensions.

Finally observe that the Thomas- $D$  formula (3.49) evidently provides a conformally invariant isomorphism

$$J^2\mathcal{E}_\Sigma[1]/S_0^2T^*\Sigma[1] \xrightarrow{\cong} \mathcal{T}\Sigma,$$

cf. (2.13). Thus our description of the tractor bundle from the introduction still applies in dimensions  $m = 1$  and  $m = 2$ , where we identify  $\mathcal{T}\Sigma$  with  $\mathcal{N}^\perp$ .

**Remark 3.17.** *We emphasise that our choice of  $\mathcal{F}_{ij}$ , in dimensions  $m = 1$  and  $m = 2$ , was indeed a choice. Any natural choice of  $\mathcal{F}_{ij}$  will lead to a corresponding tractor/Cartan connection that is canonical by dint of the naturality of  $\mathcal{F}_{ij}$ . (For the meaning of naturality in this context see [61].)*

*For example  $W_{icjd}N^{cd}$  is natural, symmetric, conformally invariant, and of weight zero – so this could be used. Similarly  $\mathring{H}_i{}^{kc}\mathring{H}_{jkc}$  for surfaces. Both vanish for curves, while observe that the symmetries and properties of the Weyl curvature imply that for surfaces*

$$W_{icjd}N^{cd} = -\frac{W_{abcd}N^{ac}N^{bd}}{2}g_{ij},$$

*while, the Cayley-Hamilton theorem implies*

$$\mathring{H}_i{}^{kc}\mathring{H}_{jkc} = \frac{\mathring{H}{}^{klc}\mathring{H}_{klc}}{2}g_{ij}.$$

*So it seems likely that naturality will force our choice for curves, and at least that  $\mathcal{F}_{(ij)_0} = 0$  for surfaces.*

*In dimension  $m = 1$  parallel transport using (3.46) is equivalent in an obvious way to a third order linear ODE along the curve. In dimension  $m = 2$  parallel tractors, with the top slot  $\sigma \in \Gamma(\mathcal{E}[1])$  non-vanishing, correspond to solutions of*

$$D_{(i}D_{j)_0}\sigma + p_{(ij)_0}\sigma = 0 \tag{3.50}$$



that also satisfy that the conformal invariant  $c_{ij}^j = D_i j - D^j p_{ij}$  is zero (if (3.50) alone holds then following the usual prolongation argument one can easily show that  $I^K = \frac{1}{2} \mathbb{D}^K \sigma$  satisfies  $D_j I^K = \sigma(D^i p_{ij} - D_j j) X^K$ , cf. [33, Section 3.4]); it follows from the above discussion that nontrivial parallel tractors only exist when the Möbius structure is locally flat, in which case they define a metric of constant Gaussian curvature on the open dense set where  $\sigma \neq 0$ .

### 3.6 Normal forms

Since  $\dim N^* \Sigma = d$ ,  $\Lambda^d N^* \Sigma$  is a line bundle. Moreover,  $\Sigma$  is oriented and it thus follows that there is a unique section  $N_{a_1 a_2 \dots a_d}$  of  $\Lambda^d N^* \Sigma[d]$  which is compatible with the orientations of  $\Sigma$  and  $M$ , and such that

$$N^{a_1 a_2 \dots a_d} N_{a_1 a_2 \dots a_d} = d! ;$$

here by *compatible with the orientations* we mean that

$$\epsilon_{a_1 a_2 \dots a_m}^\Sigma \wedge N_{a_{m+1} \dots a_n} = \epsilon_{a_1 a_2 \dots a_n},$$

where  $\epsilon^\Sigma$  and  $\epsilon$  are the weighted volume forms for  $\Sigma$  and  $M$  respectively. We call  $N_{a_1 a_2 \dots a_d}$  the *Riemannian normal form* for the submanifold  $\Sigma$ . It is not hard to show that this contains the same data as the normal projector  $N_b^a$ . Indeed, one can obtain the latter from the former via  $N_b^a = \frac{1}{(d-1)!} N^{a a_2 \dots a_d} N_{b a_2 \dots a_d}$ , cf. Proposition 3.18.

This object also has a tractor analogue. Recall that the normal tractor bundle  $\mathcal{N}^*$  is isomorphic to  $N^* \Sigma[1]$ . Thus it follows that, for any  $k$  such that  $1 \leq k \leq \text{codim} \Sigma$ , one has  $\Lambda^k \mathcal{N}^* \cong \Lambda^k N^* \Sigma[k]$ . Explicitly, for  $\nu_{a_1 a_2 \dots a_k} \in \Lambda^k N^* \Sigma[k]$ , the isomorphism is given by

$$\nu_{a_1 a_2 \dots a_k} \mapsto \nu_{a_1 a_2 \dots a_k} \mathbb{Z}_{A_1 A_2 \dots A_k}^{a_1 a_2 \dots a_k} + k \cdot \nu_{b a_2 \dots a_k} H^b \mathbb{X}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k}, \quad (3.51)$$

by taking the  $k$ -th exterior power of (3.15). (Note that in the case  $k = 1$  this map is simply the map  $N^* \Gamma[1] \rightarrow \mathcal{N}^*$  of (3.15).) Invariance of this map may independently be checked via the transformation formulae for the tractor form projectors and the mean curvature:

$$\begin{aligned} & \widehat{\nu}_{a_1 a_2 \dots a_k} \widehat{\mathbb{Z}}_{A_1 A_2 \dots A_k}^{a_1 a_2 \dots a_k} + k \cdot \widehat{\nu}_{b a_2 \dots a_k} \widehat{H}^b \widehat{\mathbb{X}}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k} \\ &= \nu_{a_1 a_2 \dots a_k} \left( \mathbb{Z}_{A_1 A_2 \dots A_k}^{a_1 a_2 \dots a_k} + k \cdot \Upsilon^{a_1} \mathbb{X}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k} \right) \\ & \quad + k \cdot \nu_{b a_2 \dots a_k} \left( H^b - N_{a_1}^b \Upsilon^{a_1} \right) \mathbb{X}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k} \\ &= \nu_{a_1 a_2 \dots a_k} \mathbb{Z}_{A_1 A_2 \dots A_k}^{a_1 a_2 \dots a_k} \\ & \quad + k \cdot \left( \nu_{b a_2 \dots a_k} H^b + \nu_{a_1 a_2 \dots a_k} \Upsilon^{a_1} - \nu_{b a_2 \dots a_k} N_{a_1}^b \Upsilon^{a_1} \right) \mathbb{X}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k} \\ &= \nu_{a_1 a_2 \dots a_k} \mathbb{Z}_{A_1 A_2 \dots A_k}^{a_1 a_2 \dots a_k} + k \cdot \nu_{b a_2 \dots a_k} H^b \mathbb{X}_{A_1 A_2 \dots A_k}^{a_2 \dots a_k}. \end{aligned}$$

In particular, it follows that there is a distinguished section of the line bundle  $\Lambda^d \mathcal{N}^*$ , where  $d = \text{codim} \Sigma$ , given by the image of the Riemannian normal form under this isomorphism. We write  $N_{A_1 A_2 \dots A_d}$  for this section and call it the *tractor normal form* for the submanifold  $\Sigma$ . From the above isomorphism, one has

$$N_{A_1 A_2 \dots A_d} = N_{a_1 a_2 \dots a_d} \mathbb{Z}_{A_1 A_2 \dots A_d}^{a_1 a_2 \dots a_d} + d \cdot N_{b a_2 \dots a_d} H^b \mathbb{X}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d}, \quad (3.52)$$

and hence

$$\begin{aligned} N_{A_1 A_2 \dots A_d} N^{A_1 A_2 \dots A_d} &= N_{a_1 a_2 \dots a_d} N^{b_1 b_2 \dots b_d} \mathbb{Z}_{A_1 A_2 \dots A_d}^{a_1 a_2 \dots a_d} \mathbb{Z}_{b_1 b_2 \dots b_d}^{A_1 A_2 \dots A_d} \\ &= N_{a_1 a_2 \dots a_d} N^{a_1 a_2 \dots a_d} \\ &= d!, \end{aligned}$$

since all other contractions of the  $\mathbb{X}$  and  $\mathbb{Z}$  projectors are zero, and where the indices of  $N^{A_1 A_2 \dots A_d}$  have been raised with the tractor metric.

Given a local orthonormal basis  $\{N_A^1, \dots, N_A^d\}$  for the normal tractor bundle, which may be constructed from a local orthonormal basis of  $N^* \Sigma[1]$ , one sees that

$$d! \cdot N_{[A_1}^1 \dots N_{A_d]}^d = N_{A_1}^1 \wedge \dots \wedge N_{A_d}^d \quad (3.53)$$

is clearly a section of  $\Lambda^d \mathcal{N}^*$  and satisfies the above normalisation condition. If the chosen basis is compatible with the orientation, then (3.53) recovers the tractor normal form  $N_{A_1 A_2 \dots A_d}$ . By construction one then has that

$$\epsilon_{A_1 A_2 \dots A_{m+2}}^\Sigma \wedge N_{A_{m+3} \dots A_{n+2}} = \epsilon_{A_1 A_2 \dots A_{n+2}},$$

where  $\epsilon^\Sigma$  and  $\epsilon$  are the tractor volume forms of  $\Sigma$  and  $M$  respectively.

Our task is now to relate the tractor normal form to the other tractor objects introduced, namely, the tractor normal projector and the tractor second fundamental form. These relationships will lay the foundation for the notion of distinguished submanifold that we will introduce in the following chapter.

First, the tractor normal projector.

**Proposition 3.18.** *The tractor projector  $N_B^A$  is equal to*

$$N_{A_2}^{A_1} = \frac{1}{(d-1)!} N^{A_1 B_2 \dots B_d} N_{A_2 B_2 \dots B_d}. \quad (3.54)$$

*Proof.* Let  $\{N_1^A, N_2^A, \dots, N_d^A\}$  be an orientation-compatible orthonormal basis for the normal tractor bundle. Then

$$N^{A_1 B_2 \dots B_d} = d! \cdot N_1^{[A_1} N_2^{B_2} \dots N_d^{B_d]}.$$

The contraction  $N^{A_1 B_2 \dots B_d} N_{A_2 B_2 \dots B_d}$  is a sum of terms of the form

$$\text{sgn}(\sigma) \text{sgn}(\tau) N_{\sigma(1)}^{A_1} N_{\sigma(2)}^{B_2} \dots N_{\sigma(d)}^{B_d} N_{A_2}^{\tau(1)} N_{B_2}^{\tau(2)} \dots N_{B_d}^{\tau(d)},$$

where  $\sigma, \tau \in \mathfrak{S}_d$ . Now, we claim that such a term will be non-zero if, and only if  $\sigma = \tau$ . Clearly, if  $\sigma = \tau$ , then

$$\text{sgn}(\sigma) \text{sgn}(\tau) N_{\sigma(1)}^{A_1} N_{\sigma(2)}^{B_2} \cdots N_{\sigma(d)}^{B_d} N_{A_2}^{\tau(1)} N_{B_2}^{\tau(2)} \cdots N_{B_d}^{\tau(d)} = N_{\sigma(1)}^{A_1} N_{A_2}^{\sigma(1)}.$$

Conversely, if  $\sigma \neq \tau$ , then there is some  $i \in \{1, 2, \dots, d\}$  such that  $\sigma(i) \neq \tau(i)$ . Then the contraction will contain

$$N_{\sigma(i)}^{B_i} N_{B_i}^{\tau(i)} = \delta_{\sigma(i)}^{\tau(i)},$$

by orthogonality, which is zero since we are assuming that  $\sigma(i) \neq \tau(i)$ . Hence the only non-zero terms in the contraction are those where the same permutation is applied to both sets of indices.

Finally, we need only count how many such terms there are. We have just established that there are  $d!$  non-zero terms. Fixing  $\sigma(1)$ , one sees that there are  $(d-1)!$  remaining possibilities for  $\sigma$ , all of which will lead to  $N_{\sigma(1)}^A N_B^{\tau(1)}$ . Thus as  $\sigma(1)$  ranges over  $\{1, 2, \dots, d\}$ , we have that

$$\begin{aligned} N^{A_1 B_2 \cdots B_d} N_{A_2 B_2 \cdots B_d} &= \sum_{\sigma \in \mathfrak{S}_d} N_{\sigma(1)}^{A_1} N_{A_2}^{\sigma(1)} \\ &= (d-1)! \cdot \left( N_1^{A_1} N_{A_2}^1 + N_2^{A_1} N_{A_2}^2 + \cdots + N_d^{A_1} N_{A_2}^d \right) \\ &= (d-1)! \cdot N_{A_2}^{A_1}. \end{aligned}$$

□

Differentiating the formula obtained in the above display leads to a relationship between the tractor second fundamental form and the derivative of the tractor normal form. An alternative route to this is via the following lemma, which we record for completeness.

**Lemma 3.19.** *The derivative of the tractor normal form expressed in the tractor projector notation is*

$$\begin{aligned} \nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d} &= \left[ \nabla_i N_{a_1 a_2 \cdots a_{d-1} a_d} + d \cdot N_{ba_2 \cdots a_{d-1} a_d} H^b \mathbf{g}_{ia_1} \right] \mathbb{Z}_{A_1 A_2 \cdots A_d}^{a_1 a_2 \cdots a_d} \\ &\quad + d \cdot \left[ \nabla_i \left( N_{ba_2 \cdots a_{d-1} a_d} H^b \right) - N_{a_1 a_2 \cdots a_{d-1} a_d} P_i^{a_1} \right] \mathbb{X}_{A_1 A_2 \cdots A_d}^{a_2 \cdots a_d}. \end{aligned} \tag{3.55}$$

*Proof.* Recall

$$N_{A_1 A_2 \cdots A_d} = N_{a_1 a_2 \cdots a_d} \mathbb{Z}_{A_1 A_2 \cdots A_d}^{a_1 a_2 \cdots a_d} + d \cdot N_{ba_2 \cdots a_d} H^b \mathbb{X}_{A_1 A_2 \cdots A_d}^{a_2 \cdots a_d}.$$

Differentiating this,

$$\begin{aligned}
\nabla_i N_{A_1 A_2 \dots A_{d-1} A_d} &= (\nabla_i N_{a_1 a_2 \dots a_d}) \mathbb{Z}_{A_1 A_2 \dots A_d}^{a_1 a_2 \dots a_d} \\
&\quad + N_{a_1 a_2 \dots a_d} \left( -d \cdot P_i^{a_1} \mathbb{X}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d} - d \cdot \delta_i^{a_1} \mathbb{Y}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d} \right) \\
&\quad + d \cdot \nabla_i \left( N_{b a_2 \dots a_d} H^b \right) \mathbb{X}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d} \\
&\quad + d \cdot N_{b a_2 \dots a_d} H^b \left( \mathbf{g}_{i a_1} \mathbb{Z}_{A_1 A_2 \dots A_d}^{a_1 a_2 \dots a_d} - (d-1) \cdot \delta_i^{a_2} \mathbb{W}_{a_1 a_2 a_3 \dots a_d}^{A_3 \dots A_d} \right) \\
&= \left[ \nabla_i N_{a_1 a_2 \dots a_{d-1} a_d} + d \cdot N_{b a_2 \dots a_{d-1} a_d} H^b \mathbf{g}_{i a_1} \right] \mathbb{Z}_{A_1 A_2 \dots A_d}^{a_1 a_2 \dots a_d} \\
&\quad + d \cdot \left[ \nabla_i \left( N_{b a_2 \dots a_{d-1} a_d} H^b \right) - N_{a_1 a_2 \dots a_{d-1} a_d} P_i^{a_1} \right] \mathbb{X}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d},
\end{aligned}$$

where we use the fact that any terms where the  $i$  index is contracted into the normal form will vanish, since  $i$  is a tangential index.  $\square$

### 3.7 The proof of Theorem 1.1

Henceforth we will only use indices to distinguish  $\bar{\mathbb{L}}$  and  $\mathbb{L}$  which (as sections of different bundles) represent the same object, i.e. we will write  $\mathbb{L}_{iA}^B = \Pi_A^J \mathbb{L}_{iJ}^B$  instead of (3.27).

Toward Theorem 1.1, first observe that Proposition 3.10 and Lemma 3.12 together give equivalence of 1 and 2 in that theorem. The following Theorem shows that 1 implies 3.

**Theorem 3.20.** *The derivative of the tractor normal form is given in terms of the tractor second fundamental form by*

$$\nabla_i N_{A_1 A_2 \dots A_{d-1} A_d} = -d \cdot \mathbb{L}_{i[A_d}^{A_0} N_{A_1 A_2 \dots A_{d-1}] A_0}. \quad (3.56)$$

*Proof.* Fix an orthonormal basis of normal tractors  $\{N_A^1, \dots, N_A^d\}$ . Recall equation (3.28) (here we are identifying  $\bar{\mathbb{L}}$  with  $\mathbb{L}$ ):

$$-\mathbb{L}_{iA_d}^{A_0} = N_B^{A_0} \nabla_i N_{A_d}^B.$$

We have already made use of this to compute the explicit expression (3.31) for the tractor second fundamental form. Now we make use of it again to compute  $N_B^{A_0} \nabla_i N_{A_d}^B$  in a different way.

Working with the orthonormal basis of normal tractors,

$$\begin{aligned}
N_B^{A_0} \nabla_i N_{A_d}^B &= \left( N_1^{A_0} N_B^1 + \dots + N_d^{A_0} N_B^d \right) \nabla_i \left( N_1^B N_{A_d}^1 + \dots + N_d^B N_{A_d}^d \right) \\
&= \left( N_1^{A_0} N_B^1 + \dots + N_d^{A_0} N_B^d \right) \left( N_{A_d}^1 \nabla_i N_1^B + N_1^B \nabla_i N_{A_d}^1 + \dots \right. \\
&\quad \left. \dots + N_{A_d}^d \nabla_i N_d^B + N_d^B \nabla_i N_{A_d}^d \right)
\end{aligned}$$

Expanding the final line of the above will yield two types of terms:

- terms of the form  $N_k^{A_0} N_B^k N_{A_d}^\ell \nabla_i N_\ell^B$ , which are zero if  $k = \ell$ ; and
- terms of the form  $N_k^{A_0} N_B^k N_\ell^B \nabla_i N_{A_d}^\ell$ , which are zero unless  $k = \ell$ , in which case it simplifies to  $N_k^{A_0} \nabla_i N_{A_d}^k$ .

Thus

$$-\mathbb{L}_{iA_d}^{A_0} = \sum_{k=1}^d N_k^{A_0} \nabla_i N_{A_d}^k + \sum_{k \neq \ell} N_k^{A_0} N_{A_d}^\ell \left( N_k^B \nabla_i N_B^\ell \right). \quad (3.57)$$

We now use this formula to compute the right-hand side of the equation (3.56).

$$\begin{aligned} -\mathbb{L}_{i[A_d}^{A_0} N_{A_1 A_2 \dots A_{d-1}] A_0} &= \left( \sum_{k=1}^d N_k^{A_0} \nabla_i N_{[A_d}^k \right) N_{A_1 A_2 \dots A_{d-1}] A_0} \\ &\quad + \sum_{k \neq \ell} \left( N_k^B \nabla_i N_B^\ell \right) N_k^{A_0} N_{[A_d}^\ell N_{A_1 A_2 \dots A_{d-1}] A_0}. \end{aligned} \quad (3.58)$$

Observe that that each term in the second sum vanishes because if  $k \neq \ell$  then clearly

$$N_k^{A_0} N_{[A_d}^\ell N_{A_1 A_2 \dots A_{d-1}] A_0} = 0.$$

Next we compute an expression for  $N_k^{A_0} N_{A_1 A_2 \dots A_{d-1} A_0}$ . It is convenient to interchange  $A_k$  and  $A_0$ , so that the two copies of  $N^k$  have the same tractor index before expanding the antisymmetrisation. This will incur a factor of  $-1$  *unless*  $d = k$ ; in this case those indices are already the same. This is the reason for the Kronecker delta term  $\delta_{k,d}$  in the following.

$$\begin{aligned} N_k^{A_0} N_{A_1 A_2 \dots A_{d-1} A_0} &= d! \cdot N_k^{A_0} N_{[A_1}^1 N_{A_2}^2 \dots N_{A_{k-1}}^{k-1} N_{A_k}^k N_{A_{k+1}}^{k+1} \dots N_{A_{d-1}}^{d-1} N_{A_0}^d] \\ &= (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot N_{[A_1}^1 N_{A_2}^2 \dots N_{A_{k-1}}^{k-1} N_{A_{k+1}}^{k+1} \dots N_{A_{d-1}}^{d-1} N_{A_k}^d]. \end{aligned} \quad (3.59)$$

For a general term in the first sum of (3.58):

$$\begin{aligned} &\left( N_k^{A_0} \nabla_i N_{[A_d}^k \right) N_{A_1 A_2 \dots A_{d-1}] A_0} \\ &= (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot \left( \nabla_i N_{[A_d}^k \right) N_{A_1}^1 N_{A_2}^2 \dots N_{A_{k-1}}^{k-1} N_{A_k}^d N_{A_{k+1}}^{k+1} \dots N_{A_{d-1}}^{d-1} \\ &= (-1)^{1-\delta_{k,d}} \cdot (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot \left( \nabla_i N_{[A_k}^k \right) N_{A_1}^1 N_{A_2}^2 \dots N_{A_{k-1}}^{k-1} N_{A_d}^d N_{A_{k+1}}^{k+1} \dots N_{A_{d-1}}^{d-1} \\ &= (d-1)! \cdot \left( \nabla_i N_{[A_k}^k \right) N_{A_1}^1 N_{A_2}^2 \dots N_{A_{k-1}}^{k-1} N_{A_d}^d N_{A_{k+1}}^{k+1} \dots N_{A_{d-1}}^{d-1} \\ &= (d-1)! \cdot N_{[A_1}^1 N_{A_2}^2 \dots N_{A_{k-1}}^{k-1} \left( \nabla_i N_{A_k}^k \right) N_{A_{k+1}}^{k+1} \dots N_{A_{d-1}}^{d-1} N_{A_d}^d. \end{aligned}$$

Note from the second to third lines we interchange  $A_d$  and  $A_k$ , which multiplies the result by  $-1$  unless  $k = d$ , so we collect another of the Kronecker delta factors – and ultimately the two cancel. Finally, it remains to sum the above terms over  $k$ :

$$\begin{aligned}
& \left( \sum_{k=1}^d N_k^{A_0} \nabla_i N_{[A_d]}^k \right) N_{A_1 A_2 \dots A_{d-1} A_0} \\
&= (d-1)! \cdot \sum_{k=1}^d N_{[A_1]}^1 N_{A_2}^2 \dots N_{A_{k-1}}^{k-1} \left( \nabla_i N_{A_k}^k \right) N_{A_{k+1}}^{k+1} \dots N_{A_{d-1}}^{d-1} N_{A_d}^d \\
&= (d-1)! \cdot \nabla_i \left( N_{[A_1]}^1 N_{A_2}^2 \dots N_{A_{d-1}}^{d-1} N_{A_d}^d \right) \\
&= \frac{1}{d} \cdot \nabla_i N_{A_1 A_2 \dots A_{d-1} A_d}.
\end{aligned}$$

Thus from equation (3.58), we have

$$\begin{aligned}
-\mathbb{L}_{i[A_d]}^{A_0} N_{A_1 A_2 \dots A_{d-1} A_0} &= \left( \sum_{k=1}^d N_k^{A_0} \nabla_i N_{[A_d]}^k \right) N_{A_1 A_2 \dots A_{d-1} A_0} \\
&= \frac{1}{d} \cdot \nabla_i N_{A_1 A_2 \dots A_{d-1} A_d},
\end{aligned}$$

whence

$$\nabla_i N_{A_1 A_2 \dots A_{d-1} A_d} = -d \cdot \mathbb{L}_{i[A_d]}^{A_0} N_{A_1 A_2 \dots A_{d-1} A_0}.$$

□

We now invert the relationship between  $\nabla N$  and  $\mathbb{L}$  to get 3 implies 1 in Theorem 1.1.

**Theorem 3.21.** *The tractor second fundamental form is given in terms of the derivative of the tractor normal form by*

$$N^{A_{d+1} A_2 \dots A_d} \nabla_i N_{A_1 A_2 \dots A_d} = -(d-1)! \cdot \mathbb{L}_{i A_1}^{A_{d+1}}. \quad (3.60)$$

*Proof.* Theorem 3.20 gives that

$$\nabla_i N_{A_1 A_2 \dots A_{d-1} A_d} = -d \cdot \mathbb{L}_{i[A_d]}^{A_0} N_{A_1 A_2 \dots A_{d-1} A_0}.$$

We will contract both sides of this display with  $N^{A_{d+1} A_2 \dots A_d}$ . First, we see that

$$\begin{aligned}
& N^{A_{d+1}A_2\cdots A_d} \mathbb{L}_{i[A_d}^{A_0} N_{A_1A_2\cdots A_{d-1}]A_0} \\
&= -N^{A_{d+1}A_2\cdots A_d} \mathbb{L}_{i[A_1}^{A_0} N_{A_dA_2\cdots A_{d-1}]A_0} \\
&= -\frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) \cdot \mathbb{L}_{iA_{\sigma(1)}}^{A_0} N_{A_{\sigma(d)}A_{\sigma(2)}\cdots A_{\sigma(d-1)}A_0} N^{A_{d+1}A_2\cdots A_d}.
\end{aligned}$$

Now, the contraction of the tractor second fundamental form and the first tractor normal form will be zero unless  $\sigma(1) = 1$ , since otherwise the normal form will be contracted into the (tangential) lower index of the tractor second fundamental form. Any permutation  $\sigma$  with  $\sigma(1) = 1$  will result in the two normal forms being contracted on all indices except one on each copy. Hence

$$\begin{aligned}
& -\frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) \cdot \mathbb{L}_{iA_{\sigma(1)}}^{A_0} N_{A_{\sigma(d)}A_{\sigma(2)}\cdots A_{\sigma(d-1)}A_0} N^{A_{d+1}A_2\cdots A_d} \\
&= \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) \mathbb{L}_{iA_{\sigma(1)}}^{A_0} N_{A_0A_{\sigma(2)}\cdots A_{\sigma(d-1)}A_{\sigma(d)}} N^{A_{d+1}A_2\cdots A_d} \\
&= \frac{1}{d!} \sum_{\sigma \in \text{Sym}\{2,\dots,d-1,d\}} \text{sgn}(\sigma) \cdot \mathbb{L}_{iA_1}^{A_0} N_{A_0A_{\sigma(2)}\cdots A_{\sigma(d-1)}A_{\sigma(d)}} N^{A_{d+1}A_2\cdots A_d} \\
&= \frac{1}{d!} \sum_{\sigma \in \text{Sym}\{2,\dots,d-1,d\}} \text{sgn}(\sigma) \cdot \text{sgn}(\sigma) \cdot \mathbb{L}_{iA_1}^{A_0} N_{A_0A_{\sigma(2)}\cdots A_{\sigma(d-1)}A_{\sigma(d)}} N^{A_{d+1}A_{\sigma(2)}\cdots A_{\sigma(d)}} \\
&= \frac{1}{d!} \sum_{\sigma \in \text{Sym}\{2,\dots,d-1,d\}} \mathbb{L}_{iA_1}^{A_0} \cdot (d-1)! \cdot N_{A_0}^{A_{d+1}} \\
&= \frac{((d-1)!)^2}{d!} \mathbb{L}_{iA_1}^{A_{d+1}} \\
&= \frac{(d-1)!}{d} \cdot \mathbb{L}_{iA_1}^{A_{d+1}}.
\end{aligned}$$

Thus, equation (3.56) implies that

$$\begin{aligned}
N^{A_{d+1}A_2\cdots A_{d-1}A_d} \nabla_i N_{A_1A_2\cdots A_{d-1}A_d} &= -d \cdot N^{A_{d+1}A_2\cdots A_{d-1}A_d} \mathbb{L}_{[A_d}^{A_0} N_{A_1A_2\cdots A_{d-1}]A_0} \\
&= -d \cdot \frac{(d-1)!}{d} \cdot \mathbb{L}_{iA_1}^{A_{d+1}} \\
&= -(d-1)! \cdot \mathbb{L}_{iA_1}^{A_{d+1}}.
\end{aligned}$$

□

The results above may also be shown by contracting the normal form into (3.55) and comparing with (3.31).

We now have all the required results to prove Theorem 1.1.

*Proof of Theorem 1.1.* As remarked above, the equivalence of 1 and 2 follows from Proposition 3.10 and Lemma 3.12. The equivalence of 1 and 3 follows from Theorems 3.20 and 3.21 above. The equivalence of 3 and 4 is a trivial consequence of the definition of the Hodge- $\star$  (see (2.27)), the fact that the volume tractor (see (2.26)) is parallel, and that  $\star\star$  is  $\pm$  the identity.  $\square$

**Remark 3.22.** *Many of the results concerning submanifold tractors that were derived in the above sections used nothing more than a local orthonormal basis for the normal bundle and the Gauß formula. Since the normal tractor bundle is isomorphic to the usual normal bundle, and we have a Gauß formula in both cases, such proofs of these results may be repeated mutatis mutandis for the Riemannian objects to yield analogous statements and formulae; the one caveat being that one should keep in mind that the connection  $\tilde{\nabla}$  on  $\mathcal{T}\Sigma$  induced from the ambient tractor connection differs in general from the submanifold tractor connection  $D$  by (3.33).*

### 3.8 Submanifold invariants

We have seen above that trace-free second fundamental form arises from using the (ambient) tractor connection acting on  $N_B^A$ . More generally the tools we have developed can be used to proliferate submanifold invariants in obvious ways. We sketch some routes.

Let us fix some submanifold  $\Sigma$ , as usual of dimension  $1 \leq m \leq n - 1$  and codimension  $d$ , in a conformal manifold  $(M, \mathbf{c})$ . Let us write  $\overline{\mathbb{D}}$  for the Thomas operator of the intrinsic conformal structure  $(\Sigma, \mathbf{c})$ . This is given by the formula (2.21) except that we couple the tractor connection  $\nabla^{\mathcal{T}\Sigma}$  to the intrinsic Levi-Civita connection  $D_i$  and replace  $n$  with  $m$ . Also in dimensions  $m = 1, 2$  we replace  $J$  with  $j$  as described in Section 3.5. In fact it is straightforward to verify this formula (2.21) provides a conformally invariant operator if we couple the Levi-Civita connection to *any* invariant connection on any vector bundle. (The key point is that verifying its conformal invariance does not involve commuting any derivatives.) To exploit this observation, we will write  $\overline{\mathbb{D}}$  also for the conformally invariant operator given by the same formula, but where the intrinsic Levi-Civita connection is coupled to any invariant connection. In practice here, the latter will be the ambient tractor connection as well as also the intrinsic tractor connection on  $\mathcal{T}\Sigma$ .

For example, along  $\Sigma$ ,  $\overline{\mathbb{D}}_B N_D^C$  is well defined and conformally invariant, as is  $\overline{\mathbb{D}}_A \overline{\mathbb{D}}_B N_D^C$ . Similarly we may instead use the normal form  $N_{F_1 \dots F_d}$ . And this comes to the main point. The collection

$$N_{F_1 \dots F_d}, \overline{\mathbb{D}}_E N_{F_1 \dots F_d}, \overline{\mathbb{D}}_C \overline{\mathbb{D}}_E N_{F_1 \dots F_d}, \overline{\mathbb{D}}_B \overline{\mathbb{D}}_C \overline{\mathbb{D}}_E N_{F_1 \dots F_d}, \dots$$



embeds the jets of the submanifold into sections of tractor bundles in a conformally invariant way, up to any desired order. These objects can then be contracted or partially contracted to produce non-linear invariants. For example

$$(\overline{\mathbb{D}}^C \overline{\mathbb{D}}^E N_B^A)(\overline{\mathbb{D}}_C \overline{\mathbb{D}}_E N_A^B)$$

is a non-trivial scalar conformal invariant of submanifolds for most dimensions  $m$ . Similarly (for  $m \geq 4$ ) we may form

$$\overline{W}^{CDEF} (\overline{\mathbb{D}}_D \overline{\mathbb{D}}_F N_B^A)(\overline{\mathbb{D}}_C \overline{\mathbb{D}}_E N_A^B),$$

where  $\overline{W}$  is the  $W$ -tractor, as defined in [48], but for the intrinsic geometry of the submanifold  $\Sigma$ . In the parlance of invariant theory such obvious complete contractions are called scalar *Weyl invariants* [6]. A slightly more subtle construction uses the idea of *quasi-Weyl invariants*, as in [48], but this will still proceed using the tools developed here. Indeed the results from [48] (for conformal invariants) suggest it is likely that these techniques would, in a suitable sense, produce almost all scalar invariants.

The construction of tensor-valued invariants is slightly more complicated, and involves ideas as here plus the use of differential splitting operators that map (in a conformally invariant way) between tensor and tractor bundles (see, e.g., Theorem 6.1 and (6.8) in Section 6). Some applications of these for the construction of hypersurface invariants are given in [8].

## 4 Characterising and generalising mean curvature, and applications

Note that for a given submanifold  $\Sigma$ , in a Riemannian manifold  $(M, g)$ , its mean curvature vector  $H^a \in \Gamma(N\Sigma[-1])$  can equivalently be captured by the *mean curvature tractor*

$$H^A := \sigma N_a^A H^a \in \Gamma(\mathcal{N})$$

via the isomorphism of Lemma 3.3 and where  $\sigma \in \Gamma(\mathcal{E}[1])$  is the scale giving  $g$ , that is  $g = \sigma^{-2}\mathbf{g}$ . With this terminology and notation, we can state the following result.

**Proposition 4.1.** *Let  $\Sigma$  be a submanifold in a Riemannian manifold  $(M, g)$ . Then*

$$H^A = N_B^A I^B. \quad (4.1)$$

*Proof.* In the scale of the metric  $g = \sigma^{-2}\mathbf{g}$  the scale tractor takes to form  $I^A = \sigma Y^A + \rho X^A$ , for some weight density  $\rho$ , or weight  $-1$ . So from formula (3.30) in Lemma 3.13, we see that  $N_B^A X^B = 0$  and

$$N_B^A I^B = \sigma H^a Z_a^A + \sigma(H^b H_b)X^A. \quad (4.2)$$

But, from (3.15), this is exactly  $H^A$ . □

Thus *minimal submanifolds*, meaning those with  $H^a = 0$ , are nicely captured by orthogonality of the scale tractor to the normal tractors, as follows (as was known in the case of hypersurfaces [52]).

**Corollary 4.2.** *In a Riemannian manifold  $(M, g)$ , let  $I$  denote the scale tractor of  $g$ . A submanifold  $\Sigma$ , of dimension  $m$ , is minimal if and only if, one of the following equivalent conditions holds*

1.  $I^A N_A^B = 0$  ;
2.  $I^{A_1} N_{A_1 A_2 \dots A_d} = 0$ ;
3.  $I^{[A_0} \star N^{A_1 A_2 \dots A_{m+2}]} = 0$ ;
4.  $I \in \Gamma(\mathcal{N}^\perp)$ ;
5.  $H^A I_A = 0$ .

**Remark 4.3.** *Corollary 4.2 here generalizes Theorem 2 from [55], as a minimal 1-dimensional submanifold in a Riemannian manifold is exactly a geodesic.*

*Note also that the corollary shows that for a minimal submanifold  $\Sigma$  the ambient scale tractor  $I_A$  can, along  $\Sigma$ , be identified with a section of the intrinsic tractor bundle  $\mathcal{T}\Sigma$  via (3.18) of Theorem 3.5.*

It is natural to say that a Riemannian submanifold has *constant mean curvature* (CMC) if the function

$$\sigma^2 H^a H_a \in \Gamma(\mathcal{E}[0]|_\Sigma)$$

is constant on  $\Sigma$ , where  $\sigma$  is the scale of the metric  $g$  used to calculate the mean curvature (the reader is cautioned that this is only standard terminology for the case of hypersurfaces; in higher codimension there are other possibilities for the definition of CMC). We will say that a  $\Sigma$  has *parallel mean curvature* if

$$\nabla_i^\perp H^b = 0,$$

or equivalently  $\nabla_i^\perp(\sigma H^b) = 0$ ; clearly this is stronger than the CMC condition. These notions are also usefully captured by tractors.

**Proposition 4.4.** *In a Riemannian manifold  $(M, g)$ , let  $I$  denote the scale tractor of  $g$ . A submanifold  $\Sigma$ :*

- *is CMC if and only if*

$$N_{AB} I^A I^B, \quad \text{or equivalently,} \quad H^A I_A$$

*is constant along  $\Sigma$ ;*

- *has parallel mean curvature if and only if*

$$\nabla_i^{\mathcal{N}} H^B = 0, \quad \text{or equivalently,} \quad N_B^A \nabla_i H^B = 0.$$

*Proof.* Continuing in notation and choices of the Proof of Proposition 4.1, the first statement follows by contracting  $I_A = \sigma Y^A + \rho X^A$  into (4.2). The second is immediate from Lemma 3.4.  $\square$

**Remark 4.5.** *Note that if a submanifold  $\Sigma$  in  $(M, g)$  has parallel mean curvature, then it is CMC as, in the scale of the metric  $g = \sigma^{-2} \mathbf{g}$ ,*

$$\nabla_i(\sigma^2 H^a H_a) = 2\sigma^2 H^a \nabla_i^\perp H_a.$$

*The converse does not hold. For example in Euclidean 3-space a round 2-circle (say in the  $x - y$ -plane) has is parallel mean curvature (and so is also CMC). But a regular spiral is CMC (by dint of its invariance under the obvious group action) but does not have parallel mean curvature.*

*Note that the stronger notion of parallel mean curvature*

$$\nabla_i(\sigma H^a) = 0$$

*implies CMC also. Thus  $|\sigma H| = \sqrt{\sigma^2 H^a H_a}$  is constant and  $\sigma H^a = |\sigma H| \hat{n}^a$  for some a unit normal along  $\Sigma$  that must be parallel. Such a parallel unit normal means that the acceleration of any curve in  $\Sigma$  is orthogonal to  $\hat{n}$ , so the second fundamental form and  $H^a$  are orthogonal to  $\hat{n}$ . But the latter obviously means  $H^a = 0$ .*

Part of the importance of Proposition 4.1, Corollary 4.2, and Proposition 4.4, is that in means that these quantities and notions at once extend to the setting of almost-Riemannian manifolds (as defined in Section 2.4). For emphasis we make this a definition.

**Definition 4.6.** *Let  $(M, \mathbf{c}, I)$  be an almost-Riemannian manifold. For any embedded submanifold  $\Sigma$  we define  $H^A = N_B^A I^B$  to be its mean curvature. Then we say it is, respectively, CMC or has parallel mean curvature if one of the conditions displayed in Proposition 4.4 holds. Similarly we say that it is minimal if any one of the equivalent conditions of Corollary 4.2 holds.*

For an almost-Riemannian manifold  $(M, \mathbf{c}, I)$ , the zero locus  $\mathcal{Z}(\sigma)$ , of  $\sigma := X^A I_A$ , is (closed and) nowhere dense. Thus, by continuity, the notions in the definition extend those on  $M \setminus \mathcal{Z}(\sigma)$ , as in the following proposition.

**Proposition 4.7.** *Let  $(M, \mathbf{c}, I)$  be an almost-Riemannian manifold and  $\sigma := X^A I_A$ . Then an embedded manifold  $\Sigma$  is minimal, CMC, or mean curvature parallel in the sense of Definition 4.6 if and only if satisfies the corresponding condition (in the non-generalised sense) on  $M \setminus \mathcal{Z}(\sigma)$  for the metric  $g := \sigma^{-2} \mathbf{g}$ .*

This perspective enables an easy recovery of the following result, which is well-known from other perspectives.

**Proposition 4.8.** *On a conformally compact manifold, any minimal submanifold that extends smoothly to the boundary meets the boundary orthogonally.*

*Proof.* On a conformally compact manifold  $\partial M = \mathcal{Z}(\sigma)$ ,

$$I^A|_{\partial M} = (\nabla^a \sigma) Z_a^A - \frac{1}{n} \Delta \sigma X^A, \quad (4.3)$$

and  $\nabla_a \sigma$  is nowhere-zero along the boundary. See Section 2.4. Thus if  $\Sigma$  meets  $\partial M$  then we have

$$I_A N_B^A = N_B^a \nabla_a \sigma \quad \text{along } \Sigma,$$

(using (3.30)) and so  $\Sigma$  minimal, meaning  $I_A N_B^A = 0$ , implies

$$N_B^a \nabla_a \sigma = 0, \quad \text{whence, } N_b^a \nabla_a \sigma = 0.$$

That is  $\nabla_a \sigma$  (the conormal to the boundary  $\partial M$ ) is orthogonal to the normal projector of  $\Sigma$ .  $\square$

Suppose now that  $(M, \mathbf{c}, I)$  is an almost-Einstein manifold. If  $\Sigma$  is minimal then, as observed above,  $I_A$  may be identified with a submanifold tractor. Since  $I_A$  is parallel for the standard tractor connection, and  $I_A$  is a submanifold tractor,  $I_A$  is also parallel for the connection  $\check{\nabla}$ :

$$\check{\nabla}_i I_J = \Pi_J^A \nabla_i (\Pi_A^K I_K) = \Pi_J^A \nabla_i I_A = 0,$$

as defined in (3.21). Therefore, from the decomposition (3.33), one sees that  $I_J$  is parallel for the submanifold tractor connection if, and only if,  $S_i^J{}_K I^K = 0$ .

Choosing a background scale to split the tractor bundles, we have that

$$\begin{aligned} S_i^J{}_K I^K &= \mathcal{F}_{ij} \left( Z^{Jj} X_K - Z_K^j X_J \right) \left( \sigma Y^K + \nabla_k \sigma Z^{Kk} - \frac{1}{n} (\Delta + J\sigma) X^K \right) \\ &= \mathcal{F}_{ij} \left( \sigma Z^{Jj} - \nabla^j \sigma X^J \right). \end{aligned}$$

Recall that on almost-Einstein manifolds the 1-jet  $j^1\sigma$  can only vanish at isolated points (see the discussion of Section 2.4, and references therein, for details). Therefore away from these points we must have  $\mathcal{F}_{ij} = 0$ , and then also at those points by continuity. Thus we have the following result.

**Proposition 4.9.** *Let  $\Sigma \hookrightarrow M$  be a minimal submanifold of an almost-Einstein manifold  $(M, \mathbf{c}, I)$ . Then  $\Sigma$  is almost-Einstein if, and only if,  $\mathcal{F}_{ij} = 0$ .*

In a conformal or Riemannian manifold a distinguished submanifold  $\Sigma$  is necessarily totally umbilic. Thus if, in a Riemannian manifold,  $\Sigma$  is distinguished and minimal then it is totally geodesic. This has a converse if the Riemannian manifold is Einstein.

**Proposition 4.10.** *Let  $(M, \mathbf{c}, I)$  be an almost-Einstein manifold. Suppose that  $\Sigma$  is a submanifold of codimension  $d$ . If  $\Sigma$  is an umbilic, minimal submanifold, then  $\Sigma$  is a distinguished submanifold.*

*Proof.* We must show any of the equivalent conditions of Theorem 1.1.

That  $\Sigma$  is minimal implies that  $H^B = 0$  on  $M$ . Since also  $\Sigma$  is totally umbilic it follows that on  $M \setminus \mathcal{Z}(\sigma)$  we have that it is totally geodesic, and hence  $\nabla_i N_{a_1 a_2 \dots a_d} = 0$ . The almost-Einstein condition implies that  $N_{ba_2 \dots a_d} P_i^b = 0$  on  $M \setminus \mathcal{Z}(\sigma)$ , where we calculate in the scale of the metric  $g^\sigma = \sigma^{-2} \mathbf{g}$ , with  $\sigma := X^A I_A$ . Combining these observations and using formula (3.56) we have that  $\nabla_i N_{A_1 A_2 \dots A_d} = 0$  on  $M \setminus \mathcal{Z}(\sigma)$ . But then by continuity  $\nabla_i N_{A_1 A_2 \dots A_d} = 0$  on  $M$ , as  $Z(\sigma)$  is nowhere dense.  $\square$

Thus if our ambient space is almost-Einstein, for submanifolds that are minimal and umbilic (so totally geodesic, in the generalised sense) first integrals may be proliferated using Corollary 1.4 (and the theory to be developed in Section 6). These conserved quantities will extend to/across singularity sets of these geometries where they exist.

## 5 Distinguished submanifolds and conformal circularity

We fix some notational conventions for this chapter. We will denote by  $\gamma$  a smooth curve in a conformal manifold  $(M, \mathbf{c})$ . By this we will here mean a smooth, regular curve  $\gamma : I \rightarrow M$  (for some interval  $I$ ). We will often identify  $\gamma$  with its image and we typically assume that this an embedded submanifold.

The symbols  $u^b$  and  $a^b$  will denote, respectively, the velocity and acceleration the curve  $\gamma$ , so  $a^b = u^a \nabla_a u^b$ . Note that the acceleration  $a^b$  depends on a choice of metric  $g \in \mathbf{c}$  and is not conformally invariant; it is easy to check that if  $\hat{g} = \Omega^2 g$  then  $\hat{a}^b = a^b - u_a u^a \Upsilon^b + 2u^a \Upsilon_a u^b$ , where  $\Upsilon_a = \Omega^{-1} \nabla_a \Omega$ . We also define  $u := \sqrt{\mathbf{g}_{ab} u^a u^b} \in \Gamma(\mathcal{E}[1]|_\gamma)$ . For some connection  $\nabla$ , we will also use the notation  $\frac{d^\nabla}{dt}$ , or  $\frac{d}{dt}$  when the meaning is clear by context, to mean  $u^a \nabla_a$ . The connection  $\nabla$  may be a Levi-Civita connection or the standard tractor connection; this should be unambiguous from context. Sometimes we will prefer to work with *weighted* versions of the velocity and acceleration vectors. These will be denoted by  $\mathbf{u}^b := u^{-1} u^b \in \Gamma(\mathcal{E}^b[-1]|_\gamma)$  and  $\mathbf{a}^b := \mathbf{u}^c \nabla_c \mathbf{u}^b \in \Gamma(\mathcal{E}^b[-2]|_\gamma)$  respectively.

### 5.1 Background on conformal circles

A smooth curve  $\gamma$  is said to be a (*projectively parametrised*) *conformal circle* if, with respect to some (equivalently any) choice of  $g \in \mathbf{c}$ , its velocity and acceleration satisfy [4]

$$u^c \nabla_c a^b = u^2 u^c P_c^b + 3u^{-2} (u_c a^c) a^b - \frac{3}{2} u^{-2} (a_c a^c) u^b - 2u^c u^d P_{cd} u^b, \quad (5.1)$$

where  $u^2 = u \cdot u = u_a u^a$  here should be understood to be *unweighted*. Equation (5.1) is a third order, conformally invariant analog of the geodesic equation in Riemannian geometry and solutions of (5.1) are sometimes referred to as *conformal geodesics* [44] (we prefer *conformal circles*). As with the geodesic equation  $u^c \nabla_c u^b = 0$ , equation (5.1) can be broken up into its tangential and normal components along the curve and any curve in  $M$  can be parametrized so that the tangential part of (5.1) holds; such a parametrisation is determined up to the action of  $\mathrm{PSL}(2, \mathbb{R})$  and a curve with such a parametrisation is said to be *projectively parametrised* [4]. (The existence of such a parametrisation, and likely the notion of conformal circles also, goes back to Élie Cartan; see [24]. For early treatments of conformal circles, see [39, 44, 85, 86, 89, 100, 101]. In the literature conformal circles are sometimes taken to be parametrised by arclength with respect to chosen metric  $g$  rather than projectively parametrised, and in this case they satisfy a slightly different equation; see, e.g. [101, Chapter VII, §2].) Asking only that the normal (to the curve) part of (5.1) holds gives a notion of conformal circles that does not depend on the parametrisation, and any such curve can be reparametrised so that (5.1) holds.

Note that a geodesic for a metric  $g \in \mathbf{c}$  need not be a conformal circle; indeed, this could not be the case since any curve  $\gamma$  in  $M$  is locally an affinely parametrised geodesic for some choice of metric  $g \in \mathbf{c}$  (see Remark 3.2). Following this line of thought, however, one sees as a direct consequence of (5.1) that a curve  $\gamma$  is a projectively parametrized conformal circle if and only if there is a metric  $g \in \mathbf{c}$  with respect to which  $\gamma$  is an affinely parametrised geodesic and  $u^c P_c^b = 0$  [4]. Note also that in the special case where one has an Einstein metric  $g$  in the conformal class it follows from (5.1) that geodesics for  $g$  are conformal circles, though the unit speed parametrization is not a projective parametrization except in the Ricci flat case.

The notion of conformal circles arises naturally from the Cartan geometric description of conformal structures in dimensions  $n \geq 3$  (and Möbius conformal structures in two dimensions) and as such it is natural that they can be simply described using tractor calculus (the corresponding calculus of associated bundles). With this in mind we now introduce some important tractor fields associated to the curve  $\gamma$ . Recall that the canonical tractor  $X^B$  can be viewed as a section of  $\mathcal{E}^B[1]$ . Hence  $u^{-1}X^B$  is an unweighted tractor along the curve and so the tractor covariant derivative of  $u^{-1}X^B$  along the curve is well defined (conformally invariant). Following [5], we define

$$U^B := u^a \nabla_a (u^{-1}X^B) \quad (5.2)$$

and

$$A^B := u^a \nabla_a U^B, \quad (5.3)$$

which we call the *velocity* and *acceleration tractors* respectively. Explicitly, one has

$$U^B \stackrel{g}{=} \begin{pmatrix} 0 \\ u^{-1}u^b \\ -u^{-3}(u_c a^c) \end{pmatrix} \quad (5.4)$$

and

$$A^B \stackrel{g}{=} \begin{pmatrix} -u \\ u^{-1}a^b - 2u^{-3}(u_c a^c)u^b \\ -u^{-3}(u_c \frac{da^c}{dt}) - u^{-3}a_c a^c + 3u^{-5}(u_c a^c)^2 - u^{-1}P_{cd}u^c u^d \end{pmatrix}. \quad (5.5)$$

It is easily checked that

$$U^B U_B = 1, \quad U^B A_B = 0 \quad (5.6)$$

and that

$$A^B A_B = 3u^{-2}a_b a^b + 2u^{-2}u_b u^c \nabla_c a^b - 6u^{-4}(u_c a^c)^2 + 2P_{ab}u^a u^b. \quad (5.7)$$

Consequently, a curve  $\gamma : I \rightarrow M$  is projectively parametrised if, and only if,  $A^B A_B = 0$ . It was then shown in [5] that a projectively parametrised curve  $\gamma : I \rightarrow M$  is a conformal geodesic if, and only if,

$$\frac{d^\nabla A^B}{dt} = 0. \quad (5.8)$$

More recently, it was shown by the second and third named authors and Taghavi-Chabert [56] that a curve  $\gamma$  is an unparametrised conformal circle if, and only if,  $d^\nabla A^B/dt$  is zero modulo  $U^B$  and  $X^B$ ; given the definitions of the velocity and acceleration tractors this is equivalent to saying that  $\gamma$  is an unparametrised conformal circle if, and only if, the 3-tractor

$$\Phi^{ABC} := 6u^{-1}X^A U^B A^C \quad (5.9)$$

is covariantly constant along  $\gamma$ . To see this we note the following: Firstly,

$$\Phi^{ABC} = 6\mathbf{u}^c X^{[A} Y^B Z^C] + 6\mathbf{u}^b \mathbf{a}^c X^A Z_b^B Z_c^C, \quad (5.10)$$

where  $\mathbf{u}^c = u^{-1}u^c$  (so that  $\mathbf{g}_{ab}\mathbf{u}^a\mathbf{u}^b = 1$ ) and  $\mathbf{a}^c = \mathbf{u}^b \nabla_b \mathbf{u}^c = u^{-2}a^c - u^{-4}(u_b a^b)u^c$ . It is then easy to show that

$$u^d \nabla_c \Phi^{ABC} = 6 \left( \mathbf{u}^d \nabla_d \mathbf{a}^c - \mathbf{u}^d P_d^c \right) \mathbf{u}^b X^{[A} Z_b^B Z_c^C]. \quad (5.11)$$

On the other hand, the requirement that the normal (to the curve) part of (5.1) holds can be written in terms of the weighted velocity and acceleration as [56, Lemma 4.9]

$$\left( \mathbf{u}^d \nabla_d \mathbf{a}^{[b} \right) \mathbf{u}^{c]} = \mathbf{u}^d P_d^{[b} \mathbf{u}^{c]}, \quad (5.12)$$

from which the claim follows.

Note that from (5.10) one can easily check that  $\Phi^{ABC}\Phi_{ABC} = 6$ , and that  $\Phi_{ABC}N^A = 0$  for any section  $N^A$  of the normal tractor bundle to  $\gamma$  (the easiest way to see the latter is to compute in a minimal scale  $g$  for  $\gamma$ , equivalently, a scale for which  $\mathbf{a}^c = 0$ ). It follows that

$$\Phi_{ABC} = \star N_{ABC}^{\gamma \hookrightarrow M}, \quad (5.13)$$

the Hodge star of the tractor normal form of  $\gamma$  as a submanifold of  $M$ .

This observation combined with the result from [56] described in the preceding paragraph shows that Theorem 1.2 follows from Theorem 1.1.

For a related approach to studying conformal circles using tractors, see [90].

## 5.2 Weak conformal circularity of submanifolds

With this background established we begin our discussion of conformal circularity.

**Definition 5.1.** *A submanifold  $\Sigma$  is weakly conformally circular if any  $M$ -conformal circle, whose 2-jet at a point lies in  $\Sigma$ , remains in  $\Sigma$ . That is, if  $\gamma$  is an  $M$ -conformal circle whose 2-jet at some point  $p$  lies in  $\Sigma$  (with  $\gamma(0) = p$ ), then  $\gamma(t) \in \Sigma$  for all  $t$ .*



In the model case of the conformal sphere (see Section 2.3) both conformal circles and totally umbilic submanifolds arise (by ray projectivisation) from the intersections of suitable linear subspaces with the forward null cone. Thus the conformal circles are all given by (transverse) intersections of totally umbilic submanifolds, and a submanifold of the conformal sphere is weakly conformally circular if and only if it is totally umbilic. It is natural to ask to what extent these facts generalize to the curved setting. In this case one quickly sees that the condition of being totally umbilic must be replaced with the stronger condition of being a distinguished submanifold (in the conformally flat case for submanifolds of dimension greater than one the vanishing of  $\mathring{I}\mathring{I}$  is equivalent to the vanishing of the tractor second fundamental form  $\mathbb{L}$ , but this is no longer true in general conformal manifolds; see Remark 5.7). If two distinguished submanifolds intersect transversally in a 1-dimensional submanifold  $\gamma$  then, since the wedge product of the two corresponding normal tractors must be parallel along  $\gamma$ ,  $\gamma$  must be a conformal circle (but, due to the sparsity of distinguished submanifolds in the curved setting, conformal circles no longer arise this way in general). An extension of this idea shows that a submanifold is weakly conformally circular if, and only if, it is distinguished. That is the content of the following theorem.

**Theorem 5.2.** *A submanifold  $\Sigma \hookrightarrow M$  is weakly conformally circular if, and only if,  $\mathbb{L}_{iJ}^C = 0$ .*

*Proof.* A one dimensional submanifold  $\Sigma$  is weakly conformally circular if, and only if, it is a conformal circle. Thus in the one dimensional case the result follows immediately from Theorem 1.2, which states that a curve is a conformal circle if, and only if, when viewed as a submanifold its tractor second fundamental form  $\mathbb{L}$  vanishes.

Suppose now that  $\Sigma$  has dimension at least 2, and is weakly conformally circular. Let  $\gamma$  be an  $M$ -conformal circle whose 2-jet at  $p \in \Sigma$  lies in  $\Sigma$ . Then by assumption  $\gamma$  remains in  $\Sigma$ . We need to introduce some notation. Let

- $N_{A_1 A_2 \dots A_d}^{\Sigma \hookrightarrow M}$  be the normal form of  $\Sigma \hookrightarrow M$ ,
- $N_{A_1 A_2 \dots A_{n-1}}^{\gamma \hookrightarrow M}$  be the normal form of  $\gamma \hookrightarrow M$ , and
- $N_{A_1 A_2 \dots A_{m-1}}^{\gamma \hookrightarrow \Sigma}$  be the normal form of  $\gamma \hookrightarrow \Sigma$ , where we are identifying the tractor bundle of  $\Sigma$  with  $\mathcal{N}^\perp$ , and hence this form is a section of  $\Lambda^{m-1} \mathcal{T}M|_\Sigma$ .

First we note some important relations between these various normal forms. First, since the curve  $\gamma$  lies in the submanifold  $\Sigma$ , we have

$$N_{\gamma \hookrightarrow \Sigma}^{A_1 B_2 \dots B_{m-1}} N_{A_1 A_2 \dots A_d}^{\Sigma \hookrightarrow M} = 0. \quad (5.14)$$

Second, by using the discussion surrounding equation (3.53), one can easily show that

$$N_{A_1 A_2 \dots A_{m-1}}^{\gamma \hookrightarrow \Sigma} \wedge N_{A_m \dots A_{n-1}}^{\Sigma \hookrightarrow M} = N_{A_1 A_2 \dots A_{n-1}}^{\gamma \hookrightarrow M}. \quad (5.15)$$

Finally, since  $\gamma$  is an  $M$ -conformal circle, it follows from Theorem 1.2 that

$$u^i \nabla_i N_{A_1 A_2 \dots A_{n-1}}^{\gamma \hookrightarrow M} = 0.$$

Therefore, using the above,

$$\left( u^i \nabla_i N_{[A_1 A_2 \dots A_{m-1}]}^{\gamma \hookrightarrow \Sigma} \right) N_{A_m \dots A_{n-1}}^{\Sigma \hookrightarrow M} + N_{[A_1 A_2 \dots A_{m-1}]}^{\gamma \hookrightarrow \Sigma} \left( u^i \nabla_i N_{A_m \dots A_{n-1}}^{\Sigma \hookrightarrow M} \right) = 0,$$

and hence

$$\left( u^i \nabla_i N_{[A_1 A_2 \dots A_{m-1}]}^{\gamma \hookrightarrow \Sigma} \right) N_{A_m \dots A_{n-1}}^{\Sigma \hookrightarrow M} + N_{[A_1 A_2 \dots A_{m-1}]}^{\gamma \hookrightarrow \Sigma} \left( -d \cdot u^i \mathbb{L}_{i A_{n-1}}^{A_0} N_{A_m \dots A_{n-2}] A_0}^{\Sigma \hookrightarrow M} \right) = 0$$

by Theorem 3.20, where  $\mathbb{L}$  is the tractor second fundamental form of  $\gamma$ . Since the downstairs tractor index on  $\mathbb{L}_{i A_{n-1}}^{A_0}$  is “tangential to  $\Sigma$ ” it is easy to see that the two terms on the left hand side of the above displayed equation lie in complementary subspaces of the bundle of tractor  $(n-1)$ -forms (the first term is in the ideal generated by  $N^{\Sigma \hookrightarrow M}$  and the second term is in the orthogonal complement to this ideal) and hence both terms must vanish. Thus, in particular,

$$N_{[A_1 A_2 \dots A_{m-1}]}^{\gamma \hookrightarrow \Sigma} u^i \mathbb{L}_{i A_{n-1}}^{A_0} N_{A_m \dots A_{n-2}] A_0}^{\Sigma \hookrightarrow M} = 0.$$

Contracting the above display with  $N_{\Sigma \hookrightarrow M}^{A_m \dots A_{n-2} B}$  (cf. the proof of Theorem 3.21) then gives

$$N_{[A_1 A_2 \dots A_{m-1}]}^{\gamma \hookrightarrow \Sigma} u^i \mathbb{L}_{i A_{n-1}}^B = 0,$$

which is equivalent to

$$u^i \mathbb{L}_{i A}^B \Pi^{\gamma \hookrightarrow \Sigma} A_C = 0, \quad (5.16)$$

where  $\Pi^{\gamma \hookrightarrow \Sigma}$  is the projector onto the rank 3 tractor bundle of the 1-manifold  $\gamma$ , viewed as a subbundle of the ambient tractor bundle along  $\gamma$ . Therefore it follows that

$$u^i \mathbb{L}_{i A}^B U^A = 0,$$

where  $U^A$  is the velocity tractor of the curve (note that  $U^A$  may be viewed as a section of the intrinsic standard tractor bundle of  $\gamma$ ; one can easily check this by working in a minimal scale for  $\gamma$ , where  $u_b a^b = 0$ ). Using Theorem 3.14 we now see that, in particular,  $\mathring{H}_{ij}^c u^i u^j = 0$ . But the above must hold for any  $M$ -conformal circle  $\gamma$ , and hence  $\mathring{H}_{ij}^c u^i u^j = 0$  for all  $u^i \in \Sigma(\mathcal{E}^i)$ , whence  $\Sigma$  is totally umbilic by polarization.

Since we have already seen that  $\mathring{H}_{ij}^c = 0$ , it suffices to show that  $N_b^c (P_i^b - \nabla_i H^b) = 0$ . Returning to (5.16), if we contract this with  $Y^A$  (or  $\Pi_I^A Y^I$ , cf. (3.20)) this gives

$$u^i N_b^c \left( P_i^b - \nabla_i H^b \right) Z_c^B + u^i H_b \left( P_i^b - \nabla_i H^b \right) X^B = 0,$$

by Theorem 3.14, since the other slots of  $\mathbb{L}$  have already been shown to be zero. Since, again, this must hold for all  $u^i \in \Sigma(\mathcal{E}^i)$  we obtain the result.

For the converse, let us consider a curve  $\gamma$  in  $\Sigma$  that satisfies

$$u^i \check{\nabla}_i N_{A_1 A_2 \dots A_{m-1}}^{\gamma \hookrightarrow \Sigma} = 0 \quad \text{and} \quad A^B A_B = 0 \quad (5.17)$$

where we have used the connection  $\check{\nabla}$ . Then, equivalently,

$$\left( u^i \nabla_i N_{[A_1 A_2 \dots A_{m-1}]}^{\gamma \hookrightarrow \Sigma} \right) N_{A_m \dots A_{n-1}}^{\Sigma \hookrightarrow M} = 0.$$

Suppose now that  $\mathbb{L}_{iJ}^C = 0$ , then

$$\left( u^i \nabla_i N_{[A_1 A_2 \dots A_{m-1}]}^{\gamma \hookrightarrow \Sigma} \right) N_{A_m \dots A_{n-1}}^{\Sigma \hookrightarrow M} + N_{[A_1 A_2 \dots A_{m-1}]}^{\gamma \hookrightarrow \Sigma} \left( -d \cdot u^i \mathbb{L}_{iA_{n-1}}^{A_0} N_{A_m \dots A_{n-2}]A_0}^{\Sigma \hookrightarrow M} \right) = 0,$$

and so

$$u^i \nabla_i N_{A_1 A_2 \dots A_{n-1}}^{\gamma \hookrightarrow M} = 0.$$

That is, if  $\gamma$  in  $\Sigma$  satisfies (5.17) and  $\mathbb{L}_{iJ}^C = 0$  then  $\gamma$  is a conformal circle for  $(M, \mathbf{c})$ . Now, it is straightforward to check that a curve satisfying (5.17) that is further projectively parametrised with respect to the conformal structure on  $M$  (meaning its  $M$ -acceleration tractor satisfies  $A^B A_B = 0$ ) is determined by its 2-jet in  $\Sigma$  at any point on its path. This follows because, by construction, a projectively parametrised curve satisfying (5.17) is characterised by a third order ordinary differential equation in any local coordinate chart (analogous to how the  $(\Sigma, \mathbf{c}_\Sigma)$ -conformal circle equation is equivalent to the requirement that the curve be  $\Sigma$ -projectively parametrised and satisfy  $u^i D_i N_{A_1 A_2 \dots A_{m-1}}^{\gamma \hookrightarrow \Sigma} = 0$ , and the  $(M, \mathbf{c})$ -conformal circle equation is equivalent to the requirement that the curve be  $M$ -projectively parametrised and satisfy  $u^a \nabla_a N_{A_1 A_2 \dots A_{n-1}}^{\gamma \hookrightarrow M} = 0$ ; cf. Remark 5.3 below). Now, suppose  $\mathbb{L}_{iJ}^C = 0$ . Then, given any 2-jet of a curve in  $\Sigma$ , the corresponding  $M$ -projectively parametrised solution of (5.17) is also an  $M$  conformal circle. Moreover, since a conformal circle in  $(M, \mathbf{c})$  is determined by its 2-jet at any point on its path, all conformal circles corresponding to two jet initial data lying in  $\Sigma$  arise this way; in particular, all such curves lie in  $\Sigma$ . That is,  $\Sigma$  is weakly conformally circular.  $\square$

**Remark 5.3.** *Here we give a version of the proof that  $\mathbb{L}_{iJ}^C = 0$  implies weak conformal circularity that avoids the use of tractor calculus. (This is along the lines of a proof of a similar result in [7], to an extent the tractor picture provides a conceptual basis for the idea.) For convenience, we work in a minimal scale  $g$ . Rather than considering curves  $\gamma$  in  $\Sigma$  solving (5.17) we will consider the curves  $\gamma$  in  $\Sigma$  solving the adapted conformal circle equation*

$$\frac{d^D a^j}{dt} = u^2 \cdot u^i P_i^j + 3u^{-2} \left( u_k a^k \right) a^j - \frac{3}{2} u^{-2} \left( a_k a^k \right) u^j - 2u^k u^l P_{kl} u^j, \quad (5.18)$$

where as usual  $P_i^j$  and  $P_{kl}$  denote the restriction of the ambient Schouten tensor to the intrinsic tangent and cotangent bundles, and  $\frac{d^D}{dt}$  denotes  $u^i D_i$  where  $D$  is

the intrinsic Levi-Civita connection for the pullback  $g_\Sigma$  of the ambient scale  $g$ . We say that  $\gamma$  is an *adapted conformal circle* if it satisfies this equation. (That equation (5.18) is equivalent to (5.17) should be clear from what follows, but we do not need this for the argument given in this remark.) Note that equation (5.18) is a third order ODE on  $\Sigma$ , and therefore the initial value problem with given 2-jet initial data has a unique solution on  $\Sigma$  for some interval centered at 0. This solution may also be viewed as a curve in  $M$ , and one may ask whether it solves a related ODE there. Since the 2-jet of  $\gamma$  is initially tangential and  $\mathbb{L}_{iJ}^C = 0$  implies in particular that  $\mathring{\mathbb{I}}_{ij}^c = 0$ , it follows that in our minimal scale  $\Pi_j^b u^j = u^b$ ,  $\Pi_j^b a^j = a^b$  and  $\Pi_j^b \frac{d^P a^j}{dt} = \frac{d^\nabla a^b}{dt}$  (the last two identities being consequences of the Gauss formula with  $\mathbb{I} = 0$ ). Thus  $u_k a^k = u_c a^c$ ,  $a_k a^k = a_c a^c$  and  $u^k u^l P_{kl} = u^c u^d P_{cd}$ . Moreover,  $\mathbb{L}_{iJ}^C = 0$  also implies (again for the minimal scale) that  $N_b^c P_i^b = 0$ , and thus one easily sees that, as a curve in  $M$ ,  $\gamma$  satisfies

$$\begin{aligned} \frac{d^\nabla a^b}{dt} &= u^2 \left( \Pi_j^b u^i P_i^j \right) + 3u^{-2} (u_c a^c) a^b - \frac{3}{2} u^{-2} (a_c a^c) u^b - 2u^c u^d P_{cd} u^b \\ &= u^2 \cdot u^c P_c^b + 3u^{-2} (u_c a^c) a^b - \frac{3}{2} u^{-2} (a_c a^c) u^b - 2u^c u^d P_{cd} u^b, \end{aligned}$$

which is exactly the (projectively parametrized)  $M$ -conformal circle equation. So if  $\mathbb{L}_{iJ}^C = 0$ , and  $\gamma$  is an adapted conformal circle, then it is an  $M$ -conformal circle, and by the uniqueness of solution to an initial value problem, the curve  $\gamma$ , which lies in  $\Sigma$ , is the unique  $M$ -conformal circle with the given initial conditions. Hence any  $M$ -conformal circle whose 2-jet at a point  $p \in \Sigma$  is tangential will remain in  $\Sigma$ , i.e.  $\Sigma$  is weakly conformally circular.

For his approach to submanifold circularity, Belgun [7] introduced a conformal invariant  $\mu \in \Gamma(T^*\Sigma \otimes N\Sigma)[-2]$  given by

$$\mu_i^c := N_b^c \left( P_i^b - \nabla_i H^b + \frac{1}{m-1} D^j \mathring{\mathbb{I}}_{ij}^b \right), \quad (5.19)$$

when  $m \neq 1$  and where the intrinsic Levi-Civita connection  $D$  is coupled to the normal connection  $\nabla^\perp$  (Belgun terms this the *mixed Schouten-Weyl tensor* since the main term is the tangential-normal part of the ambient Schouten tensor, which he calls the Schouten-Weyl tensor). In the  $m = 1$  case (i.e.  $\Sigma$  is a curve) the same formula except omitting the last term on right hand side of (5.19) defines  $\mu$ . (Recall that for a curve, one has  $\mathring{\mathbb{I}}_{ij}^c = 0$ .) In fact it is straightforward to show (by substituting the Weyl-Schouten decomposition of the ambient curvature tensor into the Codazzi equation [91]) that in the cases  $m \neq 1$ ,

$$\mu_i^c = \frac{1}{m-1} W_{ij}^{dj} N_d^c, \quad (5.20)$$

from which the conformal invariance is clear. In [7] Belgun characterises weakly conformal circular submanifolds (termed *weakly conformal geodesic* in [7]) as those

for which  $\mathring{H} = 0$  and  $\mu = 0$ . Inspecting (3.31), in Theorem 3.8, one sees immediately that this is equivalent to the vanishing of  $\mathbb{L}$ .

**Proposition 5.4.** *Let  $\Sigma \hookrightarrow M$  be a submanifold in a conformal manifold with  $\mathbb{L}_{iJ}^C$  its tractor second fundamental form. Then  $\mathbb{L}_{iJ}^C = 0$  if, and only if,  $\mathring{H}_{ij}^c = 0$  and  $\mu_i^c = 0$ .*

In fact the invariant  $\mu$  arises naturally from  $\mathbb{L}$ . The projecting part of  $\mathbb{L}$  is necessarily invariant and this is  $\mathring{H}$ . An obvious question is whether conversely  $\mathbb{L}$  is then image of a natural linear differential operator acting  $\mathring{H}$ , in which case they would be equivalent. (For example as the tractor curvature is the image of the Weyl curvature in dimensions at least 4.) This leads us to the following lemma, which exhibits a conformally invariant differential operator between the relevant bundles:

**Lemma 5.5.** *There is an invariant map  $\mathbb{M} : S_0^2 T^* \Sigma \otimes N\Sigma \rightarrow T^* \Sigma \otimes \mathcal{T}^* \Sigma \otimes \mathcal{N}$ . Written in tractor projectors, this takes the form*

$$\begin{aligned} \omega_{ij}^c \mapsto \mathbb{M}(\omega)_{ij}^C := & \omega_{ij}^c Z_J^j Z_c^C - \frac{1}{m-1} D^j \omega_{ij}^c X_J Z_c^C \\ & + H_c \omega_{ij}^c Z_J^j X^C - \frac{1}{m-1} H_c D^j \omega_{ij}^c X_J X^C, \end{aligned} \quad (5.21)$$

where again the intrinsic Levi-Civita connection  $D$  is coupled to the normal Levi-Civita connection  $\nabla^\perp$  when acting on  $\omega$ .

*Proof.* Given  $\omega_{ij}^c \in \Gamma(S_0^2 T^* \Sigma \otimes N\Sigma)$ , using (2.8) and (3.12), one computes that

$$\hat{D}^j \omega_{ij}^c = D^j \omega_{ij}^c + (m-1) \Upsilon^j \omega_{ij}^c - \Upsilon_i \omega_{kl}^c \mathbf{g}^{kl} = D^j \omega_{ij}^c + (m-1) \Upsilon^j \omega_{ij}^c \quad (5.22)$$

since  $\omega_{ij}^c$  is trace-free over the pair of indices  $(i, j)$ .

Therefore, using the above together with equations (3.14) and (2.15), and that

the  $X$  tractor is conformally invariant,

$$\begin{aligned}
& \omega_{ij}{}^c \widehat{Z}_J^j \widehat{Z}_c^C - \frac{1}{m-1} \widehat{D}^j \omega_{ij}{}^c \widehat{X}_J \widehat{Z}_c^C + \widehat{H}_c \omega_{ij}{}^c \widehat{Z}_J^j \widehat{X}^C - \frac{1}{m-1} \widehat{H}_c \widehat{D}^j \omega_{ij}{}^c \widehat{X}_J \widehat{X}^C \\
&= \omega_{ij}{}^c \left( Z_J^j + \Upsilon^j X_J \right) \left( Z_c^C + \Upsilon_c X^C \right) \\
&\quad - \frac{1}{m-1} \left( D^j \omega_{ij}{}^c + (m-1) \Upsilon^j \omega_{ij}{}^c \right) X_J \left( Z_c^C + \Upsilon_c X^C \right) \\
&\quad + \left( H_c - N_c^d \Upsilon_d \right) \omega_{ij}{}^c \left( Z_J^j + \Upsilon^j X_J \right) X_C \\
&\quad - \frac{1}{m-1} \left( H_c - N_c^d \Upsilon_d \right) \left( D^j \omega_{ij}{}^c + (m-1) \Upsilon^j \omega_{ij}{}^c \right) X_J X^C \\
&= \omega_{ij}{}^c Z_J^j Z_c^C - \frac{1}{m-1} D^j \omega_{ij}{}^c X_J Z_c^C + \left( H_c \omega_{ij}{}^c - \omega_{ij}{}^c \Upsilon_c + \omega_{ij}{}^c \Upsilon_c \right) Z_J^j X^C \\
&\quad + \left( -\frac{1}{m-1} H_c D^j \omega_{ij}{}^c - H_c \omega_{ij}{}^c \Upsilon^j + \frac{1}{m-1} \Upsilon_c D^j \omega_{ij}{}^c + \omega_{ij}{}^c \Upsilon^j \Upsilon_c - \omega_{ij}{}^c \Upsilon^j \Upsilon_c \right. \\
&\quad \left. - \frac{1}{m-1} \Upsilon_c D^j \omega_{ij}{}^c + H_c \omega_{ij}{}^c \Upsilon^j - \omega_{ij}{}^c \Upsilon^j \Upsilon_c + \omega_{ij}{}^c \Upsilon^j \Upsilon_c \right) X_J X^C \\
&= \omega_{ij}{}^c Z_J^j Z_c^C - \frac{1}{m-1} D^j \omega_{ij}{}^c X_J Z_c^C + H_c \omega_{ij}{}^c Z_J^j X^C - \frac{1}{m-1} H_c D^j \omega_{ij}{}^c X_J X^C,
\end{aligned}$$

which verifies the claimed conformal invariance of the operator  $\mathbb{M}_{cJ}{}^C$ .  $\square$

Asking whether  $\mathbb{L}$  is the image of  $\mathring{H}$  under  $\mathbb{M}$  then immediately leads to the  $\mu$  invariant:

**Theorem 5.6.** *The tensor  $\mu_i{}^c$  is equal to the projecting part of the tractor*

$$\mathbb{L}_{iJ}{}^C - \mathbb{M}(\mathring{H})_{iJ}{}^C. \quad (5.23)$$

*In particular,  $\mu_i{}^c$  is a conformal invariant of the embedding.*

*Proof.* By inspection, one sees that (5.23) has zero in the  $Z_J^j Z_c^C$  slot (since  $Z_j^K Z_D^c \mathbb{L}_{iK}{}^D = \mathring{H}_{ij}{}^c$ ) and hence projecting out the  $X_J Z_c^C$  slot must yield a conformally invariant object. Such projection is accomplished by contraction with  $Y^J Z_c^C$ , and from equations (3.31) and (5.21) one sees that this projection is equal to

$$Y^J Z_c^C \left( \mathbb{L}_{iJ}{}^C - \mathbb{M}(\mathring{H})_{iJ}{}^C \right) = N_b^c \left( P_i^b - \nabla_i H^b \right) + \frac{1}{m-1} D^j \mathring{H}_{ij}{}^c, \quad (5.24)$$

which is exactly  $\mu_i{}^c$  as defined in (5.19) (since the  $c$  index of  $D^j \mathring{H}_{ij}{}^c$  is already normal).  $\square$

In fact, the data  $(\mathring{H}_{ij}{}^c, \mu_i{}^c)$  is equivalent to the tractor second fundamental form. We have seen how to obtain  $\mathring{H}_{ij}{}^c$  and  $\mu_i{}^c$  from  $\mathbb{L}_{iJ}{}^C$ . For the reverse direction, note

that  $\mathbb{L}_{iJ}^C$  may be constructed from  $(\mathring{H}_{ij}^c, \mu_i^c)$  according to

$$\begin{aligned} (\mathring{H}_{ij}^c, \mu_i^c) \mapsto & \mathring{H}_{ij}^c Z_J^j Z_c^C + \left( \mu_i^c - \frac{1}{m-1} D^j \mathring{H}_{ij}^c \right) X_J Z_c^C \\ & + H_c \mathring{H}_{ij}^c Z_J^j X^C + H_c \left( \mu_i^c - \frac{1}{m-1} D^j \mathring{H}_{ij}^c \right) X_J X^C. \end{aligned} \quad (5.25)$$

**Remark 5.7.** *Note that the conformal invariant  $\mu$  (in codimension  $d > 1$  where it is not forced to vanish) is independent from the trace free second fundamental form  $\mathring{H}$ , and these two invariants are also independent from the Fialkow tensor  $\mathcal{F}$  (see [7, Theorem 4.22], which further shows that the conformally invariant curvature of the weighted normal bundle  $N\Sigma[-1]$  can also be independently specified). Indeed, it is not difficult to see that these three invariants capture different parts of the second order jet of the metric about the submanifold  $\Sigma$  and hence all three can be prescribed arbitrarily when extending a metric on  $\Sigma$  to a metric in a neighborhood of  $\Sigma$  in  $\Sigma \times \mathbb{R}^d$ .*

*Note further that the preceding observation establishes the abundance of conformal manifolds admitting distinguished submanifolds of any given dimension and codimension. But in a given conformal manifold  $(M, \mathbf{c})$ , outside of the dimension 1 case of conformal circles which are abundant, distinguished submanifolds (which must in particular be totally umbilic) are typically rare, the existence of the requisite parallel tractor constrains the possible compatible tractor curvature along the submanifold.*

### 5.3 Strong conformal circularity

Unlike the situation for geodesics in Riemannian geometry, in conformal geometry if a submanifold is weakly conformally circular the submanifold conformal circles need not be ambient conformal circles. This leads to the following two stronger notions of conformal circularity.

**Definition 5.8.** *Let  $\Sigma$  be a submanifold in a conformal manifold  $M$ . Then  $\Sigma$  is strongly conformally circular if any projectively-parametrised  $\Sigma$ -conformal circle is also a projectively-parametrised  $M$ -conformal circle. For the cases of submanifolds of dimensions 1 and 2, recall that the intrinsic conformal structure does not determine a conformal circle equation, however, the induced Möbius structures defined in Section 3.5 do determine a conformal circle equation (the usual conformal circle equation (5.1) with the Schouten tensor being as defined in Section 3.5), and that is the notion we are using here.*

**Definition 5.9.** *Let  $\Sigma$  be a submanifold in a conformal manifold  $M$ . Then  $\Sigma$  is conformally circular if any unparametrised  $\Sigma$ -conformal circle is also an unparametrised  $M$ -conformal circle.*

Definition 5.8 appears in [7], where the term *strongly conformally geodesic* is used. In the following two theorems we characterise these two notions of conformal circularity in terms of the basic tractor invariants  $\mathbb{L}$  and  $\mathbb{S}$  of the conformal submanifold  $\Sigma$ . The first theorem below is easily seen to be equivalent to [7, Theorem 5.4(3)], but we include proofs of both theorems for completeness.

**Theorem 5.10.** *Let  $\Sigma$  be a submanifold in a conformal manifold  $M$ . Then  $\Sigma$  is strongly conformally circular if, and only if  $\mathbb{L}_{iJ}^C = 0$  and  $\mathbb{S}_{iJK} = 0$  (i.e.  $\mathcal{F}_{ij} = 0$ ).*

**Theorem 5.11.** *Let  $\Sigma$  be a submanifold in a conformal manifold  $M$ . Then  $\Sigma$  is conformally circular if, and only if  $\mathbb{L}_{iJ}^C = 0$  and  $\mathbb{S}_{iJK} \propto \mathbf{g}_{ij} Z_{[J}^i X_{K]}$  (i.e.  $\mathcal{F}_{ij} \propto \mathbf{g}_{ij}$ ).*

Note that following our conventions for the Fialkow tensor in Section 3.5, these results still hold when  $\Sigma$  is a submanifold of dimension 2. The  $\Sigma$ -conformal circle equation is then the usual conformal circle equation (either the projectively parametrized equation (5.1) or the parametrization-independent weighted equation (5.12)) with the Schouten tensor defined in Section 3.5 playing the role of the usual Schouten tensor. For a 1-dimensional submanifold, weakly conformally circular and strongly conformally circular are equivalent (and the Fialkow tensor is defined to be zero), so in this case Theorems 5.10 and 5.11 reduce to the fact that a conformal circle is characterized by  $\mathbb{L} = 0$ .

*Proof of Theorem 5.10.* First suppose that  $\Sigma$  is strongly conformally circular. Since  $\Sigma$  is then also weakly conformally circular, we must have  $\mathbb{L}_{iJ}^C = 0$  by Theorem 5.2. In particular,  $\Sigma$  is totally umbilic ( $\mathring{H}_{ij}^c = 0$ ). Now, suppose that  $\gamma$  is a projectively parametrised  $\Sigma$ -conformal circle with initial data a given 2-jet at  $p \in \Sigma$ . Then by assumption  $\gamma$  is also an  $M$ -conformal circle with the same initial data, now viewed as the 2-jet of a curve in  $M$ . Let  $g \in \mathfrak{c}$  be a minimal scale for  $\Sigma$ . The curve  $\gamma$  must satisfy the intrinsic and ambient versions of the conformal circle equation, namely

$$\frac{d^D a^j}{dt} = u^2 \cdot u^i p_i^j + 3u^{-2} (u_k a^k) a^j - \frac{3}{2} u^{-2} (a_k a^k) u^j - 2u^k u^l p_{kl} u^j \quad (5.26)$$

and

$$\frac{d^\nabla a^b}{dt} = u^2 \cdot u^c P_c^b + 3u^{-2} (u_c a^c) a^b - \frac{3}{2} u^{-2} (a_c a^c) u^b - 2u^c u^d P_{cd} u^b \quad (5.27)$$

respectively. From the Gauß formula (3.5) and the fact that  $\Sigma$  is totally geodesic with respect to  $g$ , we have that

$$a^b = u^i \nabla_i u^b = \Pi_j^b u^i D_i u^j + \Pi_{ij}^b u^i u^j = \Pi_j^b a^j$$

and

$$\frac{d^\nabla a^b}{dt} = \Pi_j^b \frac{d^D a^j}{dt},$$



where  $a^j = \frac{d^D u^j}{dt}$ , i.e. the acceleration of the curve calculated intrinsically. Therefore, applying  $\Pi_b^j$  to (5.27) and subtracting (5.26) from the result, we see that

$$u^2 \cdot u^k P_k^j - 2u^k u^l P_{kl} u^j = u^2 \cdot u^k p_k^j - 2u^k u^l p_{kl} u^j,$$

where we have used that  $\Pi_b^j u^b = u^j$ ,  $\Pi_b^j a^b = a^j$ ,  $u_c a^c = u_k a^k$  (which follows from the Gauß formula without the need for  $g$  to be a minimal scale since  $u^c$  is tangent to the submanifold  $\Sigma$ ), and  $a^c a_c = a^k a_k$  (since  $\Sigma$  is totally geodesic with respect to  $g$ ). Contracting the above display with  $u_j$  yields

$$u^2 \cdot u_j u^k P_k^j - 2u^k u^l P_{kl} \cdot u^2 = u^2 \cdot u_j u^k p_k^j - 2u^k u^l p_{kl} \cdot u^2,$$

and hence

$$(P_{ij} - p_{ij}) u^i u^j = 0. \quad (5.28)$$

Since  $II_{ij}^c = 0$ , the term in parentheses in the above display is exactly the Fialkow tensor  $\mathcal{F}_{ij}$  from (3.36). Now, at any point  $p \in \Sigma$ , any  $u \in T_p \Sigma$  can arise as the velocity of a conformal circle, and hence (5.28) must hold for all  $u^i \in \mathcal{E}^i$ . Hence  $\mathcal{F}_{ij} = 0$ . Together with our earlier observation that strong conformal circularity implies weak conformal circularity, this establishes that  $\mathbb{L}_{iJ}^C = 0$  and  $S_{iJK} = 0$ .

Conversely, suppose that  $\mathbb{L}_{iJ}^C = 0$  and  $S_{iJK} = 0$ . Let  $\gamma$  be a projectively parametrized  $\Sigma$ -conformal circle. Then the intrinsic acceleration tractor  $A^J$  of  $\gamma$  satisfies  $\frac{d^D A^J}{dt} = 0$ . We must also show that  $\gamma$  satisfies the ambient (projectively parametrized) conformal circle equation.

For any parametrized curve in  $\Sigma$ , writing  $U^J$  for its velocity tractor, we see from explicit form of the velocity tractor (5.4) and the formula (3.18) for the isomorphism  $\Pi_J^B$  that  $U^B = \Pi_J^B U^J$ , where again we use the tangentiality of  $u^c$  together with the Gauß formula to conclude that  $u_c a^c = u_k a^k$ . Then applying the tractor Gauß formula (3.40) shows that the ambient acceleration tractor is given by

$$A^B = \frac{d^\nabla U^B}{dt} = \Pi_J^B (u^i D_i U^J + S_i^J K U^K) + u^i \mathbb{L}_{iJ}^B U^J = \Pi_J^B \frac{d^D U^J}{dt} = \Pi_J^B A^J,$$

since  $\mathbb{L} = 0$  and  $S = 0$ . Similarly, it follows that

$$\frac{d^\nabla A^B}{dt} = \Pi_J^B \frac{d^D A^J}{dt}.$$

So if moreover  $\gamma$  satisfies  $d^D A^J/dt = 0$ , then also  $d^\nabla A^B/dt = 0$ . Finally, recall that the isomorphism  $\Pi_J^B$  is also metric-preserving, and so

$$A^B A_B = A^J A_J = 0,$$

since  $A^B = \Pi_J^B A^J$  and we assumed  $\gamma$  is  $\Sigma$ -projectively parametrised. Thus, by the characterisation from [5],  $\gamma$  is a projectively-parametrised  $M$ -conformal circle, and therefore  $\Sigma$  is strongly conformally circular.  $\square$

We next prove the parametrization-independent version of Theorem 5.10, namely Theorem 5.11. As this theorem is a statement about unparametrised conformal circles, we use the 3-tractor  $\Phi$  (which equals the Hodge-star of the normal tractor form) discussed in Section 5.1.

*Proof of Theorem 5.11.* First, suppose that  $\Sigma$  is conformally circular, that is, every unparametrised  $\Sigma$ -conformal circle is an unparametrised  $M$ -conformal circle. Let  $\gamma$  be a  $\Sigma$ -conformal circle. In the previous proof, we observed that strong conformal circularity implies weak conformal circularity. Parametrisation was not used at all in this part of the proof and hence we may employ the same argument here. Thus  $\mathbb{L}_{iJ}{}^C = 0$  by Theorem 5.2.

As described in Section 5.1,  $\gamma$  determines an intrinsic 3-tractor  $\phi^{IJK} \in \mathcal{E}^{[IJK]}$  which satisfies  $X^I \phi^{JKL} = 0$  and  $u^i D_i \phi^{IJK} = 0$ , where  $X^I$  is the intrinsic canonical tractor and  $D_i$  is the intrinsic tractor connection. Explicitly,

$$\phi^{IJK} = 6u^{-1} X^I U^J A^K,$$

with  $U^J$  and  $A^K$  defined as in equations (5.2) and (5.3) respectively, using the intrinsic position tractor and tractor connection.

On the other hand, viewing  $\gamma$  as an ambient curve also defines a 3-tractor,

$$\Phi^{ABC} = 6u^{-1} X^A U^B A^C,$$

where  $X^A$  is the ambient position tractor and  $U^B$  and  $A^C$  are the velocity and acceleration tractors of  $\gamma$  as a curve in  $(M, \mathbf{c})$  (note that while  $U^B = \Pi_J^B U^J$ , we are abusing notation slightly in that the ambient acceleration tractor  $A^C$  need not equal  $\Pi_K^C A^K$ , as can be seen from the tractor Gauß formula). Since  $\Sigma$  is conformally circular,  $\gamma$  is an  $M$ -conformal circle and so  $\Phi$  must be parallel along the curve, i.e.  $u^a \nabla_a \Phi^{ABC} = 0$ .

Fix a reference metric  $g \in \mathbf{c}$ . By (5.11) the derivatives of the intrinsic and ambient 3-tractors are

$$\mathbf{u}^i D_i \phi^{IJK} = 6 \left( \mathbf{u}^i D_i \mathbf{a}^k - \mathbf{u}^l p_l^k \right) \mathbf{u}^j X^I Z_j^J Z_k^K,$$

and

$$\mathbf{u}^i \nabla_i \Phi^{ABC} = 6 \left( \mathbf{u}^i \nabla_i \mathbf{a}^c - \mathbf{u}^d P_d^c \right) \mathbf{u}^b X^A Z_b^B Z_c^C,$$

respectively. Both of the above displays are zero and hence

$$\begin{aligned} 0 &= \Pi_A^I \Pi_B^J \Pi_C^K \left( \mathbf{u}^i \nabla_i \Phi^{ABC} \right) - \mathbf{u}^i D_i \phi^{IJK} \\ &= 6 \left[ \left( \mathbf{u}^i D_i \mathbf{a}^k - \mathbf{u}^l P_l^k \right) - \left( \mathbf{u}^i D_i \mathbf{a}^k - \mathbf{u}^l p_l^k \right) \right] \mathbf{u}^j X^I Z_j^J Z_k^K \\ &= -6 \mathbf{u}^l \left( P_l^k - p_l^k \right) \mathbf{u}^j X^I Z_j^J Z_k^K, \end{aligned}$$

where we have used (3.32) and the Gauß formula in going from the first to second lines. It follows that the antisymmetric part of  $\mathbf{u}^l (P_l^k - p_l^k) \mathbf{u}^j$  equals zero. Thus

$$\mathbf{u}^l (P_l^k - p_l^k) \propto \mathbf{u}^k.$$

Since we have already seen that  $\mathring{H}_{ij}{}^c = 0$ , this means that

$$\mathbf{u}^l \mathcal{F}_l^k = \mathbf{u}^l \left( P_l^k - p_l^k + H_c \mathring{H}_l{}^{kc} + \frac{1}{2} H_c H^c \delta_l^k \right) \propto \mathbf{u}^k.$$

Now, since the (weighted) velocity of a  $\Sigma$ -conformal circle passing through a point  $p \in \Sigma$  can be any unit vector of  $T_p \Sigma[-1]$ , it follows that  $\mathcal{F}_l^k$  must equal  $f \delta_j^k$  for some smooth weight  $-2$  density on  $\Sigma$ , and hence  $\mathcal{F}_{ij} = f g_{ij}$ .

Conversely, suppose that  $\mathbb{L}_{iJ}{}^C = 0$  and  $\mathcal{F}_{ij} = f g_{ij}$  for some  $-2$  density on  $\Sigma$ , and let  $\gamma$  be a  $\Sigma$ -conformal circle. The  $\Sigma$ -conformal circle  $\gamma$  determines an intrinsic 3-tractor  $\phi^{IJK} = 6u^{-1} X^{[I} U^J A^K]$  which is parallel along  $\gamma$  for the intrinsic tractor connection. To show that  $\gamma$  is an  $M$ -conformal circle, we need to show that the ambient 3-tractor  $\Phi^{ABC} = 6u^{-1} X^{[A} U^B A^C]$  satisfies these same properties. We show this by using the conditions on the tractor second fundamental form and the difference tractor to relate the ambient  $X, U$  and  $A$  tractors to their intrinsic counterparts.

From the isomorphism of Theorem 3.5 (cf. the proof of Theorem 5.10), it follows that  $X^A = \Pi_I^A X^I$  and  $U^B = \Pi_J^B U^J$ , where  $X^I$  and  $U^J$  are the intrinsic submanifold canonical tractor and velocity tractor of  $\gamma$  respectively. From the tractor Gauß formula (3.40) we therefore have

$$\begin{aligned} A^B &= u^i \nabla_i U^B = u^i \nabla_i (\Pi_J^B U^J) \\ &= \Pi_J^B \left[ u^i D_i U^J + u^i f g_{ij} \left( Z^{Jj} X_K - Z_K^j X^J \right) U^K \right] \\ &= \Pi_J^B (A^J - u f X^J) \\ &= \Pi_J^B A^J - u f X^B, \end{aligned}$$

and

$$\begin{aligned} u^i \nabla_i A^C &= u^i \nabla_i (\Pi_J^C A^J) - u^i \nabla_i (u f X^C) \\ &= \Pi_J^C (u^i D_i A^J + f u^i (Z_i^J X_K - Z_{iK} X^J) A^K) \\ &\quad - u^i \nabla_i (u f) X^C - u f u^c Z_c^C \\ &= \Pi_J^C (u^i D_i A^J) - 2u f u^c Z_c^C + \rho X^C, \end{aligned} \tag{5.29}$$

where we have collected all the terms in the bottom slot into  $\rho$  (the exact form of  $\rho$  will not be important). Now, recall that  $U^B = u^i \nabla_i (u^{-1} X^B)$  and  $A^B = u^i \nabla_i U^B$ . Hence, using the skew-symmetry,

$$u^i \nabla_i \Phi^{ABC} = u^i \nabla_i \left( 6u^{-1} X^{[A} U^B A^C] \right) = 6u^{-1} X^{[A} U^B (u^i \nabla_i A^C)].$$

Substituting (5.29) for the derivative of the acceleration into the above it is easy to see that the  $-2ufu^c Z_c^C + \rho X^C$  terms drop out due to the skewing with  $X^A$  and with  $U^B$  (which is proportional to  $u^b Z_b^B$  modulo the canonical tractor) and thus we obtain,

$$u^i \nabla_i \Phi^{ABC} = u^i \Pi_I^A \Pi_J^B \Pi_K^C D_i \phi^{IJK}. \quad (5.30)$$

Finally, the right-hand side of (5.30) is zero, since  $\gamma$  is a  $\Sigma$ -conformal circle. Thus  $\gamma$  is an  $M$ -conformal circle.  $\square$

## 6 First integrals

Here we show that a class of solutions to a very large collection of linear differential equations provide first integrals for distinguished submanifolds. This provides a uniform framework which generalises to submanifolds (of any proper codimension) the advance for conformal circles in [56]. In [56] it is explained in detail how the ideas there extend the usual construction of first integrals for geodesics, using for example solutions of the Killing equation, Killing tensors, and Killing-Yano tensors. So we do not repeat that here.

### 6.1 Review of relevant BGG theory

The class of equations that we are interested in are the so-called (conformal) first BGG equations. This is a very large class of conformally invariant linear overdetermined PDE. It includes the conformal Killing equation, more generally the conformal Killing tensors equations of any rank, the conformal Killing-Yano equations. To understand this infinite class of equations we recall here some elements of the BGG theory. To put this into context we first recall the homogeneous model for conformal geometry, discussed in Section 2.3. The model for oriented conformal geometries of Riemannian signature is conformal  $n$ -sphere  $(S^n, \mathbf{c})$  viewed as the ray projectivisation of the forward null cone in  $(n + 2)$ -dimensional Minkowski space. The group  $G = \mathrm{SO}_0(n + 1, 1)$  acts on the forward null cone and descends to an action by conformal isometries of  $S^n$ ; the conformal  $n$ -sphere is therefore naturally viewed as a homogeneous geometry on  $G/P \cong S^n$ , for  $P$  an appropriate (parabolic) subgroup of  $G$ . Again, see, e.g., [33, 51] for a more detailed discussion.

Generalising from the model case, it is well known that a conformal manifold  $(M, \mathbf{c})$  (of dimension  $n \geq 3$ ) determines a canonical *Cartan bundle and connection* (the additional choice of a Möbius structure is required for this in dimension  $n = 2$ ). This consists of a  $P$ -principal bundle  $\mathcal{G} \rightarrow M$  equipped with a canonical *Cartan connection*  $\omega$  which is a suitably equivariant  $\mathfrak{g}$ -valued 1-form that provides a total parallelisation of  $T\mathcal{G}$ . Here  $\mathfrak{g}$  denotes the Lie algebra of  $G$ . In the case of the model,  $\mathcal{G} = G$  and  $\omega$  is the Maurer-Cartan form.

For any representation  $\mathbb{U}$  of  $P$ , one has a corresponding associated bundle  $\mathcal{G} \times_P \mathbb{U}$ . For example it follows from the equivariance properties of  $\omega$  that the tangent bundle  $TM$  can be identified with  $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$  where  $\mathfrak{p}$  is the Lie algebra of  $P$ , and the  $P$  action is induced from its adjoint action on  $\mathfrak{g}$ .

The tractor bundles are the associated bundles  $\mathcal{W} := \mathcal{G} \times_P \mathbb{W}$  where  $\mathbb{W}$  is a linear representation space of  $G$  (and hence also of  $P$  by restriction). On each of these the Cartan connection induces a linear connection  $\nabla^{\mathcal{W}}$  and this is the tractor connection for the given bundle. In particular the standard tractor bundle  $\mathcal{T}$  is  $\mathcal{W} := \mathcal{G} \times_P \mathbb{R}^{n+2}$ , with  $\mathbb{R}^{n+2}$  denoting the defining representation of  $G$ . From the latter (for example) the Cartan bundle  $\mathcal{G}$  can be recovered as an adapted frame

bundle and, on this, the Cartan connection  $\omega$  can be recovered from the tractor connection, see [17].

Now recall the bundle embedding (2.24) (with  $k = 2$ )

$$\mathbb{X} : T^*M \rightarrow \Lambda^2\mathcal{T} \subset \text{End}(\mathcal{T}),$$

where the tractor metric is used in the obvious way to identify elements of  $\Lambda^2\mathcal{T}$  with skew elements of  $\text{End}(\mathcal{T})$ . Sections of  $\text{End}(\mathcal{T})$  act on tractor bundles in the obvious tensorial way and so, via each respective  $\mathbb{X}$ , we have a canonical action of  $T^*M$  on any tractor bundle  $\mathcal{V}$  and this induces a sequence of invariant bundle maps

$$\partial^* : \Lambda^k T^*M \otimes \mathcal{V} \rightarrow \Lambda^{k-1} T^*M \otimes \mathcal{V}, \quad k = 1, \dots, n+1. \quad (6.1)$$

This is the (bundle version of the) Kostant codifferential for conformal geometry and satisfies  $\partial^* \circ \partial^* = 0$ ; so it determines subquotient bundles  $\mathcal{H}_k(M, \mathcal{V}) := \ker(\partial^*)/\text{im}(\partial^*)$  of the  $\mathcal{V}$ -valued tractor bundles  $\Lambda^k T^*M \otimes \mathcal{V}$ .

Now, for each tractor bundle  $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$ , with  $\mathbb{V}$  irreducible for  $G$ , there is a canonical differential *BGG-sequence* [23, 12],

$$\mathcal{H}_0 \xrightarrow{\mathcal{D}_0^\mathcal{V}} \mathcal{H}_1 \xrightarrow{\mathcal{D}_1^\mathcal{V}} \dots \xrightarrow{\mathcal{D}_{n-1}^\mathcal{V}} \mathcal{H}_n.$$

Here  $\mathcal{H}_k = \mathcal{H}_k(M, \mathcal{V})$  and each  $\mathcal{D}_i^\mathcal{V}$  is a linear conformally invariant differential operator.

We are, in particular, interested in the operator  $\mathcal{D}^\mathcal{V} = \mathcal{D}_0^\mathcal{V}$ , which defines an overdetermined differential system. The parabolic subgroup  $P \subset G$  determines a filtration on  $\mathbb{V}$  by  $P$ -invariant subspaces. Denoting the largest proper filtration component by  $\mathbb{V}^0 \subset \mathbb{V}$ , it is straightforward to show that  $\mathcal{H}_0$  is the quotient  $\mathcal{V}/\mathcal{V}^0$ . Here,  $\mathcal{V}^0$  is the corresponding associated bundle for  $\mathbb{V}^0$ , and we write  $\pi : \mathcal{V} \rightarrow \mathcal{H}_0$  for the natural projection. We recall here the construction of the first BGG operators  $\mathcal{D}^\mathcal{V}$ , as summarised in [19], and also the definition of the special class of so called normal solutions (cf. [80]) for these operators.

**Theorem 6.1** ([19]). *Let  $\mathbb{V}$  be an irreducible  $G$ -representation and let  $\mathcal{V} := \mathcal{G} \times_P \mathbb{V}$ . There is a unique invariant differential operator  $L : \mathcal{H}_0 \rightarrow \mathcal{V}$  such that  $\pi \circ L$  is the identity map on  $\mathcal{H}_0$  and  $\nabla \circ L$  lies in  $\ker(\partial^*) \subset T^*M \otimes \mathcal{V}$ . For  $\sigma \in \Gamma(\mathcal{H}_0)$ ,  $\mathcal{D}^\mathcal{V}\sigma$  is given by projecting  $\nabla(L(\sigma))$  to  $\Gamma(\mathcal{H}_1)$ , i.e.  $\mathcal{D}^\mathcal{V}\sigma = \pi(\nabla(L(\sigma)))$ .*

*Furthermore the bundle map  $\pi$  induces an injection from the space of parallel sections of  $\mathcal{V}$  to a subspace  $\mathfrak{N}(\mathcal{D}^\mathcal{V})$  of  $\Gamma(\mathcal{H}_0)$  which is contained in the kernel of the first BGG operator*

$$\mathcal{D}^\mathcal{V} : \mathcal{H}_0 \rightarrow \mathcal{H}_1. \quad (6.2)$$

*The operator  $L$  restricts to an isomorphism from  $\mathfrak{N}(\mathcal{D}^\mathcal{V})$  to the space of parallel tractors in  $\Gamma(\mathcal{V})$ .*

The differential operator  $L : \mathcal{H}_0 \rightarrow \mathcal{V}$ , in the Theorem, is called a *BGG splitting operator*. We sometimes denote this  $L^\mathcal{V}$  to emphasise the particular tractor bundle involved. Using the notation and setting of the Theorem, we also use the following terminology:

**Definition 6.2.** *Elements of the subspace  $\mathfrak{N}(\mathcal{D}^\mathcal{V}) \subset \Gamma(\mathcal{H}_0)$  are called normal solutions to the BGG equation  $\mathcal{D}^\mathcal{V}\sigma = 0$ .*

By definition normal solutions to (6.2) are in 1-1 correspondence with parallel sections of the corresponding tractor bundle  $\mathcal{V}$ . On geometries which are conformally flat all solutions are normal, and clearly there is  $\dim(\mathbb{V})$ -parameter family of such normal solutions locally.

For the standard tractor bundle the corresponding first BGG equation is the equation

$$\nabla_{(a}\nabla_{b)}\sigma + P_{(ab)_0}\sigma = 0,$$

on sections  $\sigma \in \Gamma(\mathcal{E}[1])$ , and all solutions are normal (on any conformal manifold admitting such solutions). However, this is not typical. In general, for solutions  $\sigma \in \Gamma(\mathcal{H}_0)$  of  $\mathcal{D}^\mathcal{V}(\sigma) = 0$ ,  $\nabla L(\sigma)$  is given by curvature terms acting on  $L(\sigma)$  (see, e.g., [15, 50, 54, 71]). Normal solutions, for which these curvature terms necessarily annihilate  $L(\sigma)$ , often correspond to interesting geometric conditions on the underlying manifold.

## 6.2 The First Integral Theorem

We work on an arbitrary conformal manifold  $(M^n, \mathbf{c})$ . Let  $\Sigma$  be an embedded submanifold of codimension  $d$ . Recall that  $\Sigma$  determines its normal form  $N_{A_1 \dots A_d} \in \Gamma(\Lambda^d \mathcal{N})$ . This is parallel if (and only if)  $\Sigma$  is distinguished. Thus if the manifold  $(M^n, \mathbf{c})$  is equipped with a parallel tractor  $S$  that can be contracted non-trivially into say  $m_0$  copies of  $N_{A_1 \dots A_d}$  to yield a function, then this scalar is necessarily constant if  $\Sigma$  is distinguished. Thus we obtain a first integral for such  $\Sigma$ . In general the parallel tractor  $S$  would not necessarily itself come from a  $G$ -irreducible representation, but rather a tensor product of such. Thus we have the following result.

As earlier, view  $\mathbb{R}^{n+2}$  as the defining representation for  $G := \mathrm{SO}(h) \cong \mathrm{SO}(p+1, q+1)$ . Define

$$\mathbb{W}(d) := \Lambda^d \mathbb{R}^{n+2} \quad d = 1, \dots, n-1.$$

For each  $d$ , this is also a representation space for  $G$ . Then we have:

**Theorem 6.3.** *Let  $\mathbb{V}_1, \dots, \mathbb{V}_k$  be irreducible representation spaces of  $G$ ,  $\mathcal{V}_i = \mathcal{G} \times_P \mathbb{V}_i$ , and  $\mathcal{D}^{\mathcal{V}_i}$ ,  $i \in \{1, \dots, k\}$  the corresponding respective first BGG operators.*

*For each  $i \in \{1, \dots, k\}$ , suppose that  $\sigma_i$  is a normal solution to the first BGG equation*

$$\mathcal{D}^{\mathcal{V}_i}\sigma_i = 0, \tag{6.3}$$

and  $m_i \in \mathbb{Z}_{\geq 0}$ . Then for each copy of the trivial  $G$ -representation  $\mathbb{R}$  in

$$(\odot^{m_0} \mathbb{W}(d)) \otimes (\odot^{m_1} \mathbb{V}_1) \otimes \cdots \otimes (\odot^{m_k} \mathbb{V}_k) \quad (6.4)$$

there is a corresponding distinguished first integral for submanifolds of codimension  $d$ .

*Proof.* The proof is an easy consequence of the reasoning above. Otherwise the formal proof is a trivial adaption of the proof of Theorem 6.1 in [56], which treats the case of curves.  $\square$

The theorem has used the normal form  $N_{A_1 \dots A_d}$  as the basis for producing first integrals. One can equivalently use its Hodge dual  $\star N_{A_1 \dots A_{m+2}}$ , or the normal projector  $N_B^A$ , or any combination of these, as by Theorem 1.1 any of these are parallel for distinguished submanifolds.

Note that to apply (6.4) of the Theorem for a given  $\Sigma$  we require normal solutions to  $k$  first BGG equations. For case of curves, several examples are given in [56], as is also the explanation of how this is linked to familiar first integrals for geodesics as obtained from Killing vectors and Killing tensors (which are solutions of projective BGG equations). Given that resource we treat just one example here.

### 6.3 First integrals from a normal conformal Killing-Yano form

We give an example to show how this machinery yields conserved quantities for distinguished submanifolds. It is easy to follow the ideas here to produce other examples, see [56] for the case of  $m = 1$ .

The space  $\mathcal{E}_{a_1[a_2 \dots a_d]}[w] = \mathcal{E}_{a_1} \otimes \mathcal{E}_{[a_2 \dots a_d]}[w]$  is completely reducible, and has the  $O(g)$ -decomposition

$$\mathcal{E}_{a_1[a_2 \dots a_d]}[w] = \mathcal{E}_{[a_1 a_2 \dots a_d]}[w] \oplus \mathcal{E}_{\{a_1[a_2 \dots a_d]\}_0}[w] \oplus \mathcal{E}_{[a_3 \dots a_d]}[w - 2], \quad (6.5)$$

where  $\mathcal{E}_{\{a_1[a_2 \dots a_d]\}_0}[w]$  consists of tensors  $s_{a_1 \dots a_d} \in \mathcal{E}_{a_1[a_2 \dots a_d]}[w]$  which are, metric trace-free, completely skew on the indices  $a_2, \dots, a_d$ , and for which  $s_{[a_1 a_2 \dots a_d]} = 0$ . A  $(d - 1)$ -form  $k_{a_2 \dots a_d} \in \Gamma(\mathcal{E}_{[a_2 \dots a_d]}[d])$  is said to be a *conformal Killing-Yano form* or simply *conformal Killing form* if it satisfies

$$\nabla_{a_1} k_{a_2 \dots a_d} = \mu_{a_1 \dots a_d} + \mathbf{g}_{a_1[a_2} \nu_{a_3 \dots a_d]}, \quad (6.6)$$

where  $\mu_{a_1 \dots a_d} \in \mathcal{E}_{[a_1 \dots a_d]}[d]$  and  $\nu_{a_3 \dots a_d} \in \mathcal{E}_{[a_3 \dots a_d]}[d - 2]$ . Equivalently,

$$\nabla_{\{a_1} k_{a_2 \dots a_d\}_0} = 0, \quad (6.7)$$

where the braces and subscript zero denote projection onto the middle factor of (6.5). This equation can be checked to be conformally invariant, and is moreover a first



BGG equation (which in this context implies conformal invariance). Thus solutions to this equation correspond bijectively to a class of sections of a certain tractor bundle. To understand this, we proceed as follows. For this equation, it is shown in [54] that the corresponding tractor bundle is  $\Lambda^d \mathcal{T}$ , and it follows from the formulae there that the BGG splitting operator  $L : \mathcal{E}_{[a_1 a_2 \dots a_d]}[d+1] \rightarrow \mathcal{E}_{[A_0 A_1 \dots A_d]}$  is:

$$\begin{aligned} L(k_{a_2 \dots a_d}) &= k_{a_2 \dots a_d} \mathbb{Y}_{A_1 \dots A_d}^{a_2 \dots a_d} + \frac{1}{d} \nabla_{a_1} k_{a_2 \dots a_d} \mathbb{Z}_{A_1 \dots A_d}^{a_1 \dots a_d} + \frac{d-1}{n-d+2} \nabla^c k_{ca_3 \dots a_d} \mathbb{W}_{A_0 A_1 A_2 \dots A_d}^{a_3 \dots a_d} \\ &\quad + \rho_{a_2 \dots a_d} \mathbb{X}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d}, \end{aligned} \tag{6.8}$$

where we do not need the details of the  $\mathbb{X}$  slot. The general theory immediately gives us the following.

**Proposition 6.4.** *Let  $k_{a_1 \dots a_{d-1}} \in \mathcal{E}_{[a_1 \dots a_{d-1}]}[d-1]$  be a normal solution to the conformal Killing-Yano equation and  $\Sigma$  a distinguished submanifold of codimension  $d$ , with corresponding tractor normal form  $N_{A_1 \dots A_d}$ . Let  $\mathbb{K}_{A_1 \dots A_d} := L(k_{a_1 \dots a_{d-1}}) \in \mathcal{E}_{[A_1 \dots A_d]}$  be the image of  $k_{a_1 \dots a_{d-1}}$  under the BGG splitting operator  $L$  of (6.8). Then the scalar function*

$$\mathbb{K}_{A_1 \dots A_d} N^{A_1 \dots A_d} \tag{6.9}$$

*is constant along  $\Sigma$ .*

*Proof.* Since  $k_{a_1 \dots a_{d-1}}$  is a normal solution, we have that  $\nabla_i \mathbb{K}_{A_1 \dots A_d} = 0$ . Moreover, since  $\Sigma$  is a distinguished submanifold,  $\nabla_i N^{A_1 \dots A_d} = 0$  by Theorem 1.1. Hence the scalar quantity  $\mathbb{K}_{A_1 \dots A_d} N^{A_1 \dots A_d}$  is constant.  $\square$

We show the non-triviality of the first integral quantity (6.9) by calculating it directly. From the explicit forms of  $\mathbb{K}_{A_1 \dots A_d}$  and  $N_{A_1 \dots A_d}$ , we see that

$$\begin{aligned} \mathbb{K}_{A_1 \dots A_d} N^{A_1 \dots A_d} &= d \cdot k_{a_1 \dots a_{d-1}} N^{cb_1 \dots b_{d-1}} H_c \cdot \mathbb{Y}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d} \mathbb{X}_{b_1 \dots b_{d-1}}^{A_1 A_2 \dots A_d} \\ &\quad + \frac{1}{d} (\nabla_{a_1} k_{a_2 \dots a_d}) N^{b_1 b_2 \dots b_d} \cdot \mathbb{Z}_{A_1 A_2 \dots A_d}^{a_1 a_2 \dots a_d} \mathbb{Z}_{b_1 b_2 \dots b_d}^{A_1 A_2 \dots A_d} \\ &= k_{a_1 \dots a_{d-1}} N^{ca_1 \dots a_{d-1}} H_c + \frac{1}{d} (\nabla_{a_1} k_{a_2 \dots a_d}) N^{a_1 a_2 \dots a_d}, \end{aligned} \tag{6.10}$$

which verifies non-triviality.

For the case of  $d = n - 1$ , meaning curves, it was seen in [56] that, for many examples, normality of the BGG solution is actually not required in order to obtain a first integral. However in general it is needed for this example, which we see as follows. If  $\Sigma$  is distinguished then  $N^{A_1 \dots A_d}$  is parallel for the tractor connection and we have

$$\nabla_i (\mathbb{K}_{A_1 \dots A_d} N^{A_1 \dots A_d}) = (\nabla_i \mathbb{K}_{A_1 \dots A_d}) N^{A_1 \dots A_d},$$

where  $\mathbb{K}_{A_1 \dots A_d} = L(k)$  for a general rank  $(d - 1)$  conformal Killing-Yano form  $k$ . Theorem 3.9 of [54] gives

$$(\nabla_c - \Psi_c) \mathbb{K}_{A_1 \dots A_d} = 0,$$

where  $\nabla_c$  is the standard tractor connection and  $\Psi_c : \mathcal{E}_{[A_1 \dots A_d]} \rightarrow \mathcal{E}_{c[A_1 \dots A_d]}$  is defined by

$$\begin{aligned} \Psi_c(\mathbb{K}_{A_1 A_2 A_3 \dots A_d}) := & -\frac{1}{2} W_{a_1 a_2 c} {}^p k_{pa_3 \dots a_d} \mathbb{Z}_{A_1 A_2 A_3 \dots A_d}^{a_1 a_2 a_3 \dots a_d} + \phi_{ca_3 \dots a_d} \mathbb{W}_{A_1 A_2 A_3 \dots A_d}^{a_3 \dots a_d} \\ & + \xi_{a_2 \dots a_d} \mathbb{X}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d}, \end{aligned} \quad (6.11)$$

where only the explicit form of the  $\mathbb{Z}$  slot will be important.

Therefore one has

$$\begin{aligned} \nabla_i (\mathbb{K}_{A_1 A_2 \dots A_d} N^{A_1 A_2 \dots A_d}) &= (\nabla_i \mathbb{K}_{A_1 A_2 \dots A_d}) N^{A_1 A_2 \dots A_d} \\ &= \Psi_i(\mathbb{K}_{A_1 A_2 \dots A_d}) N^{A_1 A_2 \dots A_d} \\ &= -\frac{1}{2} W_{a_1 a_2 i} {}^p k_{pa_3 \dots a_d} N^{b_1 b_2 b_3 \dots b_d} \cdot \mathbb{Z}_{A_1 A_2 A_3 \dots A_d}^{a_1 a_2 a_3 \dots a_d} \mathbb{Z}_{b_1 b_2 b_3 \dots b_d}^{A_1 A_2 A_3 \dots A_d} \\ &= -\frac{1}{2} W_{a_1 a_2 i} {}^p k_{pa_3 \dots a_d} N^{a_1 a_2 a_3 \dots a_d}, \end{aligned}$$

which we do not, in general, expect to vanish.

## 7 Distinguished submanifolds as curved orbits

A conformally embedded submanifold, of codimension  $d$ , determines the fundamental and equivalent objects  $N_B^A$ ,  $N_{A_1 \dots A_d}$ , and  $\star N^{A_1 \dots A_{m+2}}$  and then we have Theorem 1.1. However, for an application that follows, it is important to show that there is a characterisation of distinguished submanifolds that does not use an initial knowledge of these. The result is Theorem 1.5, that we state more explicitly here.

**Theorem 7.1.** *Let  $\Sigma \hookrightarrow M$  be a submanifold of codimension  $d$  in a conformal manifold  $(M, \mathbf{c})$ . Then  $\Sigma$  is distinguished if, and only if, there exists a nowhere-zero  $\Psi_{A_1 A_2 \dots A_d} \in \Gamma(\Lambda^d \mathcal{T}^*|_\Sigma)$  such that  $\Psi_{A_1 A_2 \dots A_d} X^{A_1} = 0$  and  $\nabla_i \Psi_{A_1 A_2 \dots A_d} = 0$  along  $\Sigma$ .*

*Proof.* If  $\Sigma$  is distinguished, then by Theorem 1.1, the tractor normal form is parallel in tangential directions. Moreover, it is clear from the definition of the tractor normal form (3.52) that  $N_{A_1 A_2 \dots A_d} X^{A_1} = 0$ . Thus we may take  $\Psi_{A_1 \dots A_d}$  to be the tractor normal form.

Conversely, suppose that we have  $\Psi \in \Gamma(\Lambda^d \mathcal{T}^*|_\Sigma)$  which satisfies  $\Psi_{A_1 A_2 \dots A_d} X^{A_1} = 0$  and  $\nabla_i \Psi_{A_1 A_2 \dots A_d} = 0$  along  $\Sigma$ . From (2.22), we know that, in a background scale,  $\Psi$  can be written

$$\begin{aligned} \Psi_{A_1 A_2 \dots A_d} &= \sigma_{a_2 \dots a_d} \mathbb{Y}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d} + \nu_{a_1 a_2 \dots a_d} \mathbb{Z}_{A_1 A_2 \dots A_d}^{a_1 a_2 \dots a_d} \\ &\quad + \phi_{a_3 \dots a_d} \mathbb{W}_{A_1 A_2 A_3 \dots A_d}^{a_3 \dots a_d} + \rho_{a_2 \dots a_d} \mathbb{X}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d}. \end{aligned}$$

But the condition  $\Psi_{A_1 \dots A_d} X^{A_1} = 0$  together with (2.18) implies that  $\sigma_{a_2 \dots a_d} = 0$  and  $\phi_{a_3 \dots a_d} = 0$ .

Moreover, if  $u^i \in \Gamma(\mathcal{E}^i)$ , the incidence relation  $\Psi_{A_1 A_2 \dots A_d} X^{A_1} = 0$  together with the parallel condition means that

$$0 = u^i \nabla_i (X^{A_1} \Psi_{A_1 A_2 \dots A_d}) = u^i Z_i^{A_1} \Psi_{A_1 A_2 \dots A_d}$$

so  $u^i Z_i^{A_1} \Psi_{A_1 A_2 \dots A_d} = 0$  for all  $u \in \Gamma(T\Sigma)$ . Expanding this, again using (2.18) and the linear independence of the  $X$  and  $Z$  projectors, one sees that  $\nu_{a_1 a_2 \dots a_d} u^{a_1} = 0$  and  $\rho_{a_2 \dots a_d} u^{a_2} = 0$ . Since  $u^i$  was an arbitrary submanifold tangent vector, we conclude that  $\nu \in (\Lambda^d N^* \Sigma)[d]$  and  $\rho \in (\Lambda^{d-1} N^* \Sigma)[d-2]$ . Thus in particular  $\nu_{a_1 a_2 \dots a_d} = f N_{a_1 a_2 \dots a_d}$ , where  $N_{a_1 a_2 \dots a_d}$  is the Riemannian normal form of  $\Sigma$  and  $f$  is a function on  $\Sigma$ .

Now note that, since  $\Psi$  is parallel,  $\Psi^{A_1 A_2 \dots A_d} \Psi_{A_1 A_2 \dots A_d}$  is constant along  $\Sigma$ . On the other hand,

$$\Psi^{A_1 A_2 \dots A_d} \Psi_{A_1 A_2 \dots A_d} = \nu^{a_1 a_2 \dots a_d} \nu_{a_1 a_2 \dots a_d} = f^2 N^{a_1 a_2 \dots a_d} N_{a_1 a_2 \dots a_d} = f^2 \cdot d!,$$

and therefore the function  $f$  is locally constant and nowhere-zero. Thus on each connected component of  $\Sigma$ ,  $\nu_{a_1 a_2 \dots a_d}$  is a constant multiple of the Riemannian normal form.

From equation (2.25), we calculate

$$0 = \nabla_i \Psi_{A_1 A_2 \dots A_d} = (f \nabla_i N_{a_1 a_2 \dots a_d} + \rho_{a_2 \dots a_d} \mathbf{g}_{i a_1}) \mathbb{Z}_{A_1 A_2 \dots A_d}^{a_1 a_2 \dots a_d} + (\nabla_i \rho_{a_2 \dots a_d} - f \cdot d \cdot N_{a_1 a_2 \dots a_d} P_i^{a_1}) \mathbb{X}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d}. \quad (7.1)$$

Now, note that the same argument that yielded equation (3.56) may be repeated replacing normal *tractors* with normal *vectors* (as per Remark 3.22) to give

$$\nabla_i N_{a_1 a_2 \dots a_d} = -d \cdot \mathbb{I}_{i[a_d}^{a_0} N_{a_1 a_2 \dots a_{d-1}]a_0}. \quad (7.2)$$

Substituting this into (7.1) gives that in particular

$$-f \cdot d \cdot \mathbb{I}_{i[a_d}^{a_0} N_{a_1 a_2 \dots a_{d-1}]a_0} + \mathbf{g}_{i[a_1} \rho_{a_2 \dots a_{d-1} a_d]} = 0. \quad (7.3)$$

Contracting the above with  $\mathbf{g}^{i a_1} = \mathbb{I}_b^i \mathbf{g}^{a_1 b}$  allows us to express  $\rho_{a_2 \dots a_d}$  explicitly. The contraction with the second fundamental form term is not completely obvious, so we compute it first:

$$\begin{aligned} \mathbf{g}^{i a_1} \mathbb{I}_{i[a_d}^{a_0} N_{a_1 a_2 \dots a_{d-1}]a_0} &= -\mathbf{g}^{i a_1} \cdot \mathbb{I}_{i[a_1}^{a_0} N_{a_d a_2 \dots a_{d-1}]a_0} \\ &= -\frac{1}{d!} \cdot \mathbf{g}^{i a_1} \sum_{\sigma \in \mathfrak{S}_d} \text{sgn } \sigma \mathbb{I}_{i a_{\sigma(1)}}^{a_0} N_{a_{\sigma(d)} a_{\sigma(2)} \dots a_{\sigma(d-1)} a_0} \\ &= -\frac{1}{d!} \cdot \mathbf{g}^{i a_1} \sum_{\substack{\sigma \in \mathfrak{S}_d, \\ \sigma(1)=1}} \text{sgn } \sigma \mathbb{I}_{i a_{\sigma(1)}}^{a_0} N_{a_{\sigma(d)} a_{\sigma(2)} \dots a_{\sigma(d-1)} a_0} \\ &= -\frac{1}{d!} \sum_{\substack{\sigma \in \mathfrak{S}_d, \\ \sigma(1)=1}} \mathbf{g}^{i a_1} \mathbb{I}_{i a_1}^{a_0} N_{a_d a_2 \dots a_{d-1} a_0} \\ &= \frac{(d-1)!}{d!} \cdot m \cdot H^{a_0} N_{a_0 a_2 \dots a_{d-1} a_d}. \end{aligned}$$

For the other term,

$$\begin{aligned} \mathbf{g}^{i a_1} \mathbf{g}_{i[a_1} \rho_{a_2 \dots a_d]} &= \mathbf{g}^{i a_1} \cdot \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \mathbf{g}_{i a_{\sigma(1)}} \rho_{a_{\sigma(2)} \dots a_{\sigma(d)}} \\ &= \mathbf{g}^{i a_1} \cdot \frac{1}{d!} \sum_{\substack{\sigma \in \mathfrak{S}_d \\ \sigma(1)=1}} \mathbf{g}_{i a_{\sigma(1)}} \rho_{a_{\sigma(2)} \dots a_{\sigma(d)}} \\ &= \frac{(d-1)!}{d!} \mathbf{g}^{i a_1} \mathbf{g}_{i a_1} \rho_{a_2 \dots a_d} \\ &= \frac{m}{d} \rho_{a_2 \dots a_d}. \end{aligned}$$

Hence from (7.3) we obtain

$$\rho_{a_2 \dots a_{d-1} a_d} = f \cdot d \cdot H^{a_0} N_{a_0 a_2 \dots a_{d-1} a_d}.$$

Thus

$$\begin{aligned}\Psi_{A_1 A_2 \dots A_d} &= f N_{a_1 a_2 \dots a_d} \mathbb{Z}_{A_1 A_2 \dots A_d}^{a_1 a_2 \dots a_d} + f \left( d \cdot H^b N_{b a_2 \dots a_d} \right) \mathbb{X}_{A_1 A_2 \dots A_d}^{a_2 \dots a_d} \\ &= f N_{A_1 A_2 \dots A_d},\end{aligned}$$

where  $N_{A_1 A_2 \dots A_d}$  is the tractor normal form.

Since the function  $f$  is locally constant and nowhere-zero,  $\nabla_i \Psi_{A_1 A_2 \dots A_d} = 0$  implies that the tractor normal form is parallel. Thus  $\Sigma$  satisfies ones of the equivalent conditions of Theorem 1.1, and is therefore a distinguished submanifold.  $\square$

Note that Theorem 1.5 follows as the tractor Hodge- $\star$  operation (2.27) commutes with the tractor covariant derivative.

The following result shows one way in which distinguished submanifolds arise as *curved orbits*, in the sense of [21]. It generalises [56, Proposition 7.1]. Some terminology: we shall say that a tractor (or vector)  $\mathbb{K}_{A_1 \dots A_d}$  is *timelike* if  $\mathbb{K}_{A_1 \dots A_d} \mathbb{K}^{A_1 \dots A_d}$  is negative, *spacelike* if this is positive and *null* if it is zero.

**Theorem 7.2.** *Suppose  $k_{a_1 \dots a_{d-1}}$  is a normal solution of the conformal Killing form equation on  $(M, \mathbf{c})$  such that the parallel tractor  $L(k_{a_1 \dots a_{d-1}})$  is simple. Then the zero locus of*

$$\mathcal{K} := (k_{a_1 \dots a_{d-1}}, \nabla^c k_{c a_2 \dots a_{d-1}}), \text{ for any } g \in \mathbf{c} \text{ with Levi-Civita connection } \nabla,$$

*is either empty, an isolated point, or a distinguished conformal submanifold of codimension  $d$ . Moreover, writing  $\mathcal{Z}(\mathcal{K})$  for this zero locus:*

- *if  $L(k_{a_1 \dots a_{d-1}})$  is timelike, then  $\mathcal{Z}(\mathcal{K})$  is necessarily empty;*
- *if  $L(k_{a_1 \dots a_{d-1}})$  is null,  $\mathcal{Z}(\mathcal{K})$  consists only of isolated points;*
- *if  $L(k_{a_1 \dots a_{d-1}})$  is spacelike, then  $\mathcal{Z}(\mathcal{K})$  is either empty or is a distinguished submanifold of codimension  $d$ .*

*Proof.* Suppose  $k_{a_1 \dots a_{d-1}} \in \mathcal{E}_{[a_1 \dots a_{d-1}]}[d]$  is a normal solution to (6.6). Then  $\mathbb{K}_{A_1 \dots A_d} := L(k_{a_1 \dots a_{d-1}})$  is parallel for the tractor connection. So  $L(k_{a_1 \dots a_{d-1}})$  is parallel and, by assumption, simple.

From equation (6.8), one sees that  $X^{A_1} \mathbb{K}_{A_1 \dots A_d} = 0$ , at some point  $p \in M$ , is exactly the condition  $(k_{a_1 \dots a_d}, \nabla^c k_{c a_2 \dots a_d}) = 0$  at the point  $p$ . On the other hand this condition  $X \lrcorner \mathbb{K} = 0$  determines a *P-type*, in the language of [21]. Thus by Theorem 2.6 of that article, it is sufficient to understand this condition on the model.

Recall that in the model case parallel tractors correspond to constant tensor fields on  $\mathbb{R}^{n+2}$  and the canonical tractor  $X^A$  is identified with the position vector field of  $\mathbb{R}^{n+2}$  along  $\mathcal{C}_+$ . Hence, in the model case, if  $\mathbb{K}_{A_1 \dots A_d}$  is a parallel simple  $d$ -cotractor and  $X \lrcorner \mathbb{K}$  is zero at some point  $p$ , then  $X \lrcorner \mathbb{K}$  is zero along a submanifold  $p$ , given as follows. The form  $\mathbb{K}$  determines in  $\mathbb{R}^{n+2}$  a unique  $(m+2)$ -plane through  $p$  (as usual

$m = n - d$  and  $(m + 2)$ -plane means a linear subspace of that dimension) consisting of the vectors  $X^A$  in  $\mathbb{R}^{n+2}$  that are in the nullity of  $\mathbb{K}_{A_1 \dots A_d}$ . The submanifold is then the ray projectivisation of the intersection of this hyperplane with the null quadric for the Minkowski signature inner product on  $\mathbb{R}^{n+2}$ .

We now treat the three cases in the statement of the theorem by considering the distinct ways that this hyperplane can intersect the null cone. First, if  $\mathbb{K}_{A_1 \dots A_d}$  is simple and timelike, then non-zero vectors in the nullity of  $\mathbb{K}_{A_1 \dots A_d}$  are spacelike. No non-zero vector in their span is null or timelike. Therefore in this case the  $(m + 2)$ -plane has no intersection with  $\mathcal{C}_+$ . Thus the zero locus  $\mathcal{Z}(\mathcal{K})$  is empty. For the second case, suppose that  $\mathbb{K}_{A_1 \dots A_d}$  is spacelike, then the hyperplane defined by vectors in the nullity of  $\mathbb{K}_{A_1 \dots A_d}$  can be spanned by one timelike vector and  $m + 1$  spacelike vectors. Such an  $(m + 2)$ -plane meets the null cone  $\mathcal{C}_+$  transversely, and hence, under ray projectivisation, will descend to a submanifold  $\mathcal{Z}(\mathcal{K})$  of  $S^n$  of codimension  $d$ . Finally, if  $\mathbb{K}_{A_1 \dots A_d}$  is null, then, using the Minkowski signature, it follows that the simple  $d$ -tractor  $\mathbb{K}$  can be obtained as the exterior product of covectors that are spacelike except for exactly one which is null. Dually, this implies that there is a collection of vectors which span the nullity  $(m + 2)$ -hyperplane that consists of a single null vector and  $m + 1$  spacelike vectors. Thus the hyperplane is tangent to the null cone, and after ray projectivisation the intersection descends to an isolated point. Now by Theorem 2.6 of [21] it then follows that on  $M$  the zero locus of the simple  $d$ -tractor  $\mathbb{K}$  will take the same form as on the model. Thus the three bullet points follow from the analysis just done, of the corresponding cases on the model, save for the very final statement that if  $\mathcal{Z}(\mathcal{K})$  is nonempty and does not just consist of isolated points, then the codimension  $d$  submanifold  $\mathcal{Z}(\mathcal{K})$  is distinguished. But this follows from Theorem 1.5.  $\square$

We note that such simple parallel tractors  $\mathbb{K}$  have arisen in the study of holonomy and generalisations of almost Einstein structures [3, 81, 82].

In the case of the model, meaning  $S^n$  with its usual conformal structure, all solutions to first BGG equations are normal. Moreover, it is easily seen that, in this setting, the space of solutions to the conformal Killing-Yano equation (6.6), of a given rank, is spanned by solutions  $k$  with  $L(k)$  satisfying the conditions of the Theorem above. It is interesting and valuable to determine the extent to which similar results hold in more general settings. In this case parts of Theorem 7.2 generalise quite easily, as we see in the following proposition.

**Proposition 7.3.** *Let  $k_{a_1 \dots a_{d-1}} \in \Gamma(\mathcal{E}_{[a_1 \dots a_{d-1}]}[2d - 2])$ , and*

$$\mathcal{K} := (k_{a_1 \dots a_{d-1}}, \nabla^c k_{ca_2 \dots a_{d-1}}), \text{ for any } g \in \mathfrak{c} \text{ with Levi-Civita connection } \nabla.$$

*Then*

1. *If  $L(k_{a_1 \dots a_{d-1}})$  is timelike, then  $\mathcal{Z}(\mathcal{K})$  is necessarily empty.*

2. If  $k_{a_1 \dots a_{d-1}}$  is such that  $L(k_{a_1 \dots a_{d-1}})$  is spacelike, simple, and satisfies (6.6) along  $\mathcal{Z}(\mathcal{K})$ , then  $\mathcal{Z}(\mathcal{K})$  is either empty or is a submanifold of codimension  $d$ .

*Proof.* From (2.22), (2.23), and (2.17) it follows that any tractor  $d$ -form  $\mathbb{K}$  satisfying  $X \lrcorner \mathbb{K} = 0$  at  $p \in M$  has  $\mathbb{K}_{A_1 \dots A_d} \mathbb{K}^{A_1 \dots A_d} \geq 0$  at  $p$ . This proves 1.

Next to consider 2 we work again in a scale  $g \in \mathfrak{c}$  and use  $g$  also to trivialise the density bundles. From the simplicity of  $L(k)$  it follows that  $k$  takes the form

$$f_1 w_1 + \dots + f_d w_d \tag{7.4}$$

where the  $w_i$ ,  $i = 1, \dots, d$ , are each simple  $(d - 1)$ -forms and the  $f_i$  are functions. Moreover, along  $\mathcal{Z}(\mathcal{K})$  (and thus also in a neighbourhood of  $\mathcal{Z}(\mathcal{K})$ ) the set  $\{w_1, \dots, w_d\}$  is pointwise linearly independent, as follows easily from the fact that the  $\mathbb{Z}$  slot  $\mu$  of  $L(k)$  is spacelike and so is, in particular, nowhere zero along  $\mathcal{Z}(\mathcal{K})$ .

Next we observe that  $\mathcal{Z}(\mathcal{K}) = \mathcal{Z}(k)$ . This follows because, from the simplicity  $L(k)$ , it follows that its  $\mathbb{W}$  slot vanishes whenever  $f_1, \dots, f_d$  vanish. But from the definition of  $L(k)$  this slot is a constant times the divergence of  $k$ . Now, from the equation (6.6) we have that

$$\nabla k = c \cdot \mu, \tag{7.5}$$

along  $\mathcal{Z}(\mathcal{K}) = \mathcal{Z}(k)$ , for a non-zero constant  $c$ .

We put these observations together. From (7.4) it follows that  $\mathcal{Z}(k)$  is the common zero set of  $f_1, \dots, f_d$ . Imposing (7.5) puts conditions on the  $f_i$  and their relation to the  $w_j$ . In particular one easily concludes that, at each point in  $\mathcal{Z}(k)$ , the equation (7.5) implies that  $\{df_1, \dots, df_d\}$  is a linearly independent set. Thus from the constant rank theorem it follows that  $\mathcal{Z}(\mathcal{K})$  is either empty or is a submanifold of codimension  $d$ .  $\square$

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