

AN INTRODUCTION TO CONFORMAL GEOMETRY AND TRACTOR CALCULUS, WITH A VIEW TO APPLICATIONS IN GENERAL RELATIVITY

SEAN N. CURRY & A. ROD GOVER

ABSTRACT. The following are expanded lecture notes for the course of eight one hour lectures given by the second author at the 2014 summer school Asymptotic Analysis in General Relativity held in Grenoble by the Institut Fourier. The first four lectures deal with conformal geometry and the conformal tractor calculus, taking as primary motivation the search for conformally invariant tensors and differential operators. The final four lectures apply the conformal tractor calculus to the study of conformally compactified geometries, motivated by the conformal treatment of infinity in general relativity.

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0. INTRODUCTION

Definition 1. A conformal n -manifold ($n \geq 3$) is the structure (M, \mathbf{c}) where

- M is an n -manifold,
- \mathbf{c} is a conformal equivalence class of signature (p, q) metrics,
i.e. $g, \hat{g} \in \mathbf{c} \stackrel{\text{def.}}{\iff} \hat{g} = \Omega^2 g$ and $C^\infty(M) \ni \Omega > 0$.

To any pseudo-Riemannian n -manifold (M, g) with $n \geq 3$ there is an associated conformal manifold $(M, [g])$ where $[g]$ is the set of all metrics \hat{g} which are smooth positive multiples of the metric g . In Riemannian signature $(p, q) = (n, 0)$ passing to the conformal manifold means geometrically that we are forgetting the notion of lengths (of tangent vectors and of curves) and retaining only the notion of angles (between tangent vectors and curves) and of ratios of lengths (of tangent vectors at a fixed point) associated to the metric g . In Lorentzian signature $(n - 1, 1)$ passing to the conformal manifold means forgetting the “spacetime interval” (analogous to length in Riemannian signature) and retaining only the light cone structure of the Lorentzian manifold. On a conformal Lorentzian manifold one also has the notion of angles between intersecting spacelike curves and the notion of orthogonality of tangent vectors at a point, but the conformal structure itself is determined by the light cone structure, justifying the use of the word *only* in the previous sentence.

The significance of conformal geometry for general relativity largely stems from the fact that the light cone structure determines the *causal structure* of spacetime (and, under some mild assumptions, the causal structure determines the light cone structure). On top of this we shall see in the lectures that the Einstein field equations admit a very nice interpretation in terms of conformal geometry.

In these notes we will develop the natural invariant calculus on conformal manifolds, the (conformal) tractor calculus, and apply it to the study of conformal invariants and of conformally compactified geometries. The course is divided into two parts consisting of four lectures each. The first four lectures deal with conformal geometry and the conformal tractor calculus, taking as primary motivation the problem of constructing conformally invariant tensors and differential operators. The tools developed for this problem however allow us to tackle much more than our original problem of invariants and invariant operators. In the final four lectures they will be applied in particular to the study of conformally compactified geometries, motivated by the conformal treatment of infinity in general relativity. Along the way we establish the connection between the conformal tractor calculus and Helmut Friedrich’s conformal field equations. We also digress for one lecture, discussing conformal hypersurface geometry, in order to facilitate the study of the relationship of the geometry of conformal infinity to that of the interior. Finally we show how the tractor calculus may be applied to treat aspects of the asymptotic analysis of boundary problems on conformally compact manifolds. For completeness an appendix has been added which covers further aspects of the conformal tractor calculus as well as discussing briefly the canonical conformal Cartan bundle and connection.

The broad philosophy behind our ensuing discussion is that conformal geometry is important not only for understanding conformal manifolds, or conformally invariant aspects of pseudo-Riemannian geometry (such as conformally invariant field equations), but that it is highly profitable to think of a pseudo-Riemannian manifold as a kind of symmetry breaking (or holonomy reduction) of a conformal manifold whenever there are any (even remotely) conformal geometry related aspects of the problem being considered. Our discussion of the conformal tractor

calculus will lead us naturally to the notions of almost Einstein and almost pseudo-Riemannian geometries, which include Einstein and pseudo-Riemannian manifolds (respectively) as well as their respective conformal compactifications (should they admit one). The general theory of Cartan holonomy reductions then enables us to put constraints on the smooth structure and the geometry of conformal infinities of Einstein manifolds, and the tractor calculus enables us to partially generalise these results to pseudo-Riemannian manifolds.

We deal exclusively with conformal manifolds of at least three dimensions in these notes. That is not to say that two dimensional conformal manifolds cannot be fitted in to the framework which we describe. However, in order to have a canonical tractor calculus on two dimensional conformal manifolds the conformal manifold needs to be equipped with an extra structure, weaker than a Riemannian structure but stronger than a conformal structure, called a *Möbius structure*. In higher dimensions a conformal structure determines a canonical Möbius structure via the construction of the canonical conformal Cartan bundle and connection (outlined in the appendix, Section A.1). In two dimensions there is no canonical Cartan bundle and connection associated to a conformal manifold, this (Möbius) structure must be imposed as an additional assumption if we wish to work with the tractor calculus. We note that to any Riemannian 2-manifold, or to any (nondegenerate) 2-dimensional submanifold of a higher dimensional conformal manifold, there is associated a canonical Möbius structure and corresponding tractor calculus.

Also left out in these notes is any discussion of conformal spin geometry. In this case there is again a canonical tractor calculus, known as *spin tractor calculus* or *local twistor calculus*, which is a refinement of the usual conformal tractor calculus in the same way that spinor calculus is a refinement of the usual tensor calculus on pseudo-Riemannian spin manifolds. The interested reader is referred to [4, 51].

0.1. Notation and conventions. We may use abstract indices, or no indices, or frame indices according to convenience. However, we will make particularly heavy use of the abstract index notation. For instance if L is a linear endomorphism of a finite dimensional vector space V then we may choose to write L using abstract indices as L^a_b (or L^b_c , or $L^{a'}_{b'}$, it makes no difference as the indices are just place holders meant to indicate tensor type, and contractions). In this case we would write a vector $v \in V$ as v^a (or v^b , or $v^{a'}$, ...) and we would write the action of L on v as $L^a_b v^b$ (repeated indices denote tensor contraction so $L^a_b v^b$ simply means $L(v)$). Similarly if $w \in V^*$ then using abstract indices we would write w as w_a (or w_b , or $w_{a'}$, ...) and $w(v)$ as $w_a v^a$, whereas the outer product $v \otimes w \in \text{End}(V)$ would be written as $v^a w_b$ (or $v^b w_c$, or $w_b v^a$, ...). A covariant 2-tensor $T \in \otimes^2 V^*$ may be written using abstract indices as T_{ab} , the symmetric part of T is then denoted by $T_{(ab)}$ and the antisymmetric part by $T_{[ab]}$, that is

$$T_{(ab)} = \frac{1}{2} (T_{ab} + T_{ba}) \quad \text{and} \quad T_{[ab]} = \frac{1}{2} (T_{ab} - T_{ba}).$$

Note that swapping the indices a and b in T_{ab} amounts to swapping the “slots” of the covariant two tensor (so that b becomes the label for the first slot and a for the

second), this gives in general a different covariant two tensor from T_{ab} whose matrix with respect to any basis for V would be the transpose of that of T_{ab} . We can similarly define the symmetric or antisymmetric part of any covariant tensor $T_{ab\dots e}$ and these are denoted by $T_{(ab\dots e)}$ and $T_{[ab\dots e]}$ respectively. We use the same bracket notation for the symmetric and antisymmetric parts of contravariant tensors. Note that we do not have to symmetrise or skew-symmetrise over all indices, for instance $T^{a[bc]}_d$ denotes

$$\frac{1}{2} (T^{abc}_d - T^{acb}_d).$$

The abstract index notation carries over in the obvious way to vector and tensor fields on a manifold. The virtue of using abstract index notation on manifolds is that it makes immediately apparent the type of tensorial object one is dealing with and its symmetries without having to bring in extraneous vector fields or 1-forms. We will commonly denote the tangent bundle of M by \mathcal{E}^a , and the cotangent bundle of M by \mathcal{E}_a . We then denote the bundle of covariant 2-tensors by \mathcal{E}_{ab} , its subbundle of symmetric 2-tensors by $\mathcal{E}_{(ab)}$, and so on. In order to avoid confusion, when working on the tangent bundle of a manifold M we will always use lower case Latin abstract indices taken from the beginning of the alphabet (a, b , etc.) whereas we will take our frame indices from a later part of the alphabet (starting from i, j , etc.).

It is common when working with abstract index notation to use the same notation \mathcal{E}^a for the tangent bundle and its space of smooth sections. Here however we have used the notation $\Gamma(\mathcal{V})$ for the space of smooth sections of a vector bundle \mathcal{V} consistently throughout, with the one exception that for a differential operator D taking sections of a vector bundle \mathcal{U} to sections of \mathcal{V} we have written

$$D : \mathcal{U} \rightarrow \mathcal{V}$$

in order to simplify the notation.

Consistent with our use of \mathcal{E}^a for TM we will often denote by \mathcal{E} the trivial \mathbb{R} -bundle over our manifold M , so that $\Gamma(\mathcal{E}) = C^\infty(M)$. When using index free notation we denote the space of vector fields on M by $\mathfrak{X}(M)$, and we use the shorthand Λ^k for $\Lambda^k T^*M$ (when the underlying manifold M is understood). Unless otherwise indicated $[\cdot, \cdot]$ is the commutator bracket acting on pairs of endomorphisms. Note that the Lie bracket arises in this way when we consider vector fields as derivations of the algebra of smooth functions. In all of the following all structures (manifolds, bundles, tensor fields, etc.) will be assumed smooth, meaning C^∞ .

0.1.1. *Coupled connections.* We assume that the reader is familiar with the notion of a linear connection on a vector bundle $\mathcal{V} \rightarrow M$ and the special case of an affine connection (being a connection on the tangent bundle of a manifold). Given a pair of vector bundles \mathcal{V} and \mathcal{V}' over the same manifold M and linear connections ∇ and ∇' defined on \mathcal{V} and \mathcal{V}' respectively, then there is a natural way to define a connection on the tensor product bundle $\mathcal{V} \otimes \mathcal{V}' \rightarrow M$. The *coupled connection* ∇^\otimes on $\mathcal{V} \otimes \mathcal{V}'$ is given on a simple section $v \otimes v'$ of $\mathcal{V} \otimes \mathcal{V}'$ by the Leibniz formula

$$\nabla_X^\otimes(v \otimes v') = (\nabla_X v) \otimes v' + v \otimes (\nabla'_X v')$$

for any $X \in \mathfrak{X}(M)$. Since $\Gamma(\mathcal{V} \otimes \mathcal{V}')$ is (locally) generated by simple sections, this formula determines the connection ∇^\otimes uniquely. In order to avoid clumsy notation we will often simply write all of our connections as ∇ when it is clear from the context which (possibly coupled) connection is being used.

0.1.2. *Associated bundles.* We assume also that the reader is familiar with the notion of a vector bundle on a smooth manifold M as well as that of an H -principal bundle over M , where H is a Lie group. If $\pi : \mathcal{G} \rightarrow M$ is a (right) H -principal bundle and \mathbb{V} is a (finite dimensional) representation of H then the *associated vector bundle* $\mathcal{G} \times_H \mathbb{V} \rightarrow M$ is the vector bundle with total space defined by

$$\mathcal{G} \times_H \mathbb{V} = \mathcal{G} \times \mathbb{V} / \sim$$

where \sim is the equivalence relation

$$(u, v) \sim (u \cdot h, h^{-1} \cdot v), \quad h \in H$$

on $\mathcal{G} \times \mathbb{V}$; the projection of $\mathcal{G} \times_H \mathbb{V}$ to M is simply defined by taking $[(u, v)]$ to $\pi(u)$. For example if \mathcal{F} is the linear frame bundle of a smooth n -manifold M then

$$TM = \mathcal{F} \times_{\mathrm{GL}(n)} \mathbb{R}^n \quad \text{and} \quad T^*M = \mathcal{F} \times_{\mathrm{GL}(n)} (\mathbb{R}^n)^*.$$

Similarly if H is contained in a larger Lie group G then one can extend any principal H -bundle $\mathcal{G} \rightarrow M$ to a principal G -bundle $\tilde{\mathcal{G}} \rightarrow M$ with total space

$$\tilde{\mathcal{G}} = \mathcal{G} \times_H G = \mathcal{G} \times G / \sim \quad \text{where} \quad (u, g) \sim (u \cdot h, h^{-1} \cdot g).$$

For example if (M, g) is a Riemannian n -manifold and \mathcal{O} denotes its orthonormal frame bundle then

$$\mathcal{F} = \mathcal{O} \times_{\mathrm{O}(n)} \mathrm{GL}(n)$$

is the Linear frame bundle of M .

1. LECTURE 1: RIEMANNIAN INVARIANTS AND INVARIANT OPERATORS

Recall that if ∇ is an affine connection then its *torsion* is the tensor field $T \in \Gamma(TM \otimes \Lambda^2 T^*M)$ defined by

$$T^\nabla(u, v) = \nabla_u v - \nabla_v u - [u, v] \quad \text{for all} \quad u, v \in \mathfrak{X}(M).$$

It is interesting that this is a tensor; by its construction one might expect a differential operator. Importantly torsion is an *invariant* of connections. On a smooth manifold the map

$$\nabla \mapsto T^\nabla$$

taking connections to their torsion depends only the smooth structure. By its construction here it is clear that it is independent of any choice of coordinates.

For any connection ∇ on a vector bundle \mathcal{V} its curvature is defined by

$$R^\nabla(u, v)W := [\nabla_u, \nabla_v]W - \nabla_{[u, v]}W \quad \text{for all} \quad u, v \in \mathfrak{X}(M), \quad \text{and} \quad W \in \Gamma(\mathcal{V}),$$

and $R^\nabla \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(\mathcal{V}))$. Again the map taking connections to their curvatures

$$\nabla \mapsto R^\nabla$$

clearly depends only the (smooth) vector bundle structure of \mathcal{V} over the smooth manifold M . The curvature is an invariant of linear connections. In particular this applies to affine connections, i.e. when $\mathcal{V} = TM$.

1.1. Ricci calculus and Weyl's invariant theory. The most familiar setting for these objects is pseudo-Riemannian geometry. In this case we obtain a beautiful local calculus that is sometimes called the Ricci calculus. Let us briefly recall how this works. Recall that a pseudo-Riemannian manifold consists of an n -manifold M equipped with a metric g of signature (p, q) , that is a section $g \in \Gamma(S^2T^*M)$ such that pointwise g is non-degenerate and of signature (p, q) . Then g canonically determines a distinguished affine connection called the Levi-Civita connection. This is the unique connection ∇ satisfying:

- $\nabla g = 0$ (metric compatibility), and
- $T^\nabla = 0$ (torsion freeness).

Thus on a smooth manifold we have a canonical map from each metric to its Levi-Civita connection

$$g \mapsto \nabla^g$$

and, as above, a canonical map which takes each Levi-Civita connection to its curvature $\nabla^g \mapsto R^{\nabla^g}$, called the *Riemannian curvature*. Composing these we get a canonical map that takes each metric to its curvature

$$g \longmapsto R^g,$$

and this map depends only on the smooth structure of M . So we say that R^g is an invariant of the pseudo-Riemannian manifold (M, g) . How can we construct more such invariants? Or “all” invariants, in some perhaps restricted sense?

The first, and perhaps most important, observation is that using the Levi-Civita connection and Riemannian curvature one can proliferate Riemannian invariants. To simplify the explanation let's fix a pseudo-Riemannian manifold (M, g) and use abstract index notation (when convenient). Then the metric is written g_{ab} and we write $R_{ab}{}^c{}_d$ for the Riemannian curvature. So if u, v, w are tangent vector fields then so is $R(u, v)w$ and this is written

$$u^a v^b R_{ab}{}^c{}_d w^d.$$

From curvature we can form the *Ricci* and *Scalar* curvatures, respectively:

$$\text{Ric}_{ab} := R_{ca}{}^c{}_b \quad \text{and} \quad \text{Sc} := g^{ab} \text{Ric}_{ab},$$

and these are invariants of (M, g) . As also are:

$$\nabla_b R_{cdef}, \quad \nabla_a \nabla_b R_{cdef},$$

which are *tensor valued invariants* and

$$\text{Ric}^{ab} \text{Ric}_{ab} \text{Sc} = |\text{Ric}|^2 \text{Sc}, \quad R_{abcd} R^{abcd} = |R|^2, \quad (\nabla_a R_{bcde}) \nabla^a R^{bcde} = |\nabla R|^2,$$

which are some *scalar valued invariants*. Here we have used the metric (and its inverse) to raise and lower indices and contractions are indicated by repeated indices.

Since this is a practical and efficient way to construct invariants, it would be useful to know: Do all local Riemannian invariants arise in this way? That is from partial or complete contractions of expressions made using g , R and its covariant derivatives $\nabla \cdots \nabla R$ (and the metric volume form vol^g , if M is oriented). We shall term invariants constructed this way *Weyl invariants*.

Before answering this one first needs to be careful about what is meant by a local invariant. For example, the following is a reasonable definition for scalar invariants.

Definition 2. A scalar Riemannian invariant P is a function which assigns to each pseudo-Riemannian n -manifold (M, g) a function $P(g)$ such that:

- (i) $P(g)$ is natural, in the sense that for any diffeomorphism $\phi : M \rightarrow M$ we have $P(\phi^*g) = \phi^*P(g)$.
- (ii) P is given by a universal polynomial expression of the nature that, given a local coordinate system (x^i) on (M, g) , $P(g)$ is given by a polynomial in the variables g_{mn} , $\partial_{i_1}g_{mn}$, \cdots , $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}g_{mn}$, $(\det g)^{-1}$, for some positive integer k .

Then with this definition, and a corresponding definition for tensor-valued invariants, it is true that all local invariants arise as *Weyl invariants*, and the result goes by the name of *Weyl's classical invariant theory*, see e.g. [1, 53]. Given this result, in the following when we mention pseudo-Riemannian invariants we will mean Weyl invariants.

1.2. Invariant operators, and analysis. In a similar way we can use the Ricci calculus to construct invariant differential operators on pseudo-Riemannian manifolds. For example the (Bochner) Laplacian is given by the formula

$$\nabla^a \nabla_a = \Delta : \mathcal{E} \longrightarrow \mathcal{E},$$

in terms of the Levi-Civita connection ∇ . There are also obvious ways to make operators with curvature in coefficients, e.g.

$$R_a{}^c{}_b{}^d \nabla_c \nabla_d : \mathcal{E} \longrightarrow \mathcal{E}_{(ab)}.$$

With suitable restrictions imposed, in analogy with the case of invariants, one can make the statement that all local natural invariant differential operators arise in this way. It is beyond our current scope to make this precise, suffice to say that when we discuss invariant differential operators on pseudo-Riemannian manifolds we will again mean operators constructed in this way.

Remark 1.1. If a manifold has a spin structure, then essentially the above is still true but there is a further ingredient involved, namely the Clifford product. This allows the construction of important operators such as the Dirac operator.

The main point here is that the ‘‘Ricci calculus’’ provides an effective and geometrically transparent route to the construction of invariants and invariant operators.

Invariants and invariant operators are the basic objects underlying the first steps (and often significantly more than just the first steps) of treating problems in general relativity and, more generally, in:

- the global analysis of manifolds;
- the study and application of geometric PDE;
- Riemannian spectral theory;
- physics and mathematical physics.

Furthermore from a purely theoretical point of view, we cannot claim to understand a geometry if we do not have a good theory of local invariants and invariant operators.

2. LECTURE 2: CONFORMAL TRANSFORMATIONS AND CONFORMAL COVARIANCE

A good theory of conformal geometry should provide some hope of treating the following closely related problems:

Problem 1. Describe a practical way to generate/construct (possibly all) local natural invariants of a conformal structure.

Problem 2. Describe a practical way to generate/construct (possibly all) natural linear differential operators that are canonical and well-defined on (i.e. are invariants of) a conformal structure.

We have not attempted to be precise in these statements, since here they are mainly for the purpose of motivation. Let us first approach these naïvely.

2.1. Conformal Transformations. Recall that for any metric g we can associate its Levi-Civita connection ∇ . Let $e_i = \frac{\partial}{\partial x^i} = \partial_i$ be a local coordinate frame and E^i its dual. Locally, any connection is determined by how it acts on a frame field. For the Levi-Civita connection the resulting *connection coefficients*

$$\Gamma^i_{jk} := E^i(\nabla_k e_j), \quad \text{where} \quad \nabla_k := \nabla_{e_k},$$

are often called the *Christoffel symbols*, and are given by the Koszul formula:

$$\Gamma^i_{jk} := \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l})$$

where $g_{ij} = g(e_i, e_j)$ and $g_{lj,k} = \partial_k g_{lj}$.

Using this formula for the Christoffel symbols we can easily compute the transformation formula for ∇ under a *conformal transformation* $g \mapsto \hat{g} = \Omega^2 g$. Let $\Upsilon_a := \Omega^{-1} \nabla_a \Omega$, $v^a \in \Gamma(\mathcal{E}^a)$ and $\omega_b \in \Gamma(\mathcal{E}_b)$. Then we have:

$$(1) \quad \nabla_a^{\hat{g}} v^b = \nabla_a v^b + \Upsilon_a v^b - \Upsilon^b v_a + \Upsilon^c v_c \delta_a^b,$$

$$(2) \quad \nabla_a^{\hat{g}} \omega_b = \nabla_a \omega_b - \Upsilon_a \omega_b - \Upsilon_b \omega_a + \Upsilon^c \omega_c g_{ab}.$$

For $\omega_b \in \Gamma(\mathcal{E}_b)$ we have from (2) that

$$\omega_b \mapsto \nabla_a \omega_b - \nabla_b \omega_a$$

is conformally invariant. But this is just the exterior derivative $\omega \mapsto d\omega$, its conformal invariance is better seen from the fact that it is defined on a smooth manifold without further structure: for $u, v \in \mathfrak{X}(M)$

$$d\omega(u, v) = u\omega(v) - v\omega(u) - \omega([u, v]).$$

Inspecting (2) it is evident that this is the only first order conformally invariant linear differential operator on T^*M that takes values in an irreducible bundle.

The Levi-Civita connection acts also on other tensor bundles. We can use the formulae (1) and (2) along with the product rule to compute the conformal transformation of the result. For example for a simple covariant 2-tensor

$$u_b \otimes w_c \quad \text{we have} \quad \widehat{\nabla}_a(u_b \otimes w_c) = (\widehat{\nabla}_a u_b) \otimes w_c + u_b \otimes (\widehat{\nabla}_a w_c).$$

Thus we can compute $\widehat{\nabla}_a(u_b \otimes w_c)$ by using (2) for each term on the right-hand-side. But locally any covariant 2-tensor is a linear combination of simple 2-tensors and so we conclude that for a covariant 2-tensor F_{bc}

$$(3) \quad \widehat{\nabla}_a F_{bc} = \nabla_a F_{bc} - 2\Upsilon_a F_{bc} - \Upsilon_b F_{ac} - \Upsilon_c F_{ba} + \Upsilon^d F_{dc} g_{ab} + \Upsilon^d F_{bd} g_{ac}.$$

By the obvious extension of this idea one quickly calculates the formula for the conformal transformation for the Levi-Civita covariant derivative of an (r, s) -tensor.

From the formula (3) we see that the completely skew part $\nabla_{[a} F_{bc]}$ is conformally invariant. In the case F is skew, in that $F_{bc} = -F_{cb}$, this recovers that dF is conformally invariant. A more interesting observation arises with the divergence $\nabla^b F_{bc}$. We have

$$\widehat{\nabla}^b F_{bc} = \widehat{g}^{ab} \widehat{\nabla}_a F_{bc},$$

and $\widehat{g}^{ab} = \Omega^{-2} g^{ab}$. Thus we obtain

$$\begin{aligned} \widehat{\nabla}^b F_{bc} &= \Omega^{-2} g^{ab} (\nabla_a F_{bc} - 2\Upsilon_a F_{bc} - \Upsilon_b F_{ac} - \Upsilon_c F_{ba} + \Upsilon^d F_{dc} g_{ab} + \Upsilon^d F_{bd} g_{ac}) \\ &= \Omega^{-2} (\nabla^b F_{bc} + (n-3)\Upsilon^d F_{dc} + \Upsilon^d F_{cd} - \Upsilon_c F_b{}^b). \end{aligned}$$

In particular then, if F is skew then $F_b{}^b = 0$ and we have simply

$$(4) \quad \widehat{\nabla}^b F_{bc} = \Omega^{-2} (\nabla^b F_{bc} + (n-4)\Upsilon^d F_{dc}).$$

So we see that something special happens in dimension 4. Combining with our earlier observation we have the following result.

Proposition 2.1. *In dimension 4 the differential operators*

$$\text{Div} : \Lambda^2 \rightarrow \Lambda^1 \quad \text{and} \quad \text{Max} : \Lambda^1 \rightarrow \Lambda^1$$

given by

$$F_{bc} \mapsto \nabla^b F_{bc} \quad \text{and} \quad u_c \mapsto \nabla^b \nabla_{[b} u_{c]}$$

respectively, are conformally covariant, in that

$$(5) \quad \widehat{\nabla}^b F_{bc} = \Omega^{-2} \nabla^b F_{bc} \quad \text{and} \quad \widehat{\nabla}^b \widehat{\nabla}_{[b} u_{c]} = \Omega^{-2} \nabla^b \nabla_{[b} u_{c]}.$$

The non-zero powers of Ω (precisely Ω^{-2}) appearing in (5) mean that these objects are only *covariant* rather than invariant. Conformal covariance is still a strong symmetry property however, as we shall see. Before we discuss that in more detail note that, for the equations, these factors of Ω make little difference:

$$\nabla^b F_{bc} = 0 \quad \Leftrightarrow \quad \widehat{\nabla}^b F_{bc} = 0$$

and

$$\nabla^b \nabla_{[b} u_{c]} = 0 \quad \Leftrightarrow \quad \widehat{\nabla}^b \widehat{\nabla}_{[b} u_{c]} = 0.$$

In this sense these equations are conformally invariant.

Remark 2.2. In fact these equations are rather important. If we add the condition that F is closed then on Lorentzian signature 4-manifolds the system

$$dF = 0 \quad \text{and} \quad \text{Div}(F) = 0$$

is the field strength formulation of the source-free *Maxwell equations* of electromagnetism. The locally equivalent equations $\text{Div}(du) = 0$ give the potential formulation of the (source-free) Maxwell equations. The conformal invariance of these has been important in Physics. ■

Returning to our search for conformally covariant operators and equations, our preliminary investigation suggests that such things might be rather rare. From (3) we see that the divergence of a 2-form is not conformally invariant except in dimension 4. In fact, in contrast to what this might suggest, there is a rich theory of conformally covariant operators. However there are some subtleties involved. Before we come to this it will be useful to examine how conformal rescaling affects the curvature.

2.2. Conformal rescaling and curvature. Using (1), (2) and the observations following these we can compute, for example, the conformal transformation formulae for the Riemannian curvature and its covariant derivatives and so forth. At the very lowest orders this provides a tractable approach to finding conformal invariants.

Let us fix a metric g . With respect to metric traces, we can decompose the curvature tensor of g into a trace-free part and a trace part in the following way:

$$R_{abcd} = \underbrace{W_{abcd}}_{\text{trace-free}} + \underbrace{2g_{c[a}P_{b]d} + 2g_{d[b}P_{a]c}}_{\text{trace part}}.$$

Here P_{ab} , so defined, is called the Schouten tensor while the tensor $W_{ab}{}^c{}_d$ is called the Weyl tensor. In dimensions $n \geq 3$ we have $\text{Ric}_{ab} = (n-2)P_{ab} + Jg_{ab}$, where $J := g^{ab}P_{ab}$. So the Schouten tensor P_{ab} is a trace modification of the Ricci tensor.

Exercise 1. Prove using (1) that under a conformal transformation $g \mapsto \widehat{g} = \Omega^2 g$, as above, the Weyl and Schouten tensors transform as follows:

$$W_{ab}{}^c{}_d{}^{\widehat{g}} = W_{ab}{}^c{}_d{}^g$$

and

$$(6) \quad P_{ab}^{\widehat{g}} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} g_{ab} \Upsilon_c \Upsilon^c.$$

Thus the Weyl curvature is a conformal invariant, while objects such as $|W|^2 := W_{abcd}W^{abcd}$ may be called *conformal covariants* because under the conformal change they pick up a power of the conformal factor $|\widehat{W}|^2 = \Omega^{-4}|W|^2$ (where for simplicity we are hatting the symbol for the object rather than the metric). We will see shortly that such objects correspond to invariants.

Here we are defining *conformal invariants* to be Riemannian invariants, that have the additional property of being unchanged under conformal transformation.

Exercise 2. Consider computing the conformal transformation of the derivatives of the curvature: ∇R , $\nabla\nabla R$, etcetera. Then possibly using this and an undetermined coefficient approach to finding conformal invariants or conformal covariants. This rapidly gets intractable.

2.3. Conformally invariant linear differential operators. We may try the corresponding approach for constructing further conformally invariant linear differential operators:

$$(7) \quad D^g : \mathcal{U} \rightarrow \mathcal{V}$$

such that $D^{\widehat{g}} = D^g$. Namely, first consider the possible Riemannian invariant linear differential operators between the bundles concerned and satisfying some order constraint. Next compute their transformation under a conformal change. Finally seek a linear combination of these that forms a conformally invariant operator. For our purposes this also defines what we mean by conformally invariant linear differential operator. For many applications we require that the domain and target bundles \mathcal{U} and \mathcal{V} are irreducible.

It turns out that irreducible tensor (or even spinor) bundles are not sufficient to deal with conformal operators. Let us see a first glimpse of this by recalling the construction of one of the most well known conformally invariant differential operators.

2.3.1. The conformal Wave operator. For analysis on pseudo-Riemannian manifolds (M, g) the Laplacian is an extremely important operator. The *Laplacian* Δ on functions, which is also called the *Laplace-Beltrami operator*, is given by

$$\Delta := \nabla^a \nabla_a : \mathcal{E} \rightarrow \mathcal{E},$$

where ∇ is the Levi-Civita connection for g .

Let us see how this behaves under conformal rescaling. For a function f , $\nabla_a f$ is simply the exterior derivative of f and so is conformally invariant. So to compute

the Laplacian for $\widehat{g} = \Omega^2 g$ we need only use (2) with $u := df$:

$$\begin{aligned}\Delta^{\widehat{g}} f &= \widehat{\nabla}^a \nabla_a f \\ &= \widehat{g}^{ab} \widehat{\nabla}_b \nabla_a f \\ &= \Omega^{-2} g^{ab} (\nabla_b \nabla_a f - \Upsilon_b \nabla_a f - \Upsilon_a \nabla_b f + g_{ab} \Upsilon^c \nabla_c f).\end{aligned}$$

So

$$(8) \quad \Delta^{\widehat{g}} f = \Omega^{-2} (\Delta f + (n-2) \Upsilon^c \nabla_c f).$$

By inspecting this formula we learn two things. Let us summarise with a proposition.

Proposition 2.3. *The Laplacian on functions is conformally covariant in dimension 2, but not in other dimensions.*

This is reminiscent of our observations surrounding the expression (4) and the Maxwell system (cf. Proposition 2.1).

We need to modify the Laplacian to have any hope of obtaining an invariant operator in other dimensions. A key idea will be to introduce a curvature into the formula for a new Laplacian. However inspecting the formulae for curvature transformation in Section 2.2 it is easily seen that this manoeuvre alone will not deal with the term $\Upsilon^c \nabla_c f$ in (8).

To eliminate the term $(n-2) \Upsilon^c \nabla_c f$ we will make what seems like a strange move (and we will explain later the mathematics behind this). We will allow the domain “function” to depend on the choice of metric in the following way: We decree that the function f on (M, g) corresponds to $\widehat{f} := \Omega^{1-\frac{n}{2}} f$ on (M, \widehat{g}) , where $\widehat{g} = \Omega^2 g$ as above. Now let us calculate $\widehat{\Delta} \widehat{f}$. First we have:

$$\begin{aligned}\nabla_a (\Omega^{1-\frac{n}{2}} f) &= \left(1 - \frac{n}{2}\right) \Omega^{1-\frac{n}{2}} \Upsilon_a f + \Omega^{1-\frac{n}{2}} df \\ &= \Omega^{1-\frac{n}{2}} \left(\nabla_a f + \left(1 - \frac{n}{2}\right) \Upsilon_a f\right).\end{aligned}$$

We use this in the next step:

$$\begin{aligned}\Delta^{\widehat{g}} \widehat{f} &= \Omega^{-2} g^{ab} \widehat{\nabla}_b \left[\Omega^{1-\frac{n}{2}} \left(\nabla_a f + \left(1 - \frac{n}{2}\right) \Upsilon_a f\right) \right] \\ &= \Omega^{-1-\frac{n}{2}} g^{ab} \left[\nabla_b \nabla_a f + \left(1 - \frac{n}{2}\right) \Upsilon_a \nabla_b f + \left(1 - \frac{n}{2}\right) \Upsilon_b \nabla_a f + \left(1 - \frac{n}{2}\right)^2 \Upsilon_a \Upsilon_b f \right. \\ &\quad + \left(1 - \frac{n}{2}\right) f \nabla_b \Upsilon_a - \Upsilon_b \nabla_a f - \left(1 - \frac{n}{2}\right) \Upsilon_b \Upsilon_a f - \Upsilon_a \nabla_b f \\ &\quad \left. - \left(1 - \frac{n}{2}\right) \Upsilon_a \Upsilon_b f + g_{ab} \left(\Upsilon^c \nabla_c f + \left(1 - \frac{n}{2}\right) \Upsilon^2 f\right) \right] \\ &= \Omega^{-1-\frac{n}{2}} \left[\Delta f + \left(1 - \frac{n}{2}\right) \left(\nabla^a \Upsilon_a + \Upsilon^2 \left(\frac{n}{2} - 1\right)\right) f \right],\end{aligned}$$

where we used (2) once again and have written Υ^2 as a shorthand for $\Upsilon^a \Upsilon_a$.

Now in the last line of the formula for $\Delta^{\widehat{g}}\widehat{f}$ the terms involving Υ do not involve derivatives of f . Thus there is hope of matching this with a curvature transformation. Indeed contracting (6) with \widehat{g}^{-1} gives

$$(9) \quad \mathbf{J}^{\widehat{g}} = \Omega^{-2} \left(\mathbf{J} - \nabla^a \Upsilon_a + \left(1 - \frac{n}{2}\right) \Upsilon^2 \right),$$

and so

$$(10) \quad \left(\Delta^{\widehat{g}} + \left(1 - \frac{n}{2}\right) \mathbf{J}^{\widehat{g}} \right) \widehat{f} = \Omega^{-1-\frac{n}{2}} \left(\Delta + \left(1 - \frac{n}{2}\right) \mathbf{J} \right) f,$$

and we have found a Laplacian operator with a symmetry under conformal rescaling. Using the relation between \mathbf{J} and the scalar curvature this is written as in the definition here.

Definition 3. On a pseudo-Riemannian manifold (M, g) the operator

$$Y^g : \mathcal{E} \rightarrow \mathcal{E} \quad \text{defined by} \quad Y^g := \Delta^g - \frac{n-2}{4(n-1)} \text{Sc}^g$$

is called the *conformal Laplacian* or, in Lorentzian signature, the *conformal wave operator*.

Remark 2.4. On Lorentzian signature manifolds it is often called the *conformal wave operator* because the leading term agrees with the operator giving usual wave equation. It seems that it was in this setting that the operator was first discovered and applied [54, 16]. On the other hand in the setting of Riemannian signature Y is often called the *Yamabe operator* because of its role in Yamabe problem of scalar curvature prescription. ■

According to our calculations above this has the following remarkable symmetry property with respect to conformal rescaling.

Proposition 2.5. *The conformal Laplacian is conformally covariant in the sense that*

$$Y^{\widehat{g}} \circ \Omega^{1-\frac{n}{2}} = \Omega^{-1-\frac{n}{2}} \circ Y^g.$$

This property of the conformal Laplacian motivates a definition.

Definition 4. On pseudo-Riemannian manifolds a natural linear differential operator P^g , on a function or tensor/spinor field is said to be a *conformally covariant operator* if for all positive functions Ω

$$P^{\widehat{g}} \circ \Omega^{w_1} = \Omega^{w_2} \circ P^g,$$

where $\widehat{g} = \Omega^2 g$, $(w_1, w_2) \in \mathbb{R} \times \mathbb{R}$, and where we view the powers of Ω as multiplication operators.

In this definition it is not meant that the domain and target bundles are necessarily the same. The example in our next exercise will be important for our later discussions.

Exercise 3. On pseudo-Riemannian manifolds $(M^{n \geq 3}, g)$ show that

$$(11) \quad \begin{aligned} A_{ab}^g : \mathcal{E} &\longrightarrow \mathcal{E}_{(ab)_0} \text{ given by} \\ f &\longmapsto \nabla_{(a} \nabla_{b)_0} f + P_{(ab)_0} f \end{aligned}$$

is conformally covariant with $(w_1, w_2) = (1, 1)$. That is if $\hat{g} = \Omega^2 g$, for some positive function Ω , then

$$A^{\hat{g}}(\Omega f) = \Omega(A^g f).$$

2.4. Conformal Geometry. Recall that we defined a conformal manifold as a manifold M equipped only with an equivalence class of conformally related metric (see Definition 1). Conformally covariant operators, as in Definition 4, have good conformal properties but (in general) fail to be invariant in the sense of (7). This is not just an aesthetic shortcoming, it means that they are not well-defined on conformal manifolds (M, \mathbf{c}) . To construct operators on (M, g) that do descend to the corresponding conformal structure $(M, \mathbf{c} = [g])$ we need the notion of conformal densities.

2.4.1. Conformal densities and the conformal metric. Let (M, \mathbf{c}) be a *conformal manifold* of signature (p, q) (with $p + q = n$). For a point $x \in M$, and two metrics g and \hat{g} from the conformal class, there is an element $s \in \mathbb{R}_+$ such that $\hat{g}_x = s^2 g_x$ (where the squaring of s is a convenient convention). Thus, we may view the conformal class as being given by a smooth ray subbundle $\mathcal{Q} \subset S^2 T^* M$, whose fibre at x is formed by the values of g_x for all metrics g in the conformal class. By construction, \mathcal{Q} has fibre \mathbb{R}_+ and the metrics in the conformal class are in bijective correspondence with smooth sections of \mathcal{Q} .

Denoting by $\pi : \mathcal{Q} \rightarrow M$ the restriction to \mathcal{Q} of the canonical projection $S^2 T^* M \rightarrow M$, we can view this as a principal bundle with structure group \mathbb{R}_+ . The usual convention is to rescale a metric g to $\hat{g} = \Omega^2 g$. This corresponds to a principal action given by $\rho^s(g_x) = s^2 g_x$ for $s \in \mathbb{R}_+$ and $g_x \in \mathcal{Q}_x$, the fibre of \mathcal{Q} over $x \in M$.

Having this, we immediately get a family of basic real line bundles $\mathcal{E}[w] \rightarrow M$ for $w \in \mathbb{R}$ by defining $\mathcal{E}[w]$ to be the associated bundle to \mathcal{Q} with respect to the action of \mathbb{R}_+ on \mathbb{R} given by $s \cdot t := s^{-w} t$. The usual correspondence between sections of an associated bundle and equivariant functions on the total space of a principal bundle then identifies the space $\Gamma(\mathcal{E}[w])$ of smooth sections of $\mathcal{E}[w]$ with the space of all smooth functions $f : \mathcal{Q} \rightarrow \mathbb{R}$ such that $f(\rho^s(g_x)) = s^w f(g_x)$ for all $s \in \mathbb{R}_+$. We shall call $\mathcal{E}[w]$ the bundle of *conformal densities of weight w* . Note that each bundle $\mathcal{E}[w]$ is trivial, and inherits an orientation from that on \mathbb{R} . We write $\mathcal{E}_+[w]$ for the ray subbundle consisting of positive elements.

If $\hat{g} = \Omega^2 g \in \mathbf{c}$ then the conformally related metrics \hat{g} and g each determine sections of \mathcal{Q} . We may pull back f via these sections and we obtain functions on M related by

$$f^{\hat{g}} = \Omega^w f^g.$$

With $w = 1 - \frac{n}{2}$ this explains the “strange move” in Section 2.3.1 for the domain function of the conformal wave operator; the conformal wave operator is really an operator on the bundle $\mathcal{E}[1 - \frac{n}{2}]$.

Although the bundle $\mathcal{E}[w]$ as we defined it depends on the choice of the conformal structure, it is naturally isomorphic to a density bundle (which is independent of the conformal structure). Recall that the bundle of α -densities is associated to the full linear frame bundle of M with respect to the 1-dimensional representation $A \mapsto |\det(A)|^{-\alpha}$ of the group $GL(n, \mathbb{R})$. In particular, 2-densities may be canonically identified with the oriented bundle $(\Lambda^n T^*M)^2$, and 1-densities are exactly the geometric objects on manifolds that may be integrated (in a coordinate-independent way).

To obtain the link with conformal densities, as defined above, recall that any metric g on M determines a nowhere vanishing 1-density, the volume density $\text{vol}(g)$. In a local frame, this density is given by $\sqrt{|\det(g_{ij})|}$, which implies that for a positive function Ω we get $\text{vol}(\Omega^2 g) = \Omega^n \text{vol}(g)$. So there is bijective a map from 1-densities to functions $\mathcal{Q} \rightarrow \mathbb{R}$ that are homogeneous of degree $-n$ given by the map

$$\phi \mapsto \phi(x) / \text{vol}(g)(x),$$

and this gives an identification of the 1-density bundle with $\mathcal{E}[-n]$ and thus an identification of $\mathcal{E}[w]$ with the bundle of $(-\frac{w}{n})$ -densities on M .

So we may think of conformal density bundles as those bundles associated to the frame bundle via 1-dimensional representations, just as tensor bundles are associated to higher rank representations. Given any vector bundle \mathcal{B} we will use the notation

$$\mathcal{B}[w] := \mathcal{B} \otimes \mathcal{E}[w],$$

and say the bundle is *weighted* of weight w . Note that $\mathcal{E}[w] \otimes \mathcal{E}[w'] = \mathcal{E}[w + w']$ and that in the above we assume that \mathcal{B} is not a density bundle itself and is unweighted (weight zero).

Clearly, sections of such weighted bundles may be viewed as homogeneous (along the fibres of \mathcal{Q}) sections of the pullback along $\pi : \mathcal{Q} \rightarrow M$. Now the tautological inclusion of $\tilde{\mathbf{g}} : \mathcal{Q} \rightarrow \pi^* S^2 T^* M$ is evidently homogeneous of degree 2, as for $(s^2 g_x, x) \in \mathcal{Q}$, we have $\tilde{\mathbf{g}}(s^2 g_x, x) = (s^2 g_x, x) \in \pi^* S^2 T^* M$. So $\tilde{\mathbf{g}}$ may be identified with a canonical section of $\mathbf{g} \in \Gamma(S^2 T^* M[2])$ which provides another description of the conformal class. We call \mathbf{g} the *conformal metric*.

Another way to recover \mathbf{g} is to observe that any metric $g \in \mathbf{c}$ is a section of \mathcal{Q} , and hence determines a section $\sigma_g \in \Gamma(\mathcal{E}_+[1])$ with the characterising property that the corresponding homogeneous function $\tilde{\sigma}_g$ on \mathcal{Q} takes the value 1 along the section g . Then

$$(12) \quad \mathbf{g} = (\sigma_g)^2 g$$

and it is easily verified that this is independent of the choice of $g \in \mathbf{c}$. Conversely it is clear that any section $\sigma \in \Gamma(\mathcal{E}_+[1])$ determines a metric via

$$g := \sigma^{-2} \mathbf{g}.$$

On a conformal manifold we call $\sigma \in \Gamma(\mathcal{E}_+[1])$, or equivalently the corresponding $g \in \mathfrak{c}$, a *choice of scale*.

A nice application of \mathbf{g} is that we can use it to raise, lower, and contract tensor indices on a conformal manifold, for example

$$\mathbf{g}_{ab} : \mathcal{E}^a \rightarrow \mathcal{E}_b[2] \quad \text{by} \quad v^a \mapsto \mathbf{g}_{ab}v^a,$$

just as we use the metric in pseudo-Riemannian geometry. Also \mathbf{g} gives the isomorphism

$$(13) \quad \otimes^n \mathbf{g} : (\Lambda^n TM)^2 \xrightarrow{\simeq} \mathcal{E}[2n].$$

2.4.2. *Some calculus with conformal densities.* Observe that a choice of scale $g \in \mathfrak{c}$ determines a connection on $\mathcal{E}[w]$ via the formula

$$(14) \quad \nabla^g \tau := \sigma^w (d(\sigma^{-w} \tau)), \quad \tau \in \Gamma(\mathcal{E}[w]),$$

where d is the exterior derivative and $\sigma \in \Gamma(\mathcal{E}_+[1])$ satisfies $\mathbf{g} = \sigma^2 g$, as $\sigma^{-w} \tau$ is a function (i.e. is a section of $\mathcal{E}[0] = \mathcal{E}$). Coupling this to the Levi-Civita connection for g (and denoting both ∇^g) we have at once that $\nabla^g \sigma = 0$ and hence

$$(15) \quad \nabla^g \mathbf{g} = 0.$$

On the other hand the Levi-Civita connection directly determines a linear connection on $\mathcal{E}[w]$ since the latter is associated to the frame bundle, as mentioned above. But (15), with (13), shows that this agrees with (14). That is, (14) is the Levi-Civita connection on $\mathcal{E}[w]$. Thus we have:

Proposition 2.6. *On a conformal manifold (M, \mathfrak{c}) the conformal metric \mathbf{g} is preserved by the Levi-Civita connection ∇^g for any $g \in \mathfrak{c}$.*

In view of Proposition 2.6 it is reasonable to use the conformal metric to raise and lower tensor indices even when working in a scale! We shall henceforth do this unless we state otherwise. This enables us to give formulae for natural conformally invariant operators acting between density bundles. For example now choosing $g \in \mathfrak{c}$ and forming the Laplacian, and trace of Schouten,

$$\Delta := \mathbf{g}^{ab} \nabla_a \nabla_b \quad \text{and} \quad \mathbf{J} := \mathbf{g}^{ab} P_{ab},$$

we have the following. The conformal Laplacian can be interpreted as a differential operator

$$Y : \mathcal{E}[1 - \frac{n}{2}] \rightarrow \mathcal{E}[-1 - \frac{n}{2}], \quad \text{given by} \quad \Delta + \left(1 - \frac{n}{2}\right) \mathbf{J},$$

that is *conformally invariant*, meaning that it is well-defined on conformal manifolds.

Similarly the operator A_{ab}^g of (11) is equivalent to a conformally invariant operator,

$$A_{ab} : \mathcal{E}[1] \rightarrow \mathcal{E}_{(ab)_0}[1].$$

2.4.3. *Conformal transformations.* The above constructions enable us to understand how conformally covariant objects may be reinterpreted as objects that descend to invariant operators on conformal manifolds. A similar result applies to curvature covariants. However we have not advanced the problem of constructing these.

It follows at once from the formula (14) that under conformal transformations $g \mapsto \hat{g} = \Omega^2 g$ the Levi-Civita connection on $\mathcal{E}[w]$ transforms by

$$(16) \quad \nabla_a^{\hat{g}} \tau = \nabla_a^g \tau + w \Upsilon_a \tau,$$

since $\sigma_{\hat{g}} = \Omega^{-1} \sigma_g$. We can combine this with the transformation formulae (1), (2), and (6) to compute the conformal transformations of weighted tensors and Riemannian invariants. However this remains a hopeless approach to finding conformal invariants.

3. LECTURE 3: PROLONGATION AND THE TRACTOR CONNECTION

If we are going to be successful at calculating conformal invariants we are going to need a better way to calculate, one which builds in conformal invariance from the start. In the next two lectures we develop such a calculus, the *conformal tractor calculus*, which can be used to proliferate conformally invariant tensor (or tractor) expressions.

For treating conformal geometry it would be clearly desirable to find an analogue of the Ricci calculus available in the pseudo-Riemannian setting. On the tangent bundle a conformal manifold (M, \mathbf{c}) has a distinguished equivalence class of connections but no distinguished connection from this class. So at first the situation does not look promising. However we will see that if we pass from the tangent bundle to a vector bundle with two more dimensions (the standard tractor bundle), then there is indeed a distinguished connection.

There are many ways to see how the tractor calculus arises on a conformal manifold; we will give a very explicit construction which facilitates calculation, but first it will be very helpful to examine how the tractor calculus arises on the flat model space of (Riemannian signature) conformal geometry, the conformal sphere.

Remark 3.1. Recall that inverse stereographic projection maps Euclidean space conformally into the sphere as a one point (conformal) compactification, so that it makes sense to think of the sphere as the conformally “flat” model of Riemannian signature conformal geometry. The real reason however is that the conformal sphere arises naturally as the geometry of a homogeneous space of Lie groups and that conformal geometry can be thought of as the geometry of curved analogues of this homogeneous geometry in the sense of Élie Cartan (see, e.g. [12]). ■

3.1. The model of conformal geometry – the conformal sphere. We now look at the conformal sphere, which is an extremely important example. We shall see that the sphere can be viewed as a homogeneous space on which is naturally endowed a conformal structure, and that the conformal tractor calculus arises

naturally from this picture. This is the model for Riemannian signature conformal geometry, but a minor variation of this applies to other signatures.

First some notation. Consider an $(n + 2)$ -dimensional real vector space $\mathbb{V} \cong \mathbb{R}^{n+2}$. Considering the equivalence relation on $\mathbb{V} \setminus \{0\}$ given by

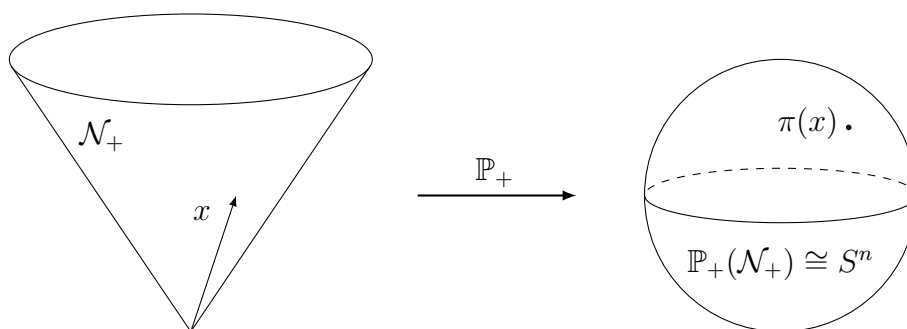
$$v \sim v', \quad \text{if and only if} \quad v' = rv \quad \text{for some} \quad r > 0$$

we now write

$$\mathbb{P}_+(\mathbb{V}) := \{[v] \mid v \in \mathbb{V}\}$$

where $[v]$ denotes the equivalence class of v . We view this as a smooth manifold by the identification with one, equivalently any, round sphere in \mathbb{V} .

Suppose now that \mathbb{V} is equipped with a non-degenerate bilinear form \mathcal{H} of signature $(n + 1, 1)$. The *null cone* \mathcal{N} of zero-length vectors forms a quadratic variety in \mathbb{V} . Choosing a time orientation, let us write \mathcal{N}_+ for the forward part of $\mathcal{N} \setminus \{0\}$. Under the the \mathbb{R}_+ -ray projectivisation of \mathbb{V} , meaning the natural map to equivalence classes $\mathbb{V} \rightarrow \mathbb{P}_+(\mathbb{V})$, the forward cone \mathcal{N}_+ is mapped to a quadric in $\mathbb{P}_+(\mathbb{V})$. This image is topologically a sphere S^n and we will write π for the submersion $\mathcal{N}_+ \rightarrow S^n = \mathbb{P}_+(\mathcal{N}_+)$.



Each point $x \in \mathcal{N}_+$ determines a positive definite inner product g_x on $T_{\pi(x)}S^n$ by $g_x(u, v) = \mathcal{H}_x(u', v')$ where $u', v' \in T_x\mathcal{N}_+$ are *lifts* of $u, v \in T_{\pi(x)}S^n$, meaning that $\pi(u') = u$, $\pi(v') = v$. For a given vector $u \in T_{\pi(x)}S^n$ two lifts to $x \in \mathcal{N}_+$ differ by a vertical vector (i.e. a vector in the kernel of $d\pi$). By differentiating the defining equation for the cone we see that any vertical vector is normal to the cone with respect to \mathcal{H} (null tangent vectors to hypersurfaces are normal), and so it follows that g_x is independent of the choices of lifts. Clearly then, each section of π determines a metric on S^n and by construction this is smooth if the section is. Evidently the metric agrees with the pull-back of \mathcal{H} via the section concerned. We may choose a coordinates X^A , $A = 0, \dots, n + 1$, for \mathbb{V} so that \mathcal{N} is the zero locus of the form $-(X^0)^2 + (X^1)^2 + \dots + (X^{n+1})^2$, in which terms the usual round sphere arises as the section $X^0 = 1$ of π .

Now, viewed as a metric on $T\mathbb{V}$, \mathcal{H} is homogeneous of degree 2 with respect to the standard Euler (or position) vector field E on \mathbb{V} , that is $\mathcal{L}_E\mathcal{H} = 2\mathcal{H}$, where \mathcal{L} denotes the Lie derivative. In particular this holds on the cone, which we note is generated by E . Write \mathbf{g} for the restriction of \mathcal{H} to vector fields in $T\mathcal{N}_+$ which are the lifts of vector fields on S^n . Note that u' is the lift of a vector field u on S^n

means that for all $x \in \mathcal{N}_+$, $d\pi(u'(x)) = u(\pi(x))$, and so $\mathcal{L}_E u' = 0$ (modulo vertical vector fields) on \mathcal{N}_+ . Thus for any pair $u, v \in \Gamma(TS^n)$, with lifts to vector fields u', v' on \mathcal{N}_+ , $\mathbf{g}(u', v')$ is a function on \mathcal{N}_+ homogeneous of degree 2, and which is independent of how the vector fields were lifted. It follows that if $s > 0$ then $g_{sx} = s^2 g_x$, for all $x \in \mathcal{N}_+$. Evidently \mathcal{N}_+ may be identified with the total space \mathcal{Q} of a bundle of conformally related metrics on $\mathbb{P}_+(\mathcal{N}_+)$. Thus $\mathbf{g}(u', v')$ may be identified with a conformal density of weight 2 on S^n . That is, this construction canonically determines a section of $\mathcal{E}_{(ab)}[2]$ that we shall also denote by \mathbf{g} . This has the property that if σ is any section of $\mathcal{E}_+[1]$ then $\sigma^{-2}\mathbf{g}$ is a metric g^σ . Obviously different sections of $\mathcal{E}_+[1]$ determine conformally related metrics and, by the last observation in the previous paragraph, there is a section σ_o of $\mathcal{E}_+[1]$ so so that g^{σ_o} is the round metric.

Thus we see that $\mathbb{P}_+(\mathcal{N}_+)$ is canonically equipped with the standard conformal structure on the sphere, but with no preferred metric from this class. Furthermore \mathbf{g} , which arises here from \mathcal{H} by restriction, is the conformal metric on $\mathbb{P}_+(\mathcal{N}_+)$. In summary then we have the following.

Lemma 3.2. *Let \mathbf{c} be the conformal class of $S^n = \mathbb{P}_+(\mathcal{N}_+)$ determined canonically by \mathcal{H} . This includes the round metric. The map $\mathcal{N}_+ \ni (\pi(x), g_x) \in \mathcal{Q}$ gives an identification of \mathcal{N}_+ with \mathcal{Q} , the bundle of conformally related metrics on (S^n, \mathbf{c}) .*

Via this identification: functions homogeneous of degree w on \mathcal{N}_+ are equivalent to functions homogeneous of degree w on \mathcal{Q} and hence correspond to conformal densities of weight w on (S^n, \mathbf{c}) ; the conformal metric \mathbf{g} on (S^n, \mathbf{c}) agrees with, and is determined by, the restriction of \mathcal{H} to the lifts of vector fields on $\mathbb{P}_+(\mathcal{N}_+)$.

The conformal sphere, as constructed here, is acted on transitively by $G = O_+(\mathcal{H}) \cong O_+(n+1, 1)$, where this is the time orientation preserving subgroup of orthogonal group preserving \mathcal{H} , $O(\mathcal{H}) \cong O(n+1, 1)$. Thus as a homogeneous space $\mathbb{P}_+(\mathcal{N}_+)$ may be identified with G/P , where P is the (parabolic) Lie subgroup of G preserving a nominated null ray in \mathcal{N}_+ .

3.1.1. *Canonical Calculus on the model.* Here we sketch briefly one way to see this on the model $\mathbb{P}_+(\mathcal{N}_+)$.

Note that as a manifold \mathbb{V} has some special structures that we have already used. In particular an origin and, from the vector space structure of \mathbb{V} , the Euler vector field E which assigns to each point $X \in \mathbb{V}$ the vector $X \in T_X\mathbb{V}$, via the canonical identification of $T_X\mathbb{V}$ with \mathbb{V} .

The vector space \mathbb{V} has an affine structure and this induces a global parallelism: the tangent space $T_x\mathbb{V}$ at any point $x \in \mathbb{V}$ may be canonically identified with \mathbb{V} . Thus, in particular, for any parameterised curve in \mathbb{V} there is a canonical notion of parallel transport along the given curve. This exactly means that viewing \mathbb{V} as a manifold, it is equipped with a canonical affine connection $\nabla^{\mathbb{V}}$. The affine structure gives more than this of course. It is isomorphic to \mathbb{R}^{n+2} with its usual affine structure, and so the tangent bundle to \mathbb{V} is trivialised by everywhere parallel tangent fields. It follows that the canonical connection $\nabla^{\mathbb{V}}$ is flat and has trivial holonomy.

Next observe that \mathcal{H} determines a signature $(n + 1, 1)$ metric on \mathbb{V} , where the latter is viewed as an affine manifold. By the definition of its promotion from bilinear form to metric, one sees at once that for any vectors U, V that are parallel on \mathbb{V} the quantity $\mathcal{H}(U, V)$ is constant. This means that \mathcal{H} is itself parallel since for any vector field W we have

$$(\nabla_W \mathcal{H})(U, V) = W \cdot \mathcal{H}(U, V) - \mathcal{H}(\nabla_W U, V) - \mathcal{H}(U, \nabla_W V) = 0.$$

The second key observation is that a restriction of these structures descend to the model. We observed above that \mathcal{N}_+ is an \mathbb{R}_+ -ray bundle over S^n . We may identify S^n with \mathcal{N}_+ / \sim where the equivalence relation is that $x \sim y$ if and only if x and y are points of the same fibre $\pi^{-1}(x')$ for some $x' \in S^n$. The restriction $T\mathbb{V}|_{\mathcal{N}_+}$ is a rank $n + 2$ vector bundle over \mathcal{N}_+ . Now we may define an equivalence relation on $T\mathbb{V}|_{\mathcal{N}_+}$ that covers the relation on \mathcal{N}_+ . Namely we decree $U_x \sim V_y$ if and only if $x, y \in \pi^{-1}(x')$ for some $x' \in S^n$, and U_x and V_y are parallel. Considering parallel transport up the fibres of π , it follows that $T\mathbb{V}|_{\mathcal{N}_+} / \sim$ is isomorphic to the restriction $T\mathbb{V}|_{\text{im}(S)}$ where S is any section of π (that $S : S^n \rightarrow \mathcal{N}_+$ is a smooth map such that $\pi \circ S = \text{id}_{S^n}$). But $\text{im}(S)$ is identified with S^n via π and it follows that $T\mathbb{V}|_{\mathcal{N}_+} / \sim$ may be viewed as a vector bundle \mathcal{T} on S^n . Furthermore it is clear from the definition of the equivalence relation on $T\mathbb{V}|_{\mathcal{N}_+}$ that \mathcal{T} is independent of S . The vector bundle \mathcal{T} on S^n is the (standard) tractor bundle of (S^n, \mathbf{c}) .

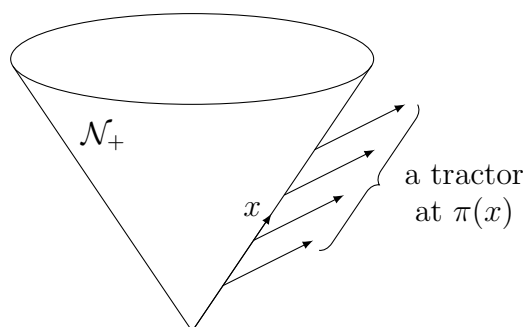


FIGURE 1. An element of $\mathcal{T}_{\pi(x)}$ corresponds to a homogeneous of degree zero vector field along the ray generated by x .

By restriction \mathcal{H} and $\nabla^{\mathbb{V}}$ determine, respectively, a (signature $(n + 1, 1)$) metric and connection on the bundle $T\mathbb{V}|_{\mathcal{N}_+}$ that we shall denote with the same notation. Since a vector field which is parallel along a curve γ in \mathcal{N}_+ may be uniquely extended to a vector field which is also parallel along every fibre of π through the curve γ , it is clear that $\nabla^{\mathbb{V}}$ canonically determines a connection on \mathcal{T} that we shall denote $\nabla^{\mathcal{T}}$. Sections $U, V \in \Gamma(\mathcal{T})$ are represented on \mathcal{N}_+ by vector fields \tilde{U}, \tilde{V} that are parallel in the direction of the fibres of $\pi : \mathcal{N}_+ \rightarrow S^n$. On the other hand \mathcal{H} is also parallel along each fibres of π and so $\mathcal{H}(\tilde{U}, \tilde{V})$ is constant on each fibre. Thus \mathcal{H} determines a signature $(n + 1, 1)$ metric h on \mathcal{T} satisfying $h(U, V) = \mathcal{H}(\tilde{U}, \tilde{V})$. What is more, since $\nabla^{\mathbb{V}} \mathcal{H} = 0$ on \mathcal{N}_+ , it follows that h is preserved by $\nabla^{\mathcal{T}}$, that is

$$\nabla^{\mathcal{T}} h = 0.$$

Summarising the situation thus far we have the following.

Theorem 3.3. *The model (S^n, \mathbf{c}) is canonically equipped with the following: a canonical rank $n + 2$ bundle \mathcal{T} ; a signature $(n + 1, 1)$ metric h on this; and a connection $\nabla^{\mathcal{T}}$ on \mathcal{T} that preserves h .*

Although we shall not go into details here it is straightforward to show the following:

Proposition 3.4. *The tractor bundle \mathcal{T} of the model (S^n, \mathbf{c}) has a composition structure*

$$(17) \quad \mathcal{T} = \mathcal{E}[1] \oplus TS^n[-1] \oplus \mathcal{E}[-1].$$

The restriction of h to the subbundle $\mathcal{T}^0 = TS^n[-1] \oplus \mathcal{E}[-1]$ induces the conformal metric $\mathbf{g} : TS^n[-1] \times TS^n[-1] \rightarrow \mathcal{E}$. Any null $Y \in \Gamma\mathcal{T}[-1]$ satisfying $h(X, Y) = 1$ determines a splitting $\mathcal{T} \xrightarrow{\cong} \mathcal{E}[1] \oplus TS^n[-1] \oplus \mathcal{E}[-1]$ such that the metric h is given by $(\sigma, \mu, \rho) \mapsto 2\sigma\rho + \mathbf{g}(\mu, \mu)$ as a quadratic form.

It is easily seen how this composition structure arises geometrically. The subbundle \mathcal{T}^0 of \mathcal{T} corresponds to the fact that $T\mathcal{N}_+$ is naturally identified with a subbundle of $T\mathbb{V}|_{\mathcal{N}_+}$. The vertical directions in there correspond to the fact that $\mathcal{E}[-1]$ is a subbundle of \mathcal{T}^0 , and the *semidirect sum* symbols \oplus record this structure.

3.1.2. *The abstract approach to the tractor connection.* An alternative way of seeing how the tractor connection arises on the flat model is via the group picture. If G/H is a homogeneous space of Lie groups, then the canonical projection $G \rightarrow G/H$ gives rise to a principal H -bundle over G/H (with total space G). If \mathbb{V} is a representation of H then one obtains a homogeneous vector bundle $\mathcal{V} := G \times_H \mathbb{V}$ over G/H whose total space is defined to be the quotient of $G \times \mathbb{V}$ by the equivalence relation

$$(g, v) \sim (gh, h^{-1} \cdot v).$$

If \mathbb{V} is in fact a representation of G , then the bundle $\mathcal{V} := G \times_H \mathbb{V}$ is trivial. The isomorphism

$$G \times_H \mathbb{V} \cong (G/H) \times \mathbb{V}$$

is given by

$$[g, v] \mapsto (gH, g \cdot v)$$

which is easily checked to be well defined. This trivialisation gives rise to a flat connection on $G \times_H \mathbb{V}$. In the case where we have the conformal sphere $G/P \cong S^n$ and \mathbb{V} is the defining representation of G (i.e. \mathbb{R}^{n+2}) then $G \times_P \mathbb{V}$ is (naturally identified with) the tractor bundle \mathcal{T} and the connection induced by the trivialisation $G \times_P \mathbb{V} \cong (G/P) \times \mathbb{V}$ is the tractor connection. The trivialisation $\mathcal{T} \cong (G/P) \times \mathbb{V}$ also immediately gives the existence of a bundle metric preserved by the tractor connection induced by the bilinear form on \mathbb{V} .

3.1.3. *The conformal model in other signatures.* With only a little more effort one can treat the case of general signature (p, q) . This again begins with a real vector space $\mathbb{V} \cong \mathbb{R}^{n+2}$, but now we equip it with a non-degenerate bilinear form \mathcal{H} of signature $(p + 1, q + 1)$, where $p + q = n$.

Writing again \mathcal{N} for the quadratic variety of vectors which have zero-length with respect to \mathcal{H} , we see that the space of null rays in $\mathcal{N} \setminus \{0\}$ has the topology of $S^p \times S^q$. This is connected, unless p or q is zero in either which case we get two copies of S^n . In any case an easy adaption of the earlier discussion shows that $\mathbb{P}_+(\mathcal{N})$ is equipped canonically with a conformal structure of signature (p, q) and on this a tractor connection preserving now a tractor metric of signature $(p + 1, q + 1)$. (One can easily check that the conformal class \mathbf{c} of $\mathbb{P}_+(\mathcal{N})$ contains a metric g which is of the form $g_{S^p} - g_{S^q}$ for some identification of $\mathbb{P}_+(\mathcal{N})$ with $S^p \times S^q$.)

As constructed here the conformal space $\mathbb{P}_+(\mathcal{N})$ is acted on transitively by $G = O(\mathcal{H}) \cong O(p+1, q+1)$, the orthogonal group preserving \mathcal{H} . As a homogeneous space $\mathbb{P}_+(\mathcal{N})$ may be identified with G/P , where P is the Lie subgroup of G preserving a nominated null ray in \mathcal{N} . We may think of this group picture as a good model for general conformal manifolds. Of course there are other possible choices of G/P (which may result in models which are only locally equivalent to the ones here), as we have already seen in the Riemannian case. See [37] for a discussion of the possible choices of (G, P) and the connection with global aspects of the conformal tractor calculus.

Remark 3.5. Note that the model space $S^1 \times S^{n-1}$ of Lorentzian signature conformal geometry is simply the quotient of the (conformal) Einstein universe $\mathbb{R} \times S^{n-1}$ by integer times 2π translations. Thus the usual embeddings of Minkowski and de Sitter space into the Einstein universe can be seen (by passing to the quotient) as conformal embeddings into the flat model space. In fact, $S^1 \times S^{n-1}$ can be seen as two copies of Minkowski space glued together along a null boundary with two cone points, or as two copies of de Sitter space glued together along a spacelike boundary with two connected components which are $(n - 1)$ -spheres. The significance of this will become clearer as we continue to develop the tractor calculus and then move on to study conformally compactified geometries. ■

3.2. Prolongation and the tractor connection. Here we construct the tractor bundle, connection, and metric on a conformal manifold of dimension at least three. We will see that the “conformal to Einstein” condition plays an important role. The tractor bundle and connection are obtained by “prolonging” the “almost Einstein equation” (11).

First we state what is meant by Einstein here.

Definition 5. In dimensions $n \geq 3$, a metric will be said to be *Einstein* if

$$\text{Ric}^g = \lambda g$$

for some function λ .

The Bianchi identities imply that, for any metric satisfying this equation, λ is constant (as we assume M connected). Throughout the following we shall assume that $n \geq 3$.

Recalling the model above, note that a parallel co-tractor I corresponds to a homogeneous polynomial which, in standard coordinates X^A on $\mathbb{V} \xrightarrow{\cong} \mathbb{R}^{n+2}$, is given by $\tilde{\sigma} = I_A X^A$. Then $\tilde{\sigma} = 1$ is a section of \mathcal{N}_+ that corresponds to the intersection of \mathcal{N}_+ with the hyperplane $I_A X^A = 1$, in \mathbb{V} . For at least some of these distinguished (conic) sections the resulting metric on S^n , $g = \tilde{\sigma}^{-2} \mathbf{g}$ (on the open set where the corresponding density $\sigma \in \Gamma(\mathcal{E}[1])$ is nowhere vanishing) is obviously Einstein. For example the round metric was already discussed. It is in fact true that all metrics obtained on open regions of S^n in this way are Einstein, as we shall shortly see.

3.3. The almost Einstein equation. Let (M, \mathbf{c}) be a conformal manifold with $\dim(M) \geq 3$. It is clear that $g^\circ \in \mathbf{c}$ is Einstein if and only if $\mathbf{P}_{(ab)_0}^{g^\circ} = 0$, and this is the link with the equation (11). As a conformally invariant equation this takes the form

$$(AE) \quad \nabla_{(a}^g \nabla_{b)_0}^g \sigma + \mathbf{P}_{(ab)_0}^g \sigma = 0,$$

where $\sigma \in \mathcal{E}_+[1]$ encodes $g^\circ = \sigma^{-2} \mathbf{g}$ and we have used the superscripts to emphasise that we have picked some metric $g \in \mathbf{c}$ in order to write the equation.

Suppose that σ is a solution with the property that it is nowhere zero. Then, without loss of generality, we may assume that σ is positive, that is $\sigma \in \mathcal{E}_+[1]$. So σ is a scale, and $g^\circ = \sigma^{-2} \mathbf{g}$ is a well-defined metric. Since (11) is conformal invariant, there is no loss if we calculate the equation (AE) in this scale. But then $\nabla^{g^\circ} \sigma = 0$. Thus we conclude that

$$\mathbf{P}_{(ab)_0} = 0.$$

Conversely suppose that $\mathbf{P}_{(ab)_0}^{g^\circ} = 0$ for some $g^\circ \in \mathbf{c}$. Then $g^\circ = \sigma^{-2} \mathbf{g}$ for some $\sigma \in \mathcal{E}_+[1]$. Therefore σ solves (AE) by the reverse of the same argument. Thus in summary we have the following, cf. [40].

Proposition 3.6. *(M, \mathbf{c}) is conformally Einstein (i.e. there is an Einstein metric g° in \mathbf{c}) if and only if there exists $\sigma \in \mathcal{E}_+[1]$ that solves (AE). If $\sigma \in \mathcal{E}_+[1]$ solves (AE) then $g^\circ := \sigma^{-2} \mathbf{g}$ is the corresponding Einstein metric.*

There are some important points to make here.

Remark 3.7. Equation (AE) is equivalent to a system of $\frac{(n+2)(n-1)}{2}$ scalar equations on one scalar variable. So it is overdetermined and we do not expect it to have solutions in general. ■

3.4. The connection determined by a conformal structure. Proposition 3.6 has the artificial feature that it is a statement about nowhere vanishing sections of $\mathcal{E}[1]$. Let us rectify this.

Definition 6. Let (M, \mathbf{c}) be a conformal manifold, of any signature, and $\sigma \in \mathcal{E}[1]$. We say that (M, \mathbf{c}, σ) is an *almost Einstein structure* if $\sigma \in \mathcal{E}[1]$ solves equation (AE).

We shall term (AE) the (conformal) almost Einstein equation.

Since the almost Einstein equation is conformally invariant it is natural to seek integrability conditions that are also conformally invariant. There is a systematic approach to this using a procedure known as prolongation that goes as follows. We fix a metric $g \in \mathbf{c}$ to facilitate the calculations. With this understood, for the most part in the following we will omit the decoration by g of the natural objects it determines; for example we shall write ∇_a rather than ∇_a^g .

As a first step observe that the equation (AE) is equivalent to the equation

$$(18) \quad \nabla_a \nabla_b \sigma + \mathbf{P}_{ab} \sigma + \mathbf{g}_{ab} \rho = 0$$

where we have introduced the new variable $\rho \in \mathcal{E}[-1]$ to absorb the trace terms. The key idea is to attempt to construct an equivalent first order closed system. We introduce $\mu_a \in \mathcal{E}_a[1]$, so our equation is replaced by the equivalent system

$$(19) \quad \nabla_a \sigma - \mu_a = 0, \quad \text{and} \quad \nabla_a \mu_b + \mathbf{P}_{ab} \sigma + \mathbf{g}_{ab} \rho = 0.$$

This system is almost closed in the sense that the derivatives of σ and μ_b are given algebraically in terms of the unknowns σ , μ_b , and ρ . However to obtain a similar result for ρ we must differentiate the system; by definition (*differential prolongation*) is precisely concerned with this process of producing higher order systems, and their consequences. Here we use notation from earlier.

The Levi-Civita covariant derivative of (18) gives

$$\nabla_a \nabla_b \nabla_c \sigma + \mathbf{g}_{bc} \nabla_a \rho + (\nabla_a \mathbf{P}_{bc}) \sigma + \mathbf{P}_{bc} \nabla_a \sigma = 0.$$

Contracting this using, respectively, \mathbf{g}^{ab} and \mathbf{g}^{bc} yields

$$\mathbf{g}^{ab} : \quad \Delta \nabla_c \sigma + \nabla_c \rho + (\nabla^a \mathbf{P}_{ac}) \sigma + \mathbf{P}^a_c \nabla_a \sigma = 0 \quad (1)$$

$$\mathbf{g}^{bc} : \quad \nabla_c \Delta \sigma + n \nabla_c \rho + (\nabla_c J) \sigma + J \nabla_c \sigma = 0 \quad (2).$$

Then the difference (2) – (1) is simply

$$(n-1) \nabla_c \rho + J \nabla_c \sigma - \mathbf{P}^a_c \nabla_a \sigma + R_{cb}{}^b{}_d \nabla^d \sigma = 0$$

where we have used the contracted Bianchi identity $\nabla^a \mathbf{P}_{ac} = \nabla_c J$ and computed the commutator $[\nabla_c, \Delta]$ acting on σ . But

$$R_{cb}{}^b{}_a \nabla^a \sigma = -R_{ca} \nabla^a \sigma = (2-n) \mathbf{P}_c{}^a \nabla_a \sigma - J \nabla_c \sigma,$$

and using this we obtain

$$(20) \quad \nabla_c \rho - \mathbf{P}_c{}^a \mu_a = 0,$$

after dividing by the overall factor $(n-1)$. So we have our closed system and, what is more, this system yields a linear connection. We discuss this now.

On a conformal manifold (M, \mathbf{c}) let us write $[\mathcal{T}]_g$ to mean the pair consisting of a direct sum bundle and $g \in \mathbf{c}$, as follows:

$$(21) \quad [\mathcal{T}]_g := (\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1], g)$$

Proposition 3.8. *On a conformal manifold (M, \mathbf{c}) , fix any metric $g \in \mathbf{c}$. There is a linear connection $\nabla^{\mathcal{T}}$ on the bundle*

$$[\mathcal{T}]_g \cong \begin{array}{c} \mathcal{E}[1] \\ \oplus \\ \mathcal{E}_a[1] \\ \oplus \\ \mathcal{E}[-1] \end{array}$$

given by

$$(22) \quad \nabla_a^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathbf{g}_{ab} \rho + \mathbf{P}_{ab} \sigma \\ \nabla_a \rho - \mathbf{P}_{ab} \mu^b \end{pmatrix}.$$

Solutions of the almost Einstein equation (AE) are in one-to-one correspondence with sections of the bundle $[\mathcal{T}]_g$ that are parallel for the connection $\nabla^{\mathcal{T}}$.

Proof. It remains only to prove that $\nabla^{\mathcal{T}}$ is a linear connection. But this is an immediate consequence of its explicit formula which we see takes the form $\nabla + \Phi$ where ∇ is the Levi-Civita connection on the direct sum bundle $[\mathcal{T}]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ and Φ is a section of $\text{End}([\mathcal{T}]_g)$. \square

We shall call $\nabla^{\mathcal{T}}$, of (22), the *(conformal) tractor connection*. Interpreted naïvely the statement in the Proposition might appear to be not very strong: we have already remarked that on a particular manifold it can be that the equation (AE) has no non-trivial solutions. However this is a universal result, and so it in fact gives an extremely useful tool for investigating the existence of solutions. Note that the connection (22) is well defined on any pseudo-Riemannian manifold. The point is that in using (22) for any application, we have immediately available the powerful theory of linear connections (e.g. parallel transport, curvature, etc.).

It is an immediate consequence of Proposition 3.8 that the almost Einstein equation (AE) can have at most $n + 2$ linearly independent solutions. However we shall see that far stronger results are available after we refine our understanding of the tractor connection (22). Furthermore this connection will be seen to have a role that extends far beyond the almost Einstein equation.

3.5. Conformal properties of the tractor connection. Although derived from a conformally invariant equation, the bundle and connection described in (21) and (22) are expressed in a way depending a priori on a choice of $g \in \mathbf{c}$, so we wish to study their conformal properties. Let us again fix some choice $g \in \mathbf{c}$, before investigating the consequences of changing this conformally.

Given the data of $(\sigma, \mu_b, \rho) \in [\mathcal{T}]_g$ at $x \in M$, it follows from the general properties of linear connections that we may always solve

$$(23) \quad \nabla^{\mathcal{T}}(\sigma, \mu_b, \rho) = 0, \quad \text{at } x \in M.$$

This imposes no restriction on either the conformal class c or the choice $g \in c$. Examining the formula (22) for the tractor connection we see that if (23) holds then, at the point x , we necessarily have

$$(24) \quad \mu_b = \nabla_b \sigma, \quad \rho = -\frac{1}{n}(\Delta \sigma + \mathbf{J}\sigma),$$

where the second equation follows by taking a \mathbf{g}^{ab} trace of the middle entry on the right hand side of (22). Thus canonically associated to the tractor connection there is the second order differential $\mathcal{E}[1] \rightarrow [\mathcal{T}]_g$ given by

$$(25) \quad [D\sigma]_g = \begin{pmatrix} n\sigma \\ n\nabla_b \sigma \\ -(\Delta \sigma + \mathbf{J}\sigma) \end{pmatrix},$$

where we have included the normalising factor n ($=\dim(M)$) for later convenience.

Recall that we know how the Levi-Civita connection, and hence also Δ and \mathbf{J} here, transform conformally. Thus it follows that $[D\sigma]_g$, or equivalently (24), determines how the variables σ , μ_b and ρ of the prolonged system must transform if they are to remain compatible with $\nabla^{\mathcal{T}}$ under conformal changes. If $\hat{g} = \Omega^2 g$, for some positive function Ω , then a brief calculation reveals

$$\nabla_{\hat{b}}^{\hat{g}} \sigma = \nabla_b^g \sigma + \Upsilon_b \sigma, \quad \text{and} \quad -\frac{1}{n}(\Delta^{\hat{g}} \sigma + \mathbf{J}^{\hat{g}} \sigma) = -\frac{1}{n}(\Delta^g \sigma + \mathbf{J}^g \sigma) - \Upsilon^b \nabla_b \sigma - \frac{1}{2} \sigma \Upsilon^b \Upsilon_b,$$

where as usual Υ denotes $d\Omega$. Thus we decree

$$\hat{\sigma} := \sigma, \quad \hat{\mu}_b := \mu_b + \Upsilon_b \sigma, \quad \hat{\rho} := \rho - \Upsilon^b \mu_b - \frac{1}{2} \sigma \Upsilon^b \Upsilon_b,$$

or, writing $\Upsilon^2 := \Upsilon^a \Upsilon_a$, this may be otherwise written using an obvious matrix notation:

$$(26) \quad [\mathcal{T}]_{\hat{g}} \ni \begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_b & \delta_b^c & 0 \\ -\frac{1}{2}\Upsilon^2 & -\Upsilon^c & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu_c \\ \rho \end{pmatrix} \sim \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \in [\mathcal{T}]_g.$$

Note that at each point $x \in M$ the transformation here is manifestly by a group action. Thus in the obvious way this defines an equivalence relation among the direct sum bundles $[\mathcal{T}]_g$ (of (21)) that covers the conformal equivalence of metrics in \mathbf{c} , and the quotient by this defines what we shall call the conformal standard tractor bundle \mathcal{T} on (M, \mathbf{c}) . More precisely we have the following definition.

Definition 7. On a conformal manifold (M, \mathbf{c}) the *standard tractor bundle* is

$$\mathcal{T} := \bigsqcup_{g \in \mathbf{c}} [\mathcal{T}]_g / \sim,$$

meaning the disjoint union of the $[\mathcal{T}]_g$ (parameterised by $g \in \mathbf{c}$) modulo equivalence relation given by (26). We shall also use the abstract index notation \mathcal{E}_A for \mathcal{T} .

We shall carry many conventions from tensor calculus over to bundles and tractor fields. For example we shall write $\mathcal{E}_{(AB)}[w]$ to mean $S^2\mathcal{T} \otimes \mathcal{E}[w]$, and so forth.

There are some immediate consequences of the Definition 7 that we should observe. First, from this definition, the next statement follows tautologically.

Proposition 3.9. *The formula (25) determines a conformally invariant differential operator*

$$D : \mathcal{E}[1] \rightarrow \mathcal{T}.$$

This operator is evidently intimately connected with the very definition of the standard tractor bundle. Because of its fundamental role we make the following definition.

Definition 8. Given a section of $\sigma \in \mathcal{E}[1]$ we shall call

$$I := \frac{1}{n}D\sigma$$

the *scale tractor* corresponding to σ .

Next observe also that from (26) it is clear that \mathcal{T} is a filtered bundle; we summarise this using a semi-direct sum notation

$$(27) \quad \mathcal{T} = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$$

meaning that \mathcal{T} has a subbundle $\mathcal{T}^1 \subset \mathcal{T}$ isomorphic to $\mathcal{E}[-1]$, $\mathcal{E}_a[1]$ is isomorphic to a subbundle of the quotient $\mathcal{T}/\mathcal{T}^1$ bundle, and $\mathcal{E}[1]$ is the final factor. We use X , to denote the bundle surjection $X : \mathcal{T} \rightarrow \mathcal{E}[1]$ or in abstract indices:

$$(28) \quad X^A : \mathcal{E}_A \rightarrow \mathcal{E}[1].$$

Note that given any metric $g \in \mathfrak{c}$ we may interpret X as the map $[\mathcal{T}]_g \rightarrow \mathcal{E}[1]$ given by

$$\begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \mapsto \sigma.$$

For our current purposes the critical result at this point is that the tractor connection $\nabla^{\mathcal{T}}$ “intertwines” with the transformation (26) in the following sense.

Exercise 4. Let $V = (\sigma, \mu_b, \rho)$, a section of $[\mathcal{T}]_g$, and $\widehat{V} = (\widehat{\sigma}, \widehat{\mu}_b, \widehat{\rho})$, a section of $[\mathcal{T}]_{\widehat{g}}$, be related by (26), where $\widehat{g} = \Omega^2 g$. Show that then

$$\begin{pmatrix} \widehat{\nabla}_a \widehat{\sigma} - \widehat{\mu}_a \\ \widehat{\nabla}_a \widehat{\mu}_b + \widehat{g}_{ab} \widehat{\rho} + \widehat{P}_{ab} \widehat{\sigma} \\ \widehat{\nabla}_a \widehat{\rho} - \widehat{P}_{ab} \widehat{\mu}^b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_b & \delta_b^c & 0 \\ -\frac{1}{2}\Upsilon^2 & -\Upsilon^c & 1 \end{pmatrix} \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ac} \rho + P_{ac} \sigma \\ \nabla_a \rho - P_{ac} \mu^c \end{pmatrix}.$$

Here $\Upsilon = d\Omega$, as usual, and for example $\widehat{\nabla}_a \widehat{\sigma} - \widehat{\mu}_a$ means $\widehat{\nabla}_a \widehat{\sigma} - \widehat{\mu}_a$.

Given a tangent vector field v^a , the exercise shows that $v^a \nabla_a^{\mathcal{T}} V$ transforms conformally as a standard tractor field, that is by (26). Whence $\nabla^{\mathcal{T}}$ descends to a well defined connection on \mathcal{T} . Let us summarise as follows.

Theorem 3.10. *Let (M, \mathbf{c}) be a conformal manifold of dimension at least 3. The formula (22) determines a conformally invariant connection*

$$\nabla^{\mathcal{T}} : \mathcal{T} \rightarrow \Lambda^1 \otimes \mathcal{T} .$$

For obvious reasons this will also be called the *conformal tractor connection* (on the standard tractor bundle \mathcal{T}); the formula (22) is henceforth regarded as the incarnation of this conformally invariant object on the realisation $[\mathcal{T}]_g$ of \mathcal{T} , as determined by the choice $g \in \mathbf{c}$.

It is important to realise that the conformal tractor connection exists canonically on any conformal manifold (of dimension at least 3). (One also has the tractor connection on 2 dimensional *Möbius* conformal manifolds). In particular its existence does not rely on solutions to the equation (AE). Nevertheless by its construction in Section 3.4 (as a prolongation of the equation (AE)), and using also equation (24), we have at once the following important property.

Theorem 3.11. *On a conformal manifold (M, \mathbf{c}) we have the following. There is a 1-1 correspondence between sections $\sigma \in \mathcal{E}[1]$, satisfying the conformal equation*

$$(AE) \quad \nabla_{(a}\nabla_{b)}\sigma + P_{(ab)}\sigma = 0,$$

and parallel standard tractors I . The mapping from almost Einstein scales to parallel tractors is given by $\sigma \mapsto \frac{1}{n}D_A\sigma$ while the inverse map is $I_A \mapsto X^A I_A$.

So a parallel tractor is necessarily a scale tractor, as in Definition 8, but in general the converse does not hold.

3.6. The tractor metric. It turns out that the tractor bundle has beautiful and important structure that is perhaps not at expected from its origins via prolongation in Section 3.4 above.

Proposition 3.12. *Let (M, \mathbf{c}) be a conformal manifold of signature (p, q) . The formula*

$$[V_A]_g = (\sigma, \mu_a, \rho) \mapsto 2\sigma\rho + \mathbf{g}^{ab}\mu_a\mu_b =: h(V, V)$$

defines, by polarisation, a signature $(p+1, q+1)$ metric on \mathcal{T} .

Proof. As a symmetric bilinear form field on the bundle $[\mathcal{T}]_g$, h takes the form

$$(29) \quad h(V', V) \stackrel{g}{=} \begin{pmatrix} \sigma' & \mu' & \rho' \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{g}^{-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu \\ \rho \end{pmatrix},$$

where $\stackrel{g}{=}$ should be read as “equals, calculating in the scale g ”. So we see that the signature is as claimed. By construction $h(V, V')$ has weight 0. It remains to check the conformal invariance. Here we use the notation from (26):

$$\begin{aligned} 2\widehat{\sigma}\widehat{\rho} + \widehat{\mu}^a\widehat{\mu}_a &= 2\sigma(\rho - \Upsilon_c\mu^c - \frac{1}{2}\Upsilon^2\sigma) + (\mu^a + \Upsilon^a\sigma)(\mu_a + \Upsilon_a\sigma) \\ &= 2\sigma\rho + \mu^a\mu_a - 2\sigma\Upsilon\mu - \sigma^2\Upsilon^2 + 2\Upsilon\mu\sigma + \Upsilon^2\sigma^2 \\ &= 2\sigma\rho + \mu^a\mu_a \end{aligned}$$

□

In the abstract index notation the tractor metric is $h_{AB} \in \Gamma(\mathcal{E}_{(AB)})$, and its inverse h^{BC} .

The standard tractor bundle, as introduced in Sections 3.4 and 3.5, would more naturally have been defined as the dual tractor bundle. But Proposition (3.12) shows that we have not damaged our development; the tractor bundle is canonically isomorphic to its dual and normally we do not distinguish these, except by the raising and lowering of abstract indices using h_{AB} .

From these considerations we see that there is no ambiguity in viewing X , of (28), as a section of $\mathcal{T} \otimes \mathcal{E}[1] \cong \mathcal{T}[1]$. In this spirit we refer to X as the *canonical tractor*. At this point it is useful to note that in view of this canonical self duality $\mathcal{T} \cong \mathcal{T}^*$, and the formula (29), we have the following result.

Proposition 3.13. *The canonical tractor X^A is null, in that*

$$h_{AB}X^AX^B = 0.$$

Furthermore $X_A = h_{AB}X^B$ gives the canonical inclusion of $\mathcal{E}[-1]$ into \mathcal{E}_A :

$$X_A : \mathcal{E}[-1] \rightarrow \mathcal{E}_A.$$

In terms of the decomposition of \mathcal{E}_A given by a choice of metric this is simply

$$\rho \mapsto \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}.$$

As another immediate application of the metric we observe the following. Given a choice of scale $\sigma \in \mathcal{E}_+[1]$, consider the corresponding scale tractor I , and in particular its squared length $h(I, I)$. This evidently has conformal weight zero and so is a function on (M, \mathbf{c}) determined only by the choice of scale σ . Explicitly we have

$$(30) \quad h(I, I) \stackrel{g}{=} \mathbf{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma) - \frac{2}{n}\sigma(\mathbf{J} + \Delta)\sigma$$

from Definition 8 with (25) and (29). Here we have calculated the right hand side in terms of some metric g in the conformal class (hence the notation $\stackrel{g}{=}$). But since σ is a scale we may, in particular, use $g := \sigma^{-2}\mathbf{g}$. Then $\nabla^g\sigma = 0$ and we find the following result.

Proposition 3.14. *On a conformal manifold (M, \mathbf{c}) , let $\sigma \in \mathcal{E}_+[1]$, and $I = \frac{1}{n}D\sigma$ the corresponding scale tractor. Then*

$$h^{AB}I_B I_C = -\frac{2}{n}\mathbf{J}^\sigma,$$

where $\mathbf{J}^\sigma := g^{ab}\mathbf{P}_{ab}$, and $g_{ab} = \sigma^{-2}\mathbf{g}_{ab}$.

Note we usually write \mathbf{J} to mean the density $\mathbf{g}^{ab}\mathbf{P}_{ab}$, as calculated in the scale g . So here $\mathbf{J}^\sigma = \sigma^2\mathbf{J}$. In a nutshell the conformal meaning of scalar curvature is that it is the length squared of the scale tractor.

Next we come to the main reason the tractor metric is important, namely because it is preserved by the tractor connection. With V as in Proposition 3.12 we have

$$\begin{aligned}\nabla_a h(V, V) &\stackrel{g}{=} 2[\rho \nabla_a \sigma + \sigma \nabla_a \rho + \mathbf{g}^{bc} \mu_b \nabla_a \mu_c] \\ &= 2[\rho(\nabla_a \sigma - \mu_a) + \sigma(\nabla_a \rho - P_{ab} \mu^b) + \mathbf{g}^{bc} \mu_b (\nabla_a \mu_c + g_{ac} \rho + P_{ac} \sigma)] \\ &= 2h(V, \nabla_a^\mathcal{T} V).\end{aligned}$$

We summarise this with also the previous result.

Theorem 3.15. *On a conformal manifold (M, \mathbf{c}) of signature (p, q) , the tractor bundle carries a canonical conformally invariant metric h of signature $(p+1, q+1)$. This is preserved by the tractor connection.*

We see at this point that the tractor calculus is beginning to look like an analogue for conformal geometry of the Ricci calculus of (pseudo-)Riemannian geometry: A metric on a manifold canonically determines a unique Levi-Civita connection on the tangent bundle preserving the metric. The analogue here is that a conformal structure of any signature (and dimension at least 3) determines canonically the standard tractor bundle \mathcal{T} equipped with the connection $\nabla^\mathcal{T}$ and a metric h preserved by $\nabla^\mathcal{T}$.

Note also that the tractor bundle, metric, and connection seem to be analogues of the corresponding structures found for model in Theorem 3.3. Especially in view of matching filtration structures: (27) should be compared with that found on the model in Proposition 17 (noting that $TS^n[-1] \cong T^*S^n[1]$). In fact the tractor connection here of Theorem 3.10, and the related structures, generalise the corresponding objects on the model. This follows by more general results in [7], or alternatively may be verified directly by computing the formula for the connection of Theorem 3.3 in terms of the Levi-Civita connection on the round sphere.

4. LECTURE 4: THE TRACTOR CURVATURE, CONFORMAL INVARIANTS AND INVARIANT OPERATORS.

Let us return briefly to our motivating problems: the construction of invariants and invariant operators.

4.1. Tractor curvature. Since the tractor connection $\nabla^\mathcal{T}$ is well defined on a conformal manifold its curvature κ on \mathcal{E}^A depends only on the conformal structure; by construction it is an invariant of that. If we couple the tractor connection with any torsion free connection (in particular with the Levi-Civita connection of any metric in the conformal class) then, according to our conventions from Lecture 1, the curvature of the tractor connection is given by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) U^C = \kappa_{ab}{}^C{}_D U^D \quad \text{for all } U^C \in \Gamma(\mathcal{E}^C),$$

where ∇ denotes the coupled connection. Now using that the tractor connection preserves the inverse metric h^{CD} we have

$$0 = (\nabla_a \nabla_b - \nabla_b \nabla_a) h^{CD} = \kappa_{ab}{}^C{}_E h^{ED} + \kappa_{ab}{}^D{}_E h^{CE}.$$

In other words, raising an index with h^{CD}

$$(31) \quad \kappa_{ab}{}^{CD} = -\kappa_{ab}{}^{DC}.$$

It is straightforward to compute κ in a scale g . We have

$$(32) \quad (\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{pmatrix} \sigma \\ \mu^c \\ \rho \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ C_{ab}{}^c & W_{ab}{}^c{}_d & 0 \\ 0 & -C_{abd} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^d \\ \rho \end{pmatrix}$$

where, recall W is the Weyl curvature and C is the Cotton tensor,

$$C_{abc} := 2\nabla_{[a} \mathbf{P}_{b]c}.$$

In the expression for κ , the zero in the top right follows from the skew symmetry (31), while the remaining zeros of the right column show that the canonical tractor X^D annihilates the curvature,

$$\kappa_{ab}{}^C{}_D X^D = 0.$$

This with the skew symmetry determines the top row of the curvature matrix. It follows from the conformal transformation properties of the tractor splittings that the central entry of the matrix is conformally invariant, and this is consistent with the appearance there of the Weyl curvature $W_{ab}{}^c{}_d$. Note that in dimension 3 this necessarily vanishes, and so it follows that then the tractor curvature is fully captured by and equivalent to the Cotton curvature C_{abc} . Again this is consistent with the well known conformal invariance of that quantity in dimension 3. Thus we have the following result.

Proposition 4.1. *The normal conformal tractor connection ∇^T is flat if and only if the conformal manifold is locally equivalent to the flat model.*

So we shall say a conformal manifold (M, \mathbf{c}) is conformally flat if $\kappa = 0$.

4.1.1. *Application: Conformally invariant obstructions to metric being conformal-to-Einstein.* Note that as an immediate application we can use the tractor curvature to easily manufacture obstructions to the existence of an Einstein metric in the conformal class \mathbf{c} .

From Proposition 3.6 and Theorem 3.11 a metric g is Einstein if and only if the corresponding scale tractor $I_A = \frac{1}{n} D_A \sigma$ is parallel, where $g = \sigma^{-2} \mathbf{g}$. Thus if g is Einstein then we have

$$(33) \quad \kappa_{ab}{}^C{}_D I^D = 0.$$

Recall that $\kappa_{ab}{}^C{}_D X^D = 0$. If at any point p the kernel of $\kappa_{ab}{}^C{}_D : \mathcal{E}^D \rightarrow \mathcal{E}_{ab}{}^C$ is exactly 1-dimensional then we say that tractor curvature has maximal rank. We have:

Proposition 4.2. *Let (M, \mathbf{c}) be a conformal manifold. If at any point $p \in M$ the tractor curvature has maximal rank then there is no Einstein metric in the conformal equivalence class \mathbf{c} .*

From this observation it is easy to manufacture conformal invariants that must vanish on an Einstein manifold, see [29].

Proof. If a metric g is Einstein then σ is a true scale and hence nowhere zero. Thus I_p^D is not parallel to X_p^D , and the result follows from (33). \square

4.2. Toward tractor calculus. Although the tractor connection and its curvature are conformally invariant it is still not evident how to easily manufacture conformal invariants. The tractor curvature takes values in $\Lambda^2(T^*M) \otimes \text{End}(\mathcal{T})$ and this not a bundle on which the tractor connection acts.

To deal with this and the related problem of constructing differential invariants of tensors and densities we need need additional tools.

The simplest of these is the Thomas-D operator. Recall that in Proposition 3.9 (cf. (25)) we constructed a differential operator $D : \mathcal{E}[1] \rightarrow \mathcal{T}$ that was, by construction, tautologically conformally invariant. This generalises. Let us write $\mathcal{E}^\Phi[w]$ to denote any tractor bundle of weight w . Then:

Proposition 4.3. *There is a conformally invariant differential operator*

$$D_A : \mathcal{E}^\Phi[w] \rightarrow \mathcal{E}_A \otimes \mathcal{E}^\Phi[w - 1],$$

defined in a scale g by

$$V \mapsto [D_A V]_g := \begin{pmatrix} (n + 2w - 2)wV \\ (n + 2w - 2)\nabla_a V \\ -(\Delta V + wJ V) \end{pmatrix}$$

where ∇ denotes the coupled tractor–Levi-Civita connection.

Proof. Under a conformal transformation $g \mapsto \hat{g} = \Omega^2 g$, $[D_A f]_g$ transforms by (26). \square

Note that this result is not as trivial as the written proof suggests since V (with any indices suppressed) is a section of any tractor bundle, and the operator is second order. In fact there are nice ways to construct the Thomas-D operator from more elementary invariant operators [8, 25].

4.2.1. Application: differential invariants of densities and weighted tractors. A key point about the Thomas-D operator is that it can be iterated. For $V \in \Gamma(\mathcal{E}^\Phi[w])$ we may form (suppressing all indices):

$$V \mapsto (V, DV, D \circ DV, D \circ D \circ DV, \dots)$$

and, for generic weights $w \in \mathbb{R}$, in a conformally invariant way this encodes the jets of the section V entirely into weighted tractor bundles. Thus we can proliferate invariants of V .

For example for $f \in \Gamma(\mathcal{E}[w])$ we can form

$$(D^A f)D_A f = -2w(n + 2w - 2)f(\Delta f + wJf) + (n + 2w - 2)^2(\nabla^a f)\nabla_a f.$$

By construction this is conformally invariant, for *any weight* w . So in fact it is a family of invariants (of densities). Similarly we may form

$(D^A D^B f) D_A B_B f = -2(n+2w-4)(n+2w-2)w(w-1)f\Delta^2 f + \text{lower order terms}$,
and so forth.

4.2.2. *Conformal Laplacian-type linear operators.* One might hope that the tractor-D operator is also effective for the construction of conformally invariant linear differential operators. In particular the construction of Laplacian type operators is important. Certainly $D^A D_A$ is by construction conformally invariant but, on any weighted tractor bundle:

$$D^A D_A = 0.$$

This is verified by a straightforward calculation, but it should not be surprising as by construction it would be invariant on, for example, densities of any weight. On the standard conformal sphere it is well-known that there is no non-trivial operator with this property.

From our earlier work we know the domain bundle of the conformal Laplacian is $\mathcal{E}[1 - \frac{n}{2}]$. Observe that for $V \in \Gamma(\mathcal{E}^\Phi[1 - \frac{n}{2}])$ we have $n + 2w - 2 = 0$, and

$$D_A f \stackrel{g}{=} \begin{pmatrix} 0 \\ 0 \\ -(\Delta + \frac{2-n}{2}J)V \end{pmatrix}, \quad \text{that is} \quad D_A V = -X_A \square V$$

where \square is the (tractor-twisted) conformal Laplacian. In particular, the proof of Proposition 4.3 was also a proof of this result:

Lemma 4.4. *The operator $\square \stackrel{g}{=} (\Delta + \frac{2-n}{2}J)$ is a conformally invariant differential operator*

$$\square : \mathcal{E}^\Phi[1 - \frac{n}{2}] \rightarrow \mathcal{E}^\Phi[-1 - \frac{n}{2}],$$

where \mathcal{E}^Φ is any tractor bundle.

This is already quite useful, as the next exercise shows.

Exercise 5. Show that if $f \in \mathcal{E}[2 - \frac{n}{2}]$ then

$$(34) \quad \square D_A f = -X_A P_4 f.$$

Thus there is a conformally invariant differential operator

$$P_4 : \mathcal{E}[2 - \frac{n}{2}] \rightarrow \mathcal{E}[-2 - \frac{n}{2}] \quad \text{where} \quad P_4 \stackrel{g}{=} \Delta^2 + \text{lower order terms}.$$

In fact this is the celebrated *Paneitz operator* discovered by Stephen Paneitz in 1983, see [46] for a reproduction of his preprint from the time. See [25, 30] for further discussion and generalisations.

An important point is that (34) does not hold if we replace f with a tractor field of the same weight! Those who complete the exercise will observe the first hint of this subtlety, in that during the calculation derivatives will need to be commuted.

4.3. Splitting tractors. Although the importance of the tractor connection stems from its conformal invariance we need efficient ways to handle the decomposition of the tractor bundle corresponding to a choice of scale.

Recall that a metric $g \in \mathbf{c}$ determines an isomorphism

$$(35) \quad \begin{array}{c} \mathcal{E}[1] \\ \oplus \\ \mathcal{E}_a[1] \\ \oplus \\ \mathcal{E}[-1] \end{array} \xrightarrow{\cong} \mathcal{E}^A, \quad \text{mapping} \quad \begin{pmatrix} \sigma \\ \mu_a \\ \rho \end{pmatrix} = [U^A]_g \mapsto U^A \in \mathcal{E}^A.$$

The inclusion of $\mathcal{E}[-1]$ into the direct sum followed by this map is just $X^A : \mathcal{E}[-1] \rightarrow \mathcal{E}^A$, as observed earlier in Proposition (3.13). This is conformally invariant. However let us now fix the notation

$$(36) \quad Z^{Aa} : \mathcal{E}_a[1] \rightarrow \mathcal{E}^A, \quad \text{and} \quad Y^A : \mathcal{E}[-1] \rightarrow \mathcal{E}^A,$$

for the other two bundle maps determined by (35). We call these (along with X^A) the *tractor projectors* and view them as bundle sections $Z^{Aa} \in \Gamma(\mathcal{E}^{Aa})[-1]$, and $Y^A \in \mathcal{E}^A[-1]$. So in summary $[U^A]_g = (\sigma, \mu_a, \rho)$ is equivalent to

$$(37) \quad U^A = Y^A \sigma + Z^{Aa} \mu_a + X^A \rho.$$

Using the formula (29) for the tractor metric it follows at once that $X^A Y_A = 1$, $Z^{Aa} Z_{Ab} = \delta_b^a$ and all other quadratic combinations of the X , Y , and Z are zero as summarised in Figure 2. Thus we also have $Y_A U^A = \rho$, $X_A U^A = \sigma$, $Z_{Ab} U^A = \mu_b$

	Y^A	Z^{Ac}	X^A
Y_A	0	0	1
Z_{Ab}	0	δ_b^c	0
X_A	1	0	0

FIGURE 2. Tractor inner product

and the metric may be decomposed into a sum of projections, $h_{AB} = Z_A^c Z_{Bc} + X_A Y_B + Y_A X_B$.

The projectors Y and Z depend on the metric $g \in \mathbf{c}$. If \hat{Y}^A and \hat{Z}^A_b are the corresponding quantities in terms of the metric $\hat{g} = \Omega^2 g$ then altogether we have

$$(38) \quad \hat{X}^A = X^A, \quad \hat{Z}^{Ab} = Z^{Ab} + \Upsilon^b X^A, \quad \hat{Y}^A = Y^A - \Upsilon_b Z^{Ab} - \frac{1}{2} \Upsilon_b \Upsilon^b X^A$$

as follows immediately from (37) and (26).

Remark 4.5. In the notation Z^A_a the tractor and tensor indices are both abstract. If we move to a concrete frame field for TM , e_1, \dots, e_n and then write Z^A_i , $i = 1, \dots, n$, for the contraction $Z^A_a e_i^a$ we come to the (weighted) tractor frame field:

$$X^A, Z^A_1, \dots, Z^A_n, Y^A.$$

This is a frame for the tractor bundle adapted to the filtration (27), as reflected in the conformal and inner product properties described in (38) and figure 2. \blacksquare

Of course tensor products of the tractor bundle are also decomposed by the isomorphism (35) and this is described by the tensor products of the projectors in an obvious way. To illustrate consider the case of the bundle of *tractor k -forms* $\Lambda^k \mathcal{T}$ (which note is non-zero for $k = 0, \dots, n + 2$.) The composition series $\mathcal{T} = \mathcal{E}[1] \oplus T^*M[-1] \oplus \mathcal{E}[-1]$ determines the composition series for $\Lambda^k \mathcal{T}$,

$$(39) \quad \Lambda^k \mathcal{T} \cong \Lambda^{k-1}[k] \oplus \begin{array}{c} \Lambda^{k-2}[k-2] \\ \oplus \\ \Lambda^k[k] \end{array} \oplus \Lambda^{k-1}[k-2].$$

Given a choice of metric g from the conformal class there is a splitting of this composition series corresponding to the splitting (35) of \mathcal{T} , and this is easily computed and dealt with using the ‘‘projectors’’ X , Y , and Z , see e.g. [26].

4.4. The connection. Just as connections are often described in terms of their action on a suitable frame field it is useful to give the tractor connection in terms of its action on the projectors X , Y , and Z . The tractor covariant derivative of a field $U^A \in \Gamma(\mathcal{E}^A)$, as in (37), is given by (22). Using the isomorphism (35) this is written

$$\nabla_a U^B = Y^B(\nabla_a \sigma - \mu_a) + Z^{Bb}(\nabla_a \mu_b + \mathbf{g}_{ab} \rho + P_{ab} \sigma) + X^B(\nabla_a \rho - P_{ab} \mu^b).$$

On the other hand applying the connection directly to U^A expanded as in (37) we have

$$\nabla_a U^B = Y^B \nabla_a \sigma + \sigma \nabla_a Y^B + Z^{Bb} \nabla_a \mu_b + \mu_b \nabla_a Z^{Bb} + X^B \nabla_a \rho + \rho \nabla_a X^B,$$

where we have used the Leibniz rule for ∇_a , viewed as the coupled tractor-Levi-Civita connection. Comparing these we obtain:

$$(40) \quad \nabla_a X^B = Z^B{}_a, \quad \nabla_a Z^B{}_b = -P_{ab} X^B - \mathbf{g}_{ab} Y^B, \quad \nabla_a Y^B = P_{ab} Z^{Bb}.$$

This gives the transport equations for the projectors (and determines these for the adapted frame as in the Remark 4.5 above). From a practical point of view the formulae (40) here enable the easy computation of the connection acting on a tractor field of any valence.

4.4.1. Application: Computing, and conformal Laplacian operators. The formulae are effective for reducing most tractor calculations to a routine task. For example suppose we want to compute $D^A X_A f = D^A(X_A f)$ for $f \in \Gamma(\mathcal{E}^\Phi[w])$, i.e. a section of any weighted tractor bundle. We calculate in some scale g . First note that

$$\begin{aligned} X^A(\Delta + (w + 1)J)X_A f &= X^A \Delta X_A f, \\ &= X^A[\Delta, X_A]f, \\ &= X^A \nabla^b[\nabla_b, X_A]f + X^A[\nabla_b, X_A] \nabla^b f \\ &= X^A(\nabla^b Z_{bA})f + 2X^A Z_{bA} \nabla^b f \\ &= X^A(-JX_A - nY_A)f \\ &= -nf, \end{aligned}$$

where we used that $X^A X_A = 0$, $X^A Z_{bA} = 0$, and $X^A Y_A = 1$. Thus we have

$$\begin{aligned} D^A X_A f &= [(w+1)(n+2w)Y^A + (n+2w)Z^{Aa}\nabla_a - X^A \Delta] X_A f \\ &= (w+1)(n+2w)f + (n+2w)Z^{aA} Z_{aA} f + n f \\ &= (w+1)(n+2w)f + (n+2w)n f + n f, \end{aligned}$$

and collecting terms we come to

$$(41) \quad D^A X_A f = (n+2w+2)(n+w)f.$$

A straightforward induction using (41) and (40) then enables us to show that:

Theorem 4.6. *On a conformal manifold (M^n, \mathbf{c}) , for any tractor bundle \mathcal{E}^Φ , and for each $k \in \mathbb{Z}_{\geq 1}$ if n is odd (or $k \in \mathbb{Z}_{\geq 1}$ and $2k < n$ if n is even) there is a conformally invariant differential operator*

$$\square_{2k} : \mathcal{E}^\Phi[k - \frac{n}{2}] \rightarrow \mathcal{E}^\Phi[-k - \frac{n}{2}]$$

of the form $\Delta^k +$ lower order terms (up to a non-zero constant factor). These are given by

$$\square_{2k} := D^{A_1} \dots D^{A_{k-1}} \square D_{A_{k-1}} \dots D_{A_1}.$$

Remark 4.7. The operators in the Theorem were first reported in [25], (and with a different proof) as part of joint work of the second author with M.G. Eastwood.

Acting on densities of weight $(k - n/2)$ there are the GJMS operators of [36]. For $k \geq 3$ these operators of Theorem 4.6 differ from the GJMS operators, as follows easily from the discussion in [30]. \blacksquare

4.5. Constructing invariants. We can now put together the above tools and proliferate curvature invariants. As a first step, following [27] we may form

$$W_{AB}{}^K{}_L := \frac{3}{n-2} D^P X_{[P} Z_A{}^a Z_B{}^b \kappa_{ab}{}^K{}_L.$$

This is conformally invariant by construction, as it is immediate from (38) that $X_{[P} Z_A{}^a Z_B{}^b$ is conformally invariant. It is exactly the object \mathbb{X}^3 which, for example, gives the conformally invariant injection

$$\mathbb{X}^3 : \Lambda^2[1] \rightarrow \Lambda^3 \mathcal{T}.$$

It turns out that the W -tractor W_{ABCD} has the symmetries of an algebraic Weyl tensor. In fact in a choice of conformal scale, W_{ABCE} is given by

$$(42) \quad \begin{aligned} &(n-4) (Z_A{}^a Z_B{}^b Z_C{}^c Z_E{}^e W_{abce} - 2Z_A{}^a Z_B{}^b X_{[C} Z_E]{}^e C_{abe} \\ &- 2X_{[A} Z_B]{}^b Z_C{}^c Z_E{}^e C_{ceb}) + 4X_{[A} Z_B]{}^b X_{[C} Z_E]{}^e B_{eb}, \end{aligned}$$

where C_{abc} is the Cotton tensor, and

$$(43) \quad B_{ab} := \nabla^c C_{cba} + \mathbf{P}^{dc} W_{dacb},$$

see [31]. Note that from (42) it follows that, in dimension 4, B_{eb} is conformally invariant. This is the *Bach tensor*.

Since W_{ABCD} takes values in a weighted tractor bundle we may apply the Thomas-D to this to capture jets of structure invariantly

$$W_{ABCD} \mapsto (W, DW, D \circ DW, D \circ D \circ DW, \dots)$$

following the idea of Section 4.2.1. Contractions of these terms then yield invariants. On odd dimensional manifolds this idea with some minor additional input produces a generating set of scalar conformal invariants [27]. An alternative (but closely related [7]) approach to the construction of conformal invariants uses the Fefferman-Graham ambient metric [3, 17]. With either approach, in even dimensions the situation is far more subtle, deeper ideas are needed, and even for the construction of scalar conformal invariants open problems remain (as mentioned in the sources referenced).

5. LECTURE 5: CONFORMAL COMPACTIFICATION OF PSEUDO-RIEMANNIAN MANIFOLDS

At this point we change directions from developing the theory of tractor calculus for conformal geometries to applying this calculus to various problems inspired by asymptotic analysis in general relativity. We will see that the conformal treatment of infinity in general relativity fits very nicely with the conformal tractor calculus, and our basic motivation will be to produce results which are useful in this setting. For the most part will work fairly generally however, and much of what is presented will be applicable in other situations involving hypersurfaces or boundaries (such as Cauchy surfaces or various kinds of horizons in general relativity, or to the study of Poincaré-Einstein metrics in differential geometry). From this point on we will adopt the convention that $d = n + 1$ (rather than n as before) is the dimension of our manifold M , so that n will be the dimension of ∂M or of a hypersurface in M .

5.1. Asymptotic flatness and conformal infinity in general relativity. It is natural in seeking to understand and describe the physics of general relativity to want to study isolated systems. In particular we want to be able to understand the mass and/or energy of the system (as well as other physical quantities) and how the system radiates gravitational energy (or how the system interacts with gravitational radiation coming in from infinity, i.e. from outside the system). Clearly it is quite unnatural to try to isolate a physical system from a spacetime by simply considering the inside of a timelike tube containing the system (events inside the tube would depend causally on events outside, and there would be no natural choice of such a tube anyway). From early on in the history of general relativity then physicists have sought to define isolated systems in terms of spacetimes which are *asymptotically flat*, that is which approach the geometry of Minkowski space in a suitable way as you approach “infinity”.

Definitions of asymptotic flatness typically involve the existence of special coordinate systems in which the metric components and other physical fields fall off sufficiently quickly as you approach infinity (infinity being defined by the coordinate system). Once a definition of asymptotically flat spacetimes is established one can talk about the *asymptotic symmetry group* of such spacetimes (which is not

in general the Poincare group) and the corresponding physical quantities such as mass and energy. One must be careful in defining the asymptotic flatness condition to not require the fields to fall off too fast (thereby excluding massive spacetimes, or gravitationally radiating ones) nor too slow (so that one loses the asymptotic symmetries needed to define physical quantities).

A condition on asymptotically flat spacetimes which came to be seen as important is the *Sachs peeling-off property*, this is the condition that along a future directed null geodesic which goes out to infinity with affine parameter λ the Weyl curvature $W^a{}_{bcd}$ satisfies

$$W = \frac{W^{(4)}}{\lambda} + \frac{W^{(3)}}{\lambda^2} + \frac{W^{(2)}}{\lambda^3} + \frac{W^{(1)}}{\lambda^4} + O\left(\frac{1}{\lambda^5}\right)$$

where each tensor $W^{(k)}$ is of special algebraic type with the tangent vector to the null geodesic as a k -fold repeated principal null direction. Typically asymptotically flat spacetimes were required to be Ricci flat near infinity, so the Weyl curvature was in fact the full curvature tensor in this region; the term $W^{(4)}$ in the asymptotic expansion was interpreted as the gravitational radiation reaching infinity.

The mass associated with this approach to isolated systems is called the *Bondi mass*, it is not a conserved quantity but satisfies a *mass loss formula* as energy is radiated away from the spacetime. The asymptotic symmetry group (which is the same abstract group for any spacetime) is known as the *BMS* (Bondi-Metzner-Sachs) *group*. There is an alternative way of defining asymptotic flatness using the 3+1 formalism. In this case one talks about the spacetime being *asymptotically flat at spatial infinity*, the corresponding mass is the *ADM* (Arnowitt-Deser-Misner) *mass*, and the asymptotic symmetry group is known as the *SPI group*. Questions of how these two notions of asymptotic flatness are related to each other can be subtle and tricky. For a good introductory survey of these issues see the chapter on asymptotic flatness in [56].

After a great deal of work in these areas, a new approach was suggested by Roger Penrose in the 1960s [47, 48, 50]. Penrose required of an asymptotically flat spacetime that the conformal structure of spacetime extend to a pair of null hypersurfaces called future and past null infinity. This mirrored the the conformal compactification of Minkowski space obtained by adding a lightcone at infinity. It was quickly shown that (i) Penrose's notion of conformal infinity satisfied the appropriate uniqueness property [22], (ii) this form of asymptotic flatness implied the peeling property [47, 50], and (iii) the corresponding group of asymptotic symmetries was the usual BMS group [23]. It is not totally surprising that this idea worked out so well: the causal (or light-cone) structure of spacetime, which is encoded by the conformal structure, had played an important role in the analysis of gravitational radiation up to that point. The conformal invariance of the zero rest mass equations for arbitrary spin particles and their peeling-off properties also fit nicely with Penrose's proposal, providing further motivation. What is really nice about this approach however is that it is both natural and coordinate free.

Let us now give the formal definition(s), following [19].

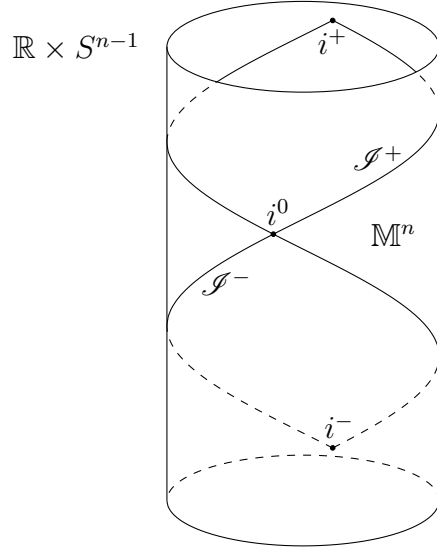


FIGURE 3. The standard conformal embedding of n -dimensional Minkowski space \mathbb{M}^n into the Einstein cylinder.

Definition 9. A smooth (time- and space-orientable) spacetime (M_+, g_+) is called *asymptotically simple* if there exists another smooth Lorentzian manifold (M, g) such that

- (i) M_+ is an open submanifold of M with smooth boundary $\partial M_+ = \mathcal{I}$;
- (ii) there exists a smooth scalar field Ω on M , such that $g = \Omega^2 g_+$ on M_+ , and so that $\Omega = 0, d\Omega \neq 0$ on \mathcal{I} ;
- (iii) every null geodesic in M_+ acquires a future and a past endpoint on \mathcal{I} .

An asymptotically simple spacetime is called *asymptotically flat* if in addition $\text{Ric}^{g_+} = 0$ in a neighbourhood of \mathcal{I} .

The Lorentz manifold (M, g) is commonly referred to as the *unphysical spacetime*, and g is called the *unphysical metric*. One can easily see that \mathcal{I} must have two connected components (assuming M_+ is connected) \mathcal{I}^+ and \mathcal{I}^- consisting of the future and past endpoints of null geodesics respectively. It is also easy to see that \mathcal{I}^+ and \mathcal{I}^- are both smooth null hypersurfaces. Usually (in the 4-dimensional setting) \mathcal{I}^+ and \mathcal{I}^- will have topology $\mathbb{S}^2 \times \mathbb{R}$, however this is not necessarily the case.

Remark 5.1. The third condition is in some cases too strong a requirement, for instance in a Schwarzschild black hole spacetime there are null geodesics which circle about the singularity forever. To include Schwarzschild and other such spacetimes one must talk about *weakly asymptotically flat* spacetimes (see [49]). ■

Remark 5.2. (For those wondering “whatever happened to the cosmological constant?”) One can also talk about *asymptotically de Sitter* spacetimes in which case the boundary hypersurface(s) will be spacelike and one asks for the spacetime to be Einstein, rather than Ricci flat, in a neighbourhood of the conformal infinity.

Similarly one can talk about *asymptotically anti-de Sitter* spacetimes, which have a timelike hypersurface as conformal infinity. ■

Just how many spacetimes satisfy this definition of asymptotic flatness, and how do we get our hands on them? These questions lead us to the discussion of Friedrich’s conformal field equations. We will see that the conformal field equations arise very naturally from the tractor picture. However, first we will discuss more generally the mathematics of conformal compactification (inspired by the conformal treatment of infinity above) for pseudo-Riemannian manifolds, and the geometric constraints placed on such “compactifications”. Again we shall see that this fits nicely within a tractor point of view.

5.2. Conformal compactification. A pseudo-Riemannian manifold (M_+, g_+) is said to *conformally compact* if M_+ may be identified with the interior of a smooth compact manifold with boundary M and there is a defining function r for the boundary $\Sigma = \partial M_+$ so that

$$g_+ = r^{-2}g \quad \text{on } M_+$$

where g is a smooth metric on M . In calling r a *defining function* for Σ we mean that r is a smooth real valued function on M such that Σ is exactly the zero locus $\mathcal{Z}(r)$ of r and furthermore that dr is non-zero at every point of Σ .

Any such g induces a metric \bar{g} on Σ , but the defining function r by $r' = f \cdot r$ where f is any non-vanishing function. This changes \bar{g} conformally and so canonically the boundary has a conformal structure determined by g_+ , but no metric. The metric g_+ is then complete and (Σ, \bar{c}) is sometimes called the conformal infinity of M_+ . Actually for our discussion here we are mainly interested in the structure and geometry of the boundary and the asymptotics of M_+ near this, so it is not really important to us that M is compact. An important problem is how to link the conformal geometry and conformal field theory of Σ to the corresponding pseudo-Riemannian objects on M_+ .

We will see that certainly this kind of “compactification” is not always possible, but it is useful in a number of settings. Clearly conformal geometry is involved. In a sense it arises in two ways (that are linked). Most obviously the boundary has a conformal structure. Secondly the interior was conformally rescaled to obtain the metric g that extends to the boundary. This suggests a strong role for conformal geometry – and it is this that we want to discuss.

Toward our subsequent discussion let us first make a first step by linking this notion of compactification to our conformal tools and notations. Let us fix a choice of r and hence g above. If $\tau \in \Gamma(\mathcal{E}[1])$ is any non-vanishing scale on M then $\sigma := r\tau$ is also a section of $\mathcal{E}[1]$ but now with zero locus $\mathcal{Z}(\sigma) = \Sigma$. This satisfies that $\nabla^g \sigma$ is nowhere zero along Σ and so we say that σ is a *defining density* for Σ . Clearly we can choose τ so that

$$g = \tau^{-2}g$$

where \mathbf{g} is the conformal metric on M . Then

$$g_+ = \sigma^{-2}\mathbf{g}.$$

Thus we may think of a conformally compact manifold, as defined above, as a conformal manifold with boundary (M, \mathbf{c}) equipped with a section $\sigma \in \Gamma(\mathcal{E}[1])$ that is a defining density for the boundary ∂M .

The key examples of conformally compactified manifolds fit nicely within this framework. Indeed, consider the Poincare ball model of hyperbolic space \mathbb{H}^d . This conformal compactification can be realised by considering the extra structure induced on the conformal sphere (S^d, \mathbf{c}) by a choice of constant spacelike tractor field I on S^d (i.e. a constant vector I in \mathbb{V}). This choice gives a symmetry reduction of the conformal group $G = \text{SO}_+(\mathcal{H})$ to the subgroup $H \cong \text{SO}_+(d, 1)$ which stabilises I . On each of the two open orbits of S^d under the action of H there is induced a hyperbolic metric, whereas the closed orbit (an n -sphere) receives only the conformal structure induced from (S^d, \mathbf{c}) . The two open orbits correspond to the two regions where the scale σ of I is positive and negative respectively, and the closed orbit is the zero locus of σ . If we choose coordinates X^A for \mathbb{V} as before, then the 1-density σ corresponds to the homogeneous degree one polynomial $\tilde{\sigma} = I_A X^A$ (dualising I using \mathcal{H}); that $\sigma^{-2}\mathbf{g}$ gives a hyperbolic metric on each of the open orbits can be seen by noting that the hyperplanes $I_A X^A = \pm 1$ intersect the future light cone \mathcal{N}_+ in hyperbolic sections, and the fact that $\mathcal{Z}(\sigma)$ receives only a conformal structure can be seen from the fact that the hyperplane $I_A X^A = 0$ intersects \mathcal{N}_+ in a subcone. Thus we see that the scale corresponding to a constant spacelike tractor I on (S^d, \mathbf{c}) gives rise to a decomposition of the conformal d -sphere into two copies of conformally compactified hyperbolic space glued along their boundaries.

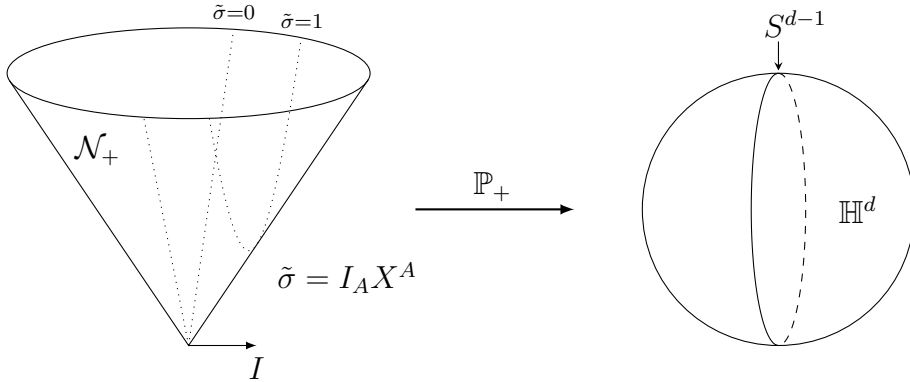


FIGURE 4. The orbit decomposition of the conformal sphere corresponding to the subgroup H of the conformal group G preserving a fixed spacelike vector I . An open orbit may be thought of as \mathbb{H}^d conformally embedded into \mathbb{S}^d .

One can repeat the construction above in the case of the Lorentzian signature model space $S^n \times S^1$ in which case one obtains a decomposition of $S^n \times S^1$ into two

copies of conformally compactified AdS^d/\mathbb{Z} glued together along their conformal infinities (where the \mathbb{Z} -action on anti-de Sitter space is given by integer times 2π time translations); this conformal compactification of AdS^d/\mathbb{Z} corresponds to the usual embedding of AdS^d into (half of) the Einstein universe after we take the quotient by \mathbb{Z} . If one instead takes a constant null tractor I on $S^n \times S^1$ then the corresponding scale σ gives decomposition of $S^n \times S^1$ into two copies of conformally compactified Minkowski space glued along their null boundaries (plus two isolated points where $\mathcal{Z}(\sigma)$ has a double cone-like singularity).

5.3. Geometry of scale. There is an interesting perspective which emerges from the above correspondence between constant (parallel) tractors and conformal compactifications. This may be motivated by Proposition 3.8 and Theorem 3.11 which linked almost Einstein scales to parallel tractors. Together they imply:

Proposition 5.3. *An Einstein manifold (M, g) is the same as a conformal manifold (M, \mathbf{c}) equipped with a parallel standard tractor I such that $\sigma := X^A I_A$ is nowhere zero: Given \mathbf{c} and I the metric is recovered by*

$$(44) \quad g = \sigma^{-2} \mathbf{g}.$$

Conversely a metric g determines $\mathbf{c} := [g]$, and $\sigma \in \mathcal{E}_+[1]$ by (44) again, now used as an expression for this variable. Then $I = \frac{1}{n} D\sigma$.

So this suggests that if we wish to draw on conformal geometry then using the package (M, \mathbf{c}, I) may give perspectives not easily seen via the (M, g) framework. Some questions arise:

- (1) First the restriction that $\sigma := X^A I_A$ is nowhere zero seems rather unnatural from this point of view. So what happens if we drop that? Then, if I is a parallel standard tractor, recall we say that (M, \mathbf{c}, I) is an almost Einstein structure. What, for example, does the zero locus $\mathcal{Z}(\sigma)$ of $\sigma = X^A I_A$ look like in this case?
- (2) Is there a sensible way to drop the Einstein condition and use this approach on general pseudo-Riemannian manifolds?

5.3.1. The zero locus of almost Einstein and ASC manifolds. In fact there is now known a way to answer the first question via a very general theory. Using “the package (M, \mathbf{c}, I) ” amounts to recovering the underlying pseudo-Riemannian structure (and its generalisations as below) as a type of *structure group reduction* of a conformal Cartan geometry. If I is parallel then this a *holonomy reduction*. On an extension of the conformal Cartan bundle to a principal bundle with fibre group $G = O(p+1, q+1)$, the parallel tractor I gives a bundle reduction to a principal bundle with fibre H where this is: (i) $O(p, q+1)$, if I is spacelike; (ii) $O(p+1, q)$, if I is timelike; and (iii) a pseudo-Euclidean group $\mathbb{R}^d \rtimes O(p, q)$, if I is null.

By the general theory of Cartan holonomy reductions [11] any such reduction yields a canonical stratification of the underlying manifold into a disjoint union of *curved orbits*. These are parametrised by $H \backslash G/P$, as are the orbits of H on G/P . Each curved orbit is an initial submanifold carrying a canonically induced Cartan

geometry of the same type as that of the corresponding orbit in the model. Furthermore this curved orbit decomposition must look locally like the decomposition of the model G/P into H -orbits. This means that in an open neighbourhood U of any point on M there is a diffeomorphism from U to the an open set U_\circ in the model that maps each curved orbit (intersected with U) diffeomorphically to the corresponding H -orbit (intersected with U_\circ) of G/P .

In our current setting P is the stabiliser in G of a null ray in $\mathbb{R}^{p+1, q+1}$ and the model G/P is isomorphic to conformal $S^d \times \{1, -1\}$, if \mathbf{c} is Riemannian, and conformal $S^p \times S^q$ otherwise. The curved orbits arise here because as we move around the manifold the algebraic relationship between the parallel object I and the canonical tractor X changes. (For general holonomy reductions of Cartan geometries the situation is a simple generalisation of this.) In particular, in Riemannian signature (or if $I^2 \neq 0$), it is easily verified using these tools that the curved orbits (and the H -orbits on the model) are distinguished by the strict sign of $\sigma = X^A I_A$, see [11, Section 3.5]. By examining these sets on the model we conclude.

Theorem 5.4. *The curved orbit decomposition of an almost Einstein manifold (M, \mathbf{c}, I) is according to the strict sign of $\sigma = I_A X^A$. The zero locus satisfies:*

- If $I^2 \neq 0$ (i.e. g° Einstein and not Ricci flat) then $\mathcal{Z}(\sigma)$ is either empty or is a smooth embedded hypersurface.
- If $I^2 = 0$ (i.e. g° Ricci flat) then $\mathcal{Z}(\sigma)$ is either empty or, after excluding isolated points from $\mathcal{Z}(\sigma)$, is a smooth embedded hypersurface.

Here g° means the metric $\sigma^{-2}\mathbf{g}$ on the open orbits (where σ is nowhere zero).

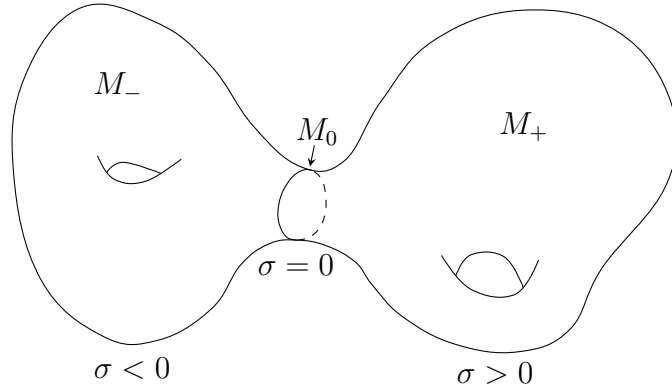


FIGURE 5. The curved orbit decomposition of an almost Einstein manifold with $\mathcal{Z}(\sigma)$ an embedded separating hypersurface.

Remark 5.5. Much more can be said using the tools mentioned. For example in the case of Riemannian signature it is easily shown that:

- If $I^2 < 0$ (i.e. g° Einstein with positive scalar curvature) then $\mathcal{Z}(\sigma)$ is empty.

- If $I^2 = 0$ (i.e. g^o Ricci flat) then $\mathcal{Z}(\sigma)$ is either empty or consists of isolated points.
- If $I^2 > 0$ (i.e. g^o Einstein with negative scalar curvature) then $\mathcal{Z}(\sigma)$ is either empty or is a smooth embedded separating hypersurface.

This holds because this is how things are locally on the flat model [24, 11]; if $I^2 < 0$ then on the model we have a round metric induced on the whole conformal sphere, the one open orbit is the whole space; if $I^2 = 0$ then on the model we are looking at the one point conformal compactification of Euclidean space given by inverse stereographic projection, and there are two orbits, one open and one an isolated point, so in the curved case $\mathcal{Z}(\sigma)$ is either empty or consists of isolated points; if $I^2 > 0$ then on the model we are looking at two copies of conformally compactified hyperbolic space glued along their boundaries as discussed earlier, in this case the closed orbit is a separating hypersurface so in the curved case $\mathcal{Z}(\sigma)$ is either empty or is a smooth embedded separating hypersurface. In the Lorentzian setting one can obtain a similar improvement of Theorem 5.4 by considering the model cases. \blacksquare

In fact similar results hold in greater generality (related to the question (2) above) and this is easily seen using the earlier tractor calculus and elementary considerations. We learn from the Einstein case above that an important role is played by the scale tractor

$$I_A = \frac{1}{d} D_A \sigma.$$

In the Einstein case this is parallel and hence non-zero everywhere. Let us drop the condition that I is parallel and for convenience say that a structure

$$(M^d, \mathbf{c}, \sigma) \quad \text{where} \quad \sigma \in \Gamma(\mathcal{E}[1])$$

is *almost pseudo-Riemannian* if the scale tractor $I_A := \frac{1}{d} D_A \sigma$ is nowhere zero. Note then that σ is non-zero on an open dense set, since $D_A \sigma$ encodes part of the 2-jet of σ . So on an almost pseudo-Riemannian manifold there is the pseudo-Riemannian metric $g^o = \sigma^{-2} \mathbf{g}$ on the same open dense set. In the following the notation I will always refer to a scale tractor, so $I = \frac{1}{d} D\sigma$, for some $\sigma \in \Gamma(\mathcal{E}[1])$. Then we often mention I instead of σ and refer to (M, \mathbf{c}, I) as an almost pseudo-Riemannian manifold.

Now recall from (30) that

$$(45) \quad I^2 \stackrel{g}{=} \mathbf{g}^{ab} (\nabla_a \sigma) (\nabla_b \sigma) - \frac{2}{d} \sigma (\mathbf{J} + \Delta) \sigma$$

where g is any metric from \mathbf{c} and ∇ its Levi-Civita connection. This is well-defined everywhere on an almost pseudo-Riemannian manifold, while according to Proposition 3.14, where σ is non-zero, it computes

$$I^2 = -\frac{2}{d} \mathbf{J}^{g^o} = -\frac{\text{Sc}^{g^o}}{d(d-1)} \quad \text{where} \quad g^o = \sigma^{-2} \mathbf{g}.$$

Thus I^2 gives a generalisation of the scalar curvature (up to a constant factor $-1/d(d-1)$); it is canonical and smoothly extends the scalar curvature to include the zero set of σ . We shall use the term *ASC manifold* (where ASC means almost scalar constant) to mean an almost pseudo-Riemannian manifold with $I^2 = \text{constant}$. Since the tractor connection preserves the tractor metric, an almost Einstein manifold is a special case, just as Einstein manifolds have constant scalar curvature.

Much of the previous theorem still holds in the almost pseudo-Riemannian setting when $I^2 \neq 0$:

Theorem 5.6. *Let (M, \mathbf{c}, I) be an almost pseudo-Riemannian manifold with $I^2 \neq 0$. Then $\mathcal{Z}(\sigma)$, if not empty, is a smooth embedded separating hypersurface. This has a spacelike (resp. timelike) normal if g has negative scalar (resp. positive) scalar curvature.*

If \mathbf{c} has Riemannian signature and $I^2 < 0$ then $\mathcal{Z}(\sigma)$ is empty.

Proof. This is an immediate consequence of (45): Along the zero locus $\mathcal{Z}(\sigma)$ of σ (assuming it is non empty) we have

$$I^2 = \mathbf{g}^{ab}(\nabla_a \sigma)(\nabla_b \sigma).$$

in particular $\nabla \sigma$ is nowhere zero on $\mathcal{Z}(\sigma)$, and so σ is a defining density. Thus $\mathcal{Z}(\sigma)$ is a smoothly embedded hypersurface by the implicit function theorem. Evidently $\mathcal{Z}(\sigma)$ separates M according to the sign of σ . Also $\nabla \sigma$ is a (weight 1) conormal field along $\mathcal{Z}(\sigma)$ so the claimed signs for the normal also follow from the display.

Finally if \mathbf{c} (and hence \mathbf{g}) has Riemannian signature, then the display shows that at any point of $\mathcal{Z}(\sigma)$ the constant I^2 must be positive. This is a contradiction if $I^2 < 0$ and so $\mathcal{Z}(\sigma) = \emptyset$. \square

Remark 5.7. Note that if M is compact then Theorem 5.6 gives a decomposition of M into conformally compact manifolds glued along their conformal infinities. Note also that if M is allowed to have boundary then we only mean that $\mathcal{Z}(\sigma)$ is separating if it is not a boundary component of M . \blacksquare

What we can conclude from all this is that almost pseudo-Riemannian manifolds (M, \mathbf{c}, I) naturally give rise to nicely conformally compactified metrics (at least in the case where $I^2 \neq 0$). If the scalar curvature of the manifold (M_+, g_+) admitting a conformal compactification (M, c) is bounded away from zero then the conformal compactification must arise from an almost pseudo-Riemannian manifold (M, \mathbf{c}, I) in this way because I^2 is then bounded away from zero on M_+ and hence I extends to be nowhere zero on M (all that we really need is that $I^2 \neq 0$ on ∂M_+). Thus we can apply the results above to the conformal compactification of (M_+, g_+) . This tells us for instance that Riemannian manifolds with negative scalar curvature bounded away from zero must have a nice smooth boundary as conformal infinity if they can be conformally compactified. In the next section we develop this kind of idea.

5.4. Constraints on possible conformal compactifications. Here we show that elementary geometric considerations restrict the topological (and geometric) possibilities for a conformal infinity.

Theorem 5.8. *Let (M_+, g_+) be a geodesically complete pseudo-Riemannian manifold and $i : M_+ \rightarrow M$ an embedding of M_+ as an open submanifold in a closed conformal manifold (M, \mathbf{c}) and σ a smooth section of $\mathcal{E}[1]$ on M so that on the image $i(M_+)$, which we identify with M_+ , we have that*

$$g_+ = \sigma^{-2} \mathbf{g}$$

where \mathbf{g} is the conformal metric on M . If the scalar curvature of g_+ is bounded away from zero then either $M_+ = M$ or the boundary points of M_+ in M form a smooth embedded hypersurface in M . This has a timelike normal field if $\text{Sc}^{g_+} > 0$ and a spacelike normal field if $\text{Sc}^{g_+} < 0$.

Proof. We can form the scale tractor

$$I_A = \frac{1}{d} D_A \sigma.$$

Then I^2 is a smooth function on M that is bounded away from zero on the open set $M_+ \subset M$. Thus $I^2 \neq 0$ on the topological closure \overline{M}_+ of M_+ in M . Now since $I^2 \neq 0$ on \overline{M}_+ we have at any point $p \in \mathcal{Z}(\sigma) \cap \overline{M}_+$ that $\nabla \sigma(p) \neq 0$. Thus the zero locus of σ (intersected with \overline{M}_+) is a regular embedded submanifold of \overline{M}_+ . By construction this lies in the set ∂M_+ of boundary points to M_+ .

On the other hand any boundary point is in the zero locus, as otherwise for any $p \in \partial M_+$ s.t. $\sigma(p) \neq 0$ there is an open neighbourhood of p in \overline{M}_+ such that σ is nowhere zero and so g extends as a metric to this neighbourhood. So then there is a geodesic from a point in M_+ that reaches p in finite time, which contradicts that M_+ is geodesically complete. \square

In particular if g_+ is Einstein with non-zero cosmological constant then I is parallel, and hence I^2 is constant and nonzero. So the above theorem applies in this case. In fact in the Einstein case we can also get results which in some aspects are stronger. For example:

Theorem 5.9. *Suppose that M_+ is an open dense submanifold in a compact connected conformal manifold (M^d, \mathbf{c}) , possibly with boundary, and that one of the following two possibilities hold: either M is a manifold with boundary ∂M and $M \setminus M_+ = \partial M$, or M is closed and $M \setminus M_+$ is contained in a smoothly embedded submanifold of M of codimension at least 2. Suppose also that g_+ is a geodesically complete Einstein, but not Ricci flat, pseudo-Riemannian metric on M_+ such that on M_+ the conformal structure $[g_+]$ coincides with the restriction of \mathbf{c} . Then, either*

- M_+ is closed and $M_+ = M$; or
- we are in the first setting with $M \setminus M_+$ the smooth n -dimensional boundary for M .

Proof. M_+ is canonically equipped with a parallel standard tractor I such that $I^2 = c \neq 0$, where c is constant.

Now $[g_+]$ on M_+ is the restriction of a smooth conformal structure \mathbf{c} on M . Thus the conformal tractor connection on M_+ is the restriction of the smooth tractor connection on M . Working locally it is straightforward to use parallel transport along a congruence of curves to give a smooth extension of I to a sufficiently small open neighborhood of any point in $M \setminus M_+$, and since M_+ is dense in M the extension is parallel and unique. (See [15] for this extension result in general for the case where $M \setminus M_+$ lies in a submanifold of dimension at least 2.) It follows that I extends as a parallel field to all of M .

It follows that $\sigma := I_A X^A$ also extends smoothly to all of M . This puts us back in the setting of Theorem 5.8, so that $M \setminus M_+$ is either empty or is an embedded hypersurface. If $M \setminus M_+$ is empty then M is closed and $M = N$. If $M \setminus M_+$ is an embedded hypersurface then, by our assumptions, we must be in the case where M is a manifold with boundary and $M \setminus M_+ = \partial M$ (since an embedded hypersurface cannot be contained in a submanifold of codimension two). \square

Remark 5.10. In Theorem 5.8 we had to assume that there was a smooth globally defined conformal 1 density σ which glued the metric on M_+ smoothly to the conformal structure of M . In Theorem 5.9 we are able to drop this condition because we are requiring that the metric g_+ on M_+ is Einstein with conformal structure agreeing with that on M , allowing us to recover the smooth global 1 density σ by parallel extension of the scale tractor for g_+ to $M \setminus M_+$. However, it is not hard to see that one must place some restriction on the submanifold $M \setminus M_+$ in order for a parallel extension to exist, this is where the condition that $M \setminus M_+$ must lie in a submanifold of codimension two comes in for the case where M is closed. \blacksquare

5.5. Friedrich's conformal field equations and tractors. In this section we briefly discuss Friedrich's conformal field equations and their applications. We will see that the equations can be very easily arrived at using tractor calculus, and that the tractor point of view is also nicely compatible with the way the equations are used in applications.

5.5.1. The equations derived. Suppose that (M_+, g_+) is an asymptotically flat spacetime with corresponding unphysical spacetime (M, g) and conformal factor Ω . (We may assume that $M = M_+ \cup \mathcal{I}$.) Then since (M_+, g_+) is Ricci flat near the conformal infinity \mathcal{I} the conformal scale tractor I defined on M_+ corresponding to the metric g_+ is parallel (and null) near \mathcal{I} and thus has a natural extension (via parallel transport, or simply by taking the limit) to all of M . Let us assume for simplicity that (M_+, g_+) is globally Ricci flat, then the naturally extended tractor I is globally parallel on the non-physical spacetime (M, g) . If we write out the equation $\nabla_a I^B = 0$ on M in slots using the decomposition of the standard tractor bundle of $(M, [g])$ induced by g then we get the system of equations

$$\begin{aligned}\nabla_a \sigma &= \mu_a \\ \nabla_a \mu_b &= -\sigma P_{ab} - \rho g_{ab} \\ \nabla_a \rho &= P_{ab} \mu^b\end{aligned}$$

where $I \stackrel{g}{=} (\sigma, \mu_a, \rho)$. If we also trivialise the conformal density bundles using g then σ becomes Ω , \mathbf{g} becomes g , and we recognise the above three equations as the first three equations in what are commonly known as *Friedrich's conformal field equations*. A fourth member of the conformal field equations can be obtained by writing $I_A I^A = 0$ out as

$$2\sigma\rho + \mu_a \mu^a = 0.$$

Next we observe that $\nabla_a I^B = 0$ clearly implies $\kappa_{ab}{}^C{}_D I^D = 0$, and the projecting part of this equation gives the *conformal C-space equation*

$$\nabla_b P_{ac} - \nabla_a P_{bc} = W_{abcd} \sigma^{-1} \mu^d$$

after multiplication by σ^{-1} . (The “bottom slot” of $\kappa_{ab}{}^C{}_D I^D = 0$ taken w.r.t. g is $(\nabla_b P_{ac} - \nabla_a P_{bc}) \mu^c = 0$ which also follows from contracting the above displayed equation with μ^c .) The contracted Bianchi identity states that

$$\nabla^d W_{abcd} = \nabla_b P_{ac} - \nabla_a P_{bc}$$

and from this and the previous display it follows that

$$\nabla^d (\sigma^{-1} W_{abcd}) = 0$$

(where we have used that $\nabla_a \sigma = \mu_a$). It can be shown [19] that the field $\sigma^{-1} W_{ab}{}^c{}_d$ is regular at \mathcal{I} (recall the Sachs peeling property), and we will write this field as $K_{ab}{}^c{}_d$. If we substitute $K_{abcd} = g_{ce} K_{ab}{}^e{}_d$ for $\sigma^{-1} W_{abcd}$ in the C-space equation and in the last equation displayed above we get the remaining two equations from Friedrich's conformal system

$$\nabla_b P_{ac} - \nabla_a P_{bc} = K_{abcd} \mu^d$$

and

$$\nabla^a K_{abcd} = 0,$$

which hold not only on M_+ but on all of M . This pair of equations encode not only the fact that (M_+, g_+) is Cotton flat (both equations being in some sense conformal C-space equations), but also by their difference they encode the contracted Bianchi identity (and hence the full Bianchi identity if M_+ is 4-dimensional [21]).

The equations from Friedrich's conformal system are thus seen to be elementary consequences of the system of four tractor equations

$$\begin{aligned}\nabla_a I^B &= 0 \\ I^A I_A &= 0 \\ \kappa_{ab}{}^C{}_D I^D &= 0 \\ \nabla_{[a} \kappa_{bc]DE} &= 0.\end{aligned}$$

Remark 5.11. There are in fact more equations that could be considered as a part of Friedrich’s conformal system, however these simply define the connection and the curvature terms used in the equations we have given. These equations would be the expressions $\nabla g = 0$ and $T^\nabla = 0$ satisfied by the Levi-Civita connection of g , and the decomposition of the Riemannian curvature tensor of ∇ as

$$R_{abcd} = \sigma K_{abcd} + 2P_{a[c}g_{d]b} - 2P_{b[c}g_{d]a}$$

which serves to define the curvature tensors K_{abcd} and P_{ab} (along with the appropriate symmetry conditions on K_{abcd} and P_{ab} as well as the condition that $g^{ac}K_{abcd} = 0$).

Note that we can easily allow for a nonzero cosmological constant in our tractor system by taking $I^A I_A$ to be a constant rather than simply zero. ■

5.5.2. *The purpose of the equations.* What are the conformal Field equations for? The answer is that one seeks solutions to them in the same way that one seeks solutions to Einstein’s field equations. A global solution to the conformal field equations gives a conformally compactified solution of Einstein’s field equations. (In fact solutions of the conformal field equations may extend to regions “beyond infinity” where Ω becomes negative.) We should note one can very easily allow for a non zero cosmological constant and even for non zero matter fields (especially ones with nice conformal behaviour) in the conformal field equations. The conformal field equations are then a tool for obtaining and investigating isolated systems in general relativity.

Like Einstein’s field equations, the conformal field equations have an initial value formulation where initial data is specified on a Cauchy hypersurface. But when working with the conformal field equations is that it is also natural to prescribe data on the conformal infinity; when initial data is prescribed on (part of) \mathcal{I}^- as well as on an ingoing null hypersurface which meets \mathcal{I}^- transversally we have the *characteristic initial value problem*. When initial data is specified on a spacelike hypersurface which meets conformal infinity transversally we have the *hyperboloidal initial value problem*. (The name comes from the fact that spacelike hyperboloid in Minkowski space are the prime examples of such initial data hypersurfaces.) If the hyperboloidal initial data hypersurface meets \mathcal{I}^- (rather than \mathcal{I}^+) then one should also prescribe data on the part of \mathcal{I}^- to the future of the hypersurface, giving rise to an initial-boundary value problem.

What is being sought in the study of these various geometric PDE problems? Firstly information about when spacetimes will admit a conformal infinity and what kind of smoothness it might have. Secondly information about gravitational radiation produced by various gravitational systems as well as the way that such systems interact with gravitational radiation (scattering properties). Thirdly, one is obviously interested in the end in having a general understanding of the solutions of the conformal field equations (though this is a very hard problem). These problems have been studied a good deal, both analytically and numerically, however there are many questions left to be answered. For helpful overviews of this work consult [19, 21].

5.5.3. *Regularity.* An important feature of the conformal field equations is that they are *regular* at infinity. From the tractor calculus point of view this is obvious since the conformal structure extends to the conformal infinity so the tractor system displayed above cannot break down there, but one can also easily see this from the usual form of the equations. The significance of this is highlighted when one considers what field equations one might naively expect to use in this setting: the most obvious guess would be to rewrite $\text{Ric}^{g^+} = 0$ in terms of the Ricci tensor of the unphysical metric g and its Levi-Civita connection yielding

$$\text{Ric}_{ab} + \frac{d-2}{\Omega} \nabla_a \nabla_b \Omega - \mathbf{g}_{ab} \left(\frac{1}{\Omega} \nabla^c \nabla_c \Omega - \frac{d-1}{\Omega^2} (\nabla^c \Omega) \nabla_c \Omega \right) = 0$$

which degenerates as $\Omega \rightarrow 0$.

It is worth noting the significance of the regularity of the conformal field equations in the area of numerical relativity. The appeal of “conformal compactification” to numerical relativists should be obvious: it means that one is dealing with finite domains.

5.5.4. *Different reductions and different forms of the equations.* The conformal field equations are a system of geometric partial differential equations. In order to study them analytically or numerically they need to be reduced to a classical system of PDE; this involves introducing coordinates and adding conditions on various fields to pin down the natural gauge freedom in the equations (gauge fixing). There is a significant amount of freedom in how one reduces the system. If this process is done carefully with the conformal field equations one can obtain (in 4 dimensions) a symmetric hyperbolic system of evolution equations together with an elliptic system of constraint equations (reflecting the fact that the conformal field equations are overdetermined). The constraint equations can be taken as conditions on initial data for the Cauchy problem for the conformal field equations, whereas the evolution equations can be taken as prescribing how such data will evolve off the Cauchy hypersurface. For further discussion of the reduction process and a demonstration of how the conformal field equations can be reduced see [19].

One may employ different reductions of the conformal field equations in different settings and for different purposes. Indeed, the form of Friedrich’s conformal field equations which we presented above is by no means the only form of the equations that is used, and the significance of the other forms is that they allow for a still broader range of different reductions. As an example of this, note that one could instead have employed the splitting of the tractor bundle induced by a Weyl connection [8] to obtain a system of conformal field equations in a similar to what we did above; this would result in a different form of the equations which have different gauge freedoms and from which we can obtain different reduced systems; the equations which one would obtain this way are what Friedrich calls the *general conformal field equations* whereas the equations we presented above are referred to as the *metric conformal field equations* [21]. Friedrich also frequently casts the equations in spinor form, and now we have seen that the conformal field equations

take a very simple tractor form

$$\begin{aligned}\nabla_a I^B &= 0 \\ I^A I_A &= 0 \\ \kappa_{ab}{}^C{}_D I^D &= 0 \\ \nabla_{[a} \kappa_{bc]DE} &= 0\end{aligned}$$

(where we also need to impose the appropriate conditions on ∇ and κ as variables). It may indeed prove profitable to see the various reductions of the conformal field equations as being reductions of this tractor system. The gauge freedom(s) in Friedrich’s field equations can be seen as coming from the splitting of the tractor equations into tensor equations (along with freedom to choose coordinates). Friedrich’s use of conformal geodesics [21] in constructing coordinates as part of the reduction process also fits quite nicely with the tractor picture (see, e.g., [42, 44]).

Remark 5.12. In [20] Fraudiener and Sparling observed that in the 4-dimensional (spin) case Friedrich’s conformal field equations could be recast in terms of local twistors involving the so called “infinity twistor” $I^{\alpha\beta}$. In this case the bundle of real bitwistors is canonically isomorphic to the standard tractor bundle, and the “infinity twistor” $I^{\alpha\beta}$ corresponds (up to a constant factor) to the scale tractor I^A , so that the local twistor system in [20] is closely related to the tractor system above. Indeed the connection is so close (the modern approach to conformal tractors having developed out of study of the local twistor calculus [2]) that we may consider the local twistor formulation to be the origin of the above tractor system. The conformal field equations have also been presented in terms of tractors by Christian Lübbe [42]. Lübbe and Tod have applied the tractor calculus to the study of conformal gauge singularities in general relativity (see, e.g., [44]). ■

6. LECTURE 6: CONFORMAL HYPERSURFACES

In order to progress in our study of conformal compactification we first need to spend some time considering the geometry of embedded hypersurfaces in conformal manifolds (or of boundaries to conformal manifolds if you like). In particular we will need to examine how the standard tractor bundle (and connection, etc.) of the hypersurface with its induced conformal structure is related to the standard tractor bundle (and connection, etc.) of the ambient space. Although our main motivation is the study of conformal compactification for this lecture we will consider conformal hypersurfaces more generally since there are many other important kinds of hypersurface which turn up in general relativity. We will however restrict ourselves to the case of nondegenerate hypersurfaces, in particular we do not give a treatment of null hypersurfaces here.

6.1. Conformal hypersurfaces. By a *hypersurface* Σ in a manifold M we mean a smoothly embedded codimension 1 submanifold of M . We recall some facts concerning hypersurfaces in a conformal manifold (M^d, c) , $d \geq 3$. In fact we wish to include the case that Σ might be a boundary component. In the case of a pseudo-Riemannian (or conformal pseudo-Riemannian) manifold with boundary

then, without further comment, we will assume that the conformal structure extends smoothly to the boundary.

Here we shall restrict to hypersurfaces Σ with the property that the any conormal field along Σ is nowhere null (i.e. to nondegenerate hypersurfaces). In this case the restriction of any metric $g \in \mathbf{c}$ gives a metric \bar{g} on Σ . Different metrics, among the metrics so obtained, are related conformally and so the conformal class \mathbf{c} determines a conformal structure $\bar{\mathbf{c}}$ on Σ . To distinguish from the ambient objects, we shall overline the corresponding objects intrinsic to this conformal structure. For example \bar{g} denotes the conformal metric on Σ .

Then by working locally we may assume that there is a section $n_a \in \Gamma(\mathcal{E}_a[1])$ on M such that, along Σ , n_a is a conormal satisfying $|n|_{\mathbf{g}}^2 := \mathbf{g}^{ab}n_a n_b = \pm 1$. This means that if $g = \sigma^{-2}\mathbf{g}$ is any metric in \mathbf{c} then $n_a^g = \sigma^{-1}n_a$ is an extension to M of a unit conormal to Σ in (M, g) . So $n_a|_{\Sigma}$ is the conformally invariant version of a unit conormal in pseudo-Riemannian geometry; n_a must have conformal weight 1 since \mathbf{g}^{-1} has conformal weight -2 .

We choose to work with the weighted (extended) conormal field n_a even in the presence of a metric g from the conformal class. Thus we end up with a weight 1 *second fundamental form* L_{ab} by restricting, along Σ , $\nabla_a n_b$ to $T\Sigma \times T\Sigma \subset (TM \times TM)|_{\Sigma}$, where $\nabla = \nabla^g$. (Here we are viewing $T^*\Sigma$ as the subbundle of $T^*M|_{\Sigma}$ orthogonal to n^a .) Explicitly L_{ab} is given by

$$L_{ab} := \nabla_a n_b \mp n_a n^c \nabla_c n_b \quad \text{along } \Sigma,$$

since $|n|_{\mathbf{g}}^2$ is constant along Σ . From this formula, it is easily verified that L_{ab} is independent of how n_a is extended off Σ . It is timely to note that L_{ab} harbours a hypersurface conformal invariant: Using the formulae (2) and (16) we compute that under a conformal rescaling, $g \mapsto \hat{g} = e^{2\omega}g$, L_{ab} transforms according to

$$L_{ab}^{\hat{g}} = L_{ab}^g + \bar{g}_{ab} \Upsilon_c n^c,$$

where as usual Υ is the exterior derivative of ω (which is equal to the log exterior derivative $\Omega^{-1}d\Omega$ of $\Omega = e^\omega$) and we use this notation below without further mention. Thus we see easily the following well-known result:

Proposition 6.1. *The trace-free part of the second fundamental form*

$$\mathring{L}_{ab} = L_{ab} - H\bar{g}_{ab}, \quad \text{where,} \quad H := \frac{1}{d-1} \bar{g}^{cd} L_{cd}$$

is conformally invariant.

The averaged trace of L ($= L^g$), denoted H above, is the *mean curvature* of Σ . Evidently this a conformal -1 -density and under a conformal rescaling, $g \mapsto \hat{g} = e^{2\omega}g$, H^g transforms to $H^{\hat{g}} = H^g + n^a \Upsilon_a$.

Thus we obtain a conformally invariant section N of $\mathcal{T}|_{\Sigma}$

$$N_A \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ -H^g \end{pmatrix},$$

and from (29) $h(N, N) = \pm 1$ along Σ ; the conformal invariance follows because the right-hand-side of the display transforms according to the ‘‘tractor characterising’’ transformation (26). Obviously N is independent of any choices in the extension of n_a off Σ . This is the *normal tractor* of [2] and may be viewed as a tractor bundle analogue of the unit conormal field from the theory of pseudo-Riemannian hypersurfaces.

6.1.1. *Umbilicity.* A point p in a hypersurface is said to be an *umbilic point* if, at that point, the trace-free part \mathring{L} of the second fundamental form is zero. Evidently this is a conformally invariant condition. A hypersurface is *totally umbilic* if this holds at all points. As an easy first application of the normal tractor we recall that it leads to a nice characterisation of the the umbilicity condition.

Differentiating N tangentially along Σ using $\nabla^{\mathcal{T}}$, we obtain the following result.

Lemma 6.2.

$$(46) \quad \mathbb{L}_{aB} := \underline{\nabla}_a N_B \stackrel{g_{cb}}{=} \begin{pmatrix} 0 \\ \mathring{L}_{ab} \\ -\frac{1}{d-2} \nabla^b \mathring{L}_{ab} \end{pmatrix}$$

where $\underline{\nabla}$ is the pullback to Σ of the ambient tractor connection.

Proof. Using the formula (22) for the tractor connection, we have

$$\nabla_c N_B \stackrel{g}{=} \begin{pmatrix} -n_c \\ \nabla_c n_b - \mathbf{g}_{cb} H \\ -\nabla_c H - P_{cb} n^b \end{pmatrix}.$$

Thus applying the orthogonal $TM|_{\Sigma} \rightarrow T\Sigma$ projector $\Pi_a^c := (\delta_a^c \mp n^c n_a)$ we obtain immediately the top two terms on the right-hand-side of (46). The remaining term follows after using the hypersurface Codazzi equation for pseudo-Riemannian geometry

$$\bar{\nabla}_a L_{bc} - \bar{\nabla}_b L_{ac} = \Pi_a^{a'} \Pi_b^{b'} R_{a'b'cd} n^d,$$

where $\bar{\nabla}$ denotes the Levi-Civita connection for the metric \bar{g} induced by g (for full details see [55, 32]). \square

Thus we recover the following result.

Proposition 6.3. [2] *Along a conformal hypersurface Σ , the normal tractor N is parallel, with respect to $\nabla^{\mathcal{T}}$, if and only if the hypersurface Σ is totally umbilic.*

6.1.2. *Conformal calculus for hypersurfaces.* The local calculus for hypersurfaces in Riemannian geometry is to a large extent straightforward because there is a particularly simple formula, known as the *Gauss formula*, which relates the ambient Levi-Civita connection to the Levi-Civita connection of the induced metric. It is natural to ask if we have the same with our conformal tractor calculus.

Given an ambient metric g we write $\underline{\nabla}$ to denote the pullback of the ambient Levi-Civita connection along the embedding of the hypersurface Σ , i.e. the

ambient connection differentiating sections of $TM|_\Sigma$ in directions tangent to the hypersurface. Then the Gauss formula may be expressed as

$$\underline{\nabla}_a v^b = \bar{\nabla}_a v^b \mp n^b L_{ac} v^c$$

for any tangent vector field v to Σ (thought of as a section v^a of $\mathcal{E}^a|_\Sigma$ satisfying $v^a n_a = 0$). We now turn towards deriving a tractor analogue of this.

Note that by dint of its conformal structure the hypersurface $(\Sigma, \bar{\mathbf{c}})$ has its own *intrinsic* tractor calculus, and in particular a rank $d + 1$ standard tractor bundle $\bar{\mathcal{T}}$. Before we could hope to address the question above we need to relate $\bar{\mathcal{T}}$ to the ambient tractor bundle \mathcal{T} .

First observe that, along Σ , \mathcal{T} has a natural rank $(d + 1)$ -subbundle, namely N^\perp the orthogonal complement to N_B . As noted in [5, 38], there is a canonical (conformally invariant) isomorphism

$$(47) \quad N^\perp \xrightarrow{\simeq} \bar{\mathcal{T}}.$$

Calculating in a scale g on M the tractor bundle \mathcal{T} , and hence also N^\perp , decomposes into a triple. Then the mapping of the isomorphism is

$$(48) \quad [N^\perp]_g \ni \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu_b \mp H n_b \sigma \\ \rho \pm \frac{1}{2} H^2 \sigma \end{pmatrix} \in [\bar{\mathcal{T}}]_{\bar{g}}$$

where, as usual, H denotes the mean curvature of Σ in the scale g and \bar{g} is the pullback of g to Σ . Since (σ, μ_b, ρ) is a section of $[N^\perp]_g$ we have $n^a \mu_a = H\sigma$. Using this one easily verifies that the mapping is conformally invariant: If we transform to $\hat{g} = e^{2\omega} g$, $\omega \in \mathcal{E}$, then (σ, μ_b, ρ) transforms according to (26). Using that $\hat{H} = H + n^a \Upsilon_a$ one calculates that the image of (σ, μ_b, ρ) (under the map displayed) transforms by the intrinsic version of (26), that is by (26) except where Υ_a is replaced by $\bar{\Upsilon}_a = \Upsilon_a \mp n_a n^b \Upsilon_b$ (which on Σ agrees with $\bar{d}\omega$, the tangential derivative of ω). This signals that the explicit map displayed in (48) descends to a conformally invariant map (47). We henceforth use this to identify N^\perp with $\bar{\mathcal{T}}$, and write $\text{Proj}_\Sigma : \mathcal{T}|_\Sigma \rightarrow \bar{\mathcal{T}}$ for the orthogonal projection afforded by N (or using abstract indices $\Pi_B^A = \delta_B^A \mp N^A N_B$).

It follows easily from (48) that the tractor metric \bar{h} on $\bar{\mathcal{T}}$ agrees with the restriction of the ambient tractor metric h to N^\perp . In summary we have:

Theorem 6.4. *Let (M^d, \mathbf{c}) conformal manifold of dimension $d \geq 4$ and Σ a regular hypersurface in M . Then, with $\bar{\mathcal{T}}$ denoting the intrinsic tractor bundle of the induced conformal structure \mathbf{c}_Σ , there is a canonical isomorphism*

$$\bar{\mathcal{T}} \rightarrow N^\perp.$$

Furthermore the tractor metric of \mathbf{c}_Σ coincides with the pullback of the ambient tractor metric, under this map.

Henceforth we shall simply identify $\bar{\mathcal{T}}$ and N^\perp .

Remark 6.5. Note that if Σ is minimal in the scale g , that is if $H^g = 0$, then the isomorphism (48) is simply

$$(49) \quad [N^\perp]_g \ni \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \in [\overline{\mathcal{T}}]_{\overline{g}}.$$

Moreover, it is easy to see that one can always find such a minimal scale g for Σ . Let $g = \sigma^{-2}\mathbf{g}$ be any metric in \mathbf{c} and let $\omega := \mp s\sigma H^g$, where s is a normalised defining function for Σ (at least in a neighbourhood of Σ) and H^g has been extended off Σ arbitrarily. Then if $\hat{g} = e^{2\omega}g$ we have that, along Σ ,

$$H^{\hat{g}} = H^g + n^a \Upsilon_a = H^g + n^a \nabla_a \omega = 0$$

since by assumption $\nabla_a s = \sigma^{-1}n_a$ along Σ and $s|_\Sigma = 0$. ■

Now we have two connections on $\overline{\mathcal{T}}$ that we may compare, namely the intrinsic tractor connection $\overline{\nabla}^{\overline{\mathcal{T}}}$, meaning the normal tractor connection determined by conformal structure (Σ, \mathbf{c}) , and the *projected ambient tractor connection* $\overset{\vee}{\nabla}$. On $U \in \Gamma(\overline{\mathcal{T}})$ the latter is defined by

$$\overset{\vee}{\nabla}_a U^B := \Pi_C^B(\Pi_a^c \nabla_c U^C) \quad \text{along } \Sigma,$$

where we view $U \in \Gamma(\overline{\mathcal{T}})$ as a section of N^\perp , and make an arbitrary smooth extension of this to a section of \mathcal{T} in a neighbourhood (in M) of Σ . It is then easily verified that $\overset{\vee}{\nabla}$ is a connection on $\overline{\mathcal{T}}$. By construction it is conformally invariant. Thus the difference between this and the intrinsic tractor connection is some canonical conformally invariant section of $T^*\Sigma \otimes \text{End}(\overline{\mathcal{T}})$.

The difference between the projected ambient and the intrinsic tractor connections can be expressed using the tractor contorsion $S_a{}^B{}_C$ defined by the equation

$$(50) \quad \overset{\vee}{\nabla}_a V^B = \overline{\nabla}_a V^B \mp S_a{}^B{}_C V^C$$

where $V^B \in \Gamma(\overline{\mathcal{E}}^B)$ is an intrinsic tractor. The intrinsic tractor contorsion can be computed explicitly in any scale g to take the form

$$(51) \quad S_a{}^B{}_C \stackrel{g}{=} \bar{X}^B \bar{Z}_C^c \mathcal{F}_{ac} - \bar{Z}^B{}_b \bar{X}_C \mathcal{F}_a{}^b$$

or in other words

$$(52) \quad S_{aBC} = \bar{\mathbb{X}}_{BC}{}^c \mathcal{F}_{ac},$$

where evidently \mathcal{F}_{ac} must be some conformal invariant of hypersurfaces (here \bar{X}^B and $\bar{Z}^B{}_b$ are standard tractor projectors for Σ and $\bar{\mathbb{X}}_{BC}{}^c = 2\bar{X}_{[B}\bar{Z}_{C]}^c$, which is conformally invariant). In fact the details of the computation (see [52, 55]) reveal this is the *Fialkow tensor* (cf. [18])

$$(53) \quad \mathcal{F}_{ab} = \frac{1}{n-2} \left(W_{abcd} n^c n^d + \mathring{L}_{ab}^2 - \frac{|\mathring{L}|^2}{2(n-1)} \bar{\mathbf{g}}_{ab} \right),$$

where $\mathring{L}_{ab}^2 := \mathring{L}_a^c \mathring{L}_{cb}$ and $n = d - 1$. Altogether we have,

$$\begin{aligned}
 \nabla_a V^B &= \Pi^B{}_C \nabla_a V^C \pm N^B N_C \nabla_a V^C \\
 (54) \qquad &= \check{\nabla}_a V^B \mp N^B \mathbb{L}_{aC} V^C \\
 &= \bar{\nabla}_a V^B - S_a{}^B{}_C V^C \mp N^B \mathbb{L}_{aC} V^C
 \end{aligned}$$

for an intrinsic tractor $V^B \in \overline{\mathcal{T}}^B$. The tractor Gauss formula is therefore

$$(55) \qquad \nabla_a V^B = \bar{\nabla}_a V^B \mp S_a{}^B{}_C V^C \mp N^B \mathbb{L}_{aC} V^C$$

for any $V^B \in \overline{\mathcal{T}}^B$. Here we see that the object

$$\mathbb{L}_{aB} := \nabla_a N_B$$

is a tractor analogue of the second fundamental form, we shall therefore call it the *tractor shape form*.

These results provide the first steps in a calculus for conformal hypersurfaces that is somewhat analogous to the local invariant calculus for Riemannian hypersurfaces. In particular it can be used to proliferate hypersurface conformal invariants and conformally invariant operators [55, 32]. We will apply this calculus to the study of conformal infinities in the final two lectures.

7. LECTURE 7: GEOMETRY OF CONFORMAL INFINITY

Here we apply the results of the previous lecture on conformal hypersurfaces to the geometry of conformal infinity.

7.1. Geometry of conformal infinity and its embedding. We return now to the study of conformally compact geometries. We will consider in particular those which near the conformal infinity are asymptotically of constant nonzero scalar curvature. By imposing a constant dilation we may assume that I^2 approaches ± 1 .

We begin by observing that the normal tractor is linked, in an essential way, to the ambient geometry off the hypersurface $\Sigma := \mathcal{Z}(\sigma)$.

Proposition 7.1. *Let (M^d, \mathbf{c}, I) be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$ and $I^2 = \pm 1 + \sigma^2 f$ for some smooth (weight -2) density f . Then by Theorem 5.6 Σ is a smoothly embedded hypersurface and, with N denoting the normal tractor for Σ , we have $N = I|_{\Sigma}$.*

Proof. For simplicity let us first assume $I^2 = \pm 1$ (so $f = 0$ and the structure is ASC). As usual let us write $\sigma := h(X, I)$. By definition

$$I_A = \frac{1}{d} D_A \sigma \stackrel{g}{=} \begin{pmatrix} \sigma \\ \nabla_a \sigma \\ -\frac{1}{d} (\Delta \sigma + J\sigma) \end{pmatrix},$$

where $g \in \mathfrak{c}$ and ∇ denotes its Levi-Civita connection. Let us write $n_a := \nabla_a \sigma$. Along Σ we have $\sigma = 0$, therefore

$$I|_{\Sigma} \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ -\frac{1}{d}\Delta\sigma \end{pmatrix},$$

along Σ . Clearly then $|n|_g^2 = \pm 1$, along Σ , since $I^2 = \pm 1$. So $n_a|_{\Sigma}$ is a conformal weight 1 conormal field for Σ .

Next we calculate the mean curvature $H = H^g$ in terms of σ . Recall $(d-1)H = \nabla^a n_a \mp n^a n^b \nabla_b n_a$, on Σ . We calculate the right hand side in a neighbourhood of Σ . Since $n_a = \nabla_a \sigma$, we have $\nabla^a n_a = \Delta\sigma$. On the other hand

$$n^a n^b \nabla_b n_a = \frac{1}{2} n^b \nabla_b (n^a n_a) = \frac{1}{2} n^b \nabla_b (\pm 1 + \frac{2}{d} \sigma \Delta\sigma + \frac{2}{d} J\sigma^2),$$

where we used that $|\frac{1}{d}D\sigma|^2 = \pm 1$ and so $n^a n_a = \pm 1 + \frac{2}{d} \sigma \Delta\sigma + \frac{2}{d} J\sigma^2$. Now along Σ we have $\pm 1 = n^a n_a = n^a \nabla_a \sigma$, and so there this simplifies to

$$n^a n^b \nabla_b n_a = \pm \frac{1}{d} \Delta\sigma.$$

Putting these results together, we have

$$(d-1)H = \frac{1}{d}(d-1)\Delta\sigma \quad \Rightarrow \quad H = \frac{1}{d}\Delta\sigma \quad \text{along } \Sigma.$$

Thus

$$I|_{\Sigma} \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ -H \end{pmatrix},$$

as claimed. Now note that if we repeat the calculation with $I^2 = \pm 1 + \sigma^2 f$ then the result still holds, as in the calculation this relation was differentiated just once. \square

Corollary 7.2. *Let (M^d, \mathfrak{c}, I) be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$, and that is asymptotically Einstein in the sense that $I^2|_{\Sigma} = \pm 1$, and $\nabla_a I_B = \sigma f_{aB}$ for some smooth (weight -1) tractor valued 1-form f_{aB} . Then Σ is a totally umbilic hypersurface.*

Proof. The assumptions on the scale tractor I imply that $I^2 = \pm 1 + \sigma^2 f$ for some smooth function f (since we assume \mathfrak{c} and σ smooth). Thus it follows from Proposition 7.1 above that, along the singularity hypersurface, I agrees with the normal tractor N . Thus N is parallel along Σ and so, from Proposition 6.3, Σ is totally umbilic. \square

Note that a hypersurface is totally umbilic if and only if is conformally totally geodesic: if Σ is totally umbilic then locally it is straightforward to find a metric $g \in \mathfrak{c}$ so that $H^g = 0$ whence $L_{ab}^g = 0$ (this was demonstrated in Remark 6.5). In this scale any geodesic on the submanifold Σ , with its induced metric \bar{g} , is also a geodesic of the ambient (M, g) . So the condition of being totally umbilic is a strong matching of the conformal structures.

In fact for a conformally compact metric that is asymptotically Einstein, as in Corollary 7.2 above, there is an even stronger compatibility (involving a higher order of contact) between the geometry of \mathbf{c} and $\bar{\mathbf{c}}$. First a preliminary result.

Proposition 7.3. *Let $(M^{d \geq 4}, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$, and that is asymptotically Einstein in the sense that $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma^2 f_{aB}$ for some smooth (weight -2) tractor valued 1-form field f_{aB} . Then the Weyl curvature $W_{ab}{}^c{}_d$ satisfies*

$$W_{ab}{}^c{}_d n^d = 0, \quad \text{along } \Sigma,$$

where n^d is the normal field.

Proof. Since, along Σ , I_B is parallel to the given order we have that the tractor curvature satisfies

$$\kappa_{ab}{}^C{}_D I^D = \kappa_{ab}{}^C{}_D N^D = 0 \quad \text{along } \Sigma,$$

and from the formulae for κ and N the result is immediate. \square

Theorem 7.4. *Let $(M^{d \geq 4}, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$, and that is asymptotically Einstein in the sense that $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma^2 f_{aB}$ for some smooth (weight -2) tractor valued 1-form field f_{aB} . Then the tractor connection of (M, \mathbf{c}) preserves the intrinsic tractor bundle of Σ , where the latter is viewed as a subbundle of the ambient tractors: $\mathcal{T}_\Sigma \subset \mathcal{T}$. Furthermore the restriction of the parallel transport of $\nabla^\mathcal{T}$ coincides with the intrinsic tractor parallel transport of $\nabla^{\mathcal{T}_\Sigma}$.*

Proof. From Theorem 6.4 the tractor bundle $\bar{\mathcal{T}}$ of (Σ, \mathbf{c}) may be identified with the subbundle N^\perp in $\mathcal{T}|_\Sigma$, consisting of standard tractors orthogonal to the normal tractor N . Since $N = I|_\Sigma$ is parallel along Σ this subbundle is preserved by the ambient tractor connection and the projected ambient tractor connection $\check{\nabla}$ coincides with restriction to $\bar{\mathcal{T}}$ of the pullback of the ambient tractor connection:

$$\check{\nabla}_a = \Pi_a^b \nabla_b^\mathcal{T} \quad \text{on } \bar{\mathcal{T}} \subset \mathcal{T}.$$

Thus the result follows from the tractor Gauss formula if the Fialkow tensor

$$\mathcal{F}_{ab} = \frac{1}{n-2} \left(W_{abcd} n^c n^d + \mathring{L}_{ab}^2 - \frac{|\mathring{L}|^2}{2(n-1)} \bar{\mathbf{g}}_{ab} \right),$$

of (53) vanishes. Now from Corollary 7.2 Σ is totally umbilic, and so $\mathring{L}_{ab} = 0$, while Proposition 7.3 states that $W_{abcd} n^d = 0$ along Σ . Thus $\mathcal{F}_{ab} = 0$. \square

Remark 7.5. In the case of an almost Einstein manifold the Theorem 7.4 can also be seen to follow directly from the general theory of [11], see Theorem 3.5. \blacksquare

8. LECTURE 8: BOUNDARY CALCULUS AND ASYMPTOTIC ANALYSIS

In the previous lectures we have seen that for conformally compact manifolds with a nondegenerate conformal infinity there are nice tools for studying the link between the conformal structure on the conformal infinity and its relation to the ambient pseudo-Riemannian structure. These showed for example that conditions

like asymptotically ASC and then asymptotically Einstein lead to a higher order of contact between the ambient and boundary conformal structures.

Here we show that the same tools, which are canonical to the geometry of almost-pseudo Riemannian manifolds, lead first to an interesting canonical calculus along the boundary and then to canonical boundary problems that can be partly solved by these tools. Further details of the boundary calculus for conformally compactified manifolds presented in this lecture, and on applications, can be found in [28, 33, 34, 35].

8.1. The canonical degenerate Laplacian. On an almost pseudo-Riemannian manifold (M, \mathbf{c}, I) there is a canonical degenerate Laplacian type differential operator, namely

$$I \cdot D := I^A D_A.$$

This acts on any weighted tractor bundle, preserving its tensor type but lowering the weight:

$$I \cdot D : \mathcal{E}^\Phi[w] \rightarrow \mathcal{E}^\Phi[w - 1].$$

In the following it will be useful to define the *weight operator* \mathbf{w} . On sections of a conformal density bundle this is just the linear operator that returns the weight. So if $\tau \in \Gamma(\mathcal{E}[w_0])$ then

$$\mathbf{w} \tau = w_0 \tau.$$

Then this is extended in obvious way to weighted tensor or tractor bundles. So if \mathcal{B} is some vector bundle of conformal weight zero then \mathbf{w} acts as the zero operator on its sections and then if $\beta \in \Gamma(\mathcal{B}[w_0])$ we have

$$\mathbf{w} \beta = w_0 \beta.$$

Now expanding $I \cdot D$ in terms of some background metric $g \in \mathbf{c}$, we have

$$I \cdot D \stackrel{g}{=} \begin{pmatrix} -\frac{1}{d}(\Delta\sigma + \mathbf{J}\sigma) & \nabla^a \sigma & \sigma \end{pmatrix} \begin{pmatrix} \mathbf{w}(d + 2\mathbf{w} - 2) \\ \nabla_a(d + 2\mathbf{w} - 2) \\ -(\Delta + \mathbf{J}\mathbf{w}) \end{pmatrix}.$$

As an operator on any density or weighted tractor bundle of weight w each occurrence of \mathbf{w} evaluates to w . So then

$$(56) \quad I \cdot D \stackrel{g}{=} -\sigma\Delta + (d + 2w - 2)[(\nabla^a \sigma)\nabla_a - \frac{w}{d}(\Delta\sigma)] - \frac{2w}{d}(d + w - 1)\sigma\mathbf{J}$$

on $\mathcal{E}^\Phi[w]$. Now if we calculate in the metric $g_+ = \sigma^{-2}\mathbf{g}$, away from the zero locus of σ , and trivialisise the densities accordingly, then σ is represented by 1 in the trivialisisation, and we have

$$I \cdot D \stackrel{g_\pm}{=} -\left(\Delta^{g_+} + \frac{2w(d + w - 1)}{d}\mathbf{J}^{g_+}\right).$$

In particular if g_+ satisfies $\mathbf{J}^{g_+} = \mp \frac{d}{2}$ (i.e. $\text{Sc}^{g_+} = \mp d(d-1)$) or equivalently $I^2 = \pm 1$) then, relabeling $d + w - 1 =: s$ and $d - 1 =: n$, we have

$$(57) \quad I \cdot D \stackrel{g_\pm}{=} -(\Delta^{g_+} \pm s(n - s)).$$

On the other hand, looking again to (56), we see that $I \cdot D$ degenerates along the conformal infinity $\Sigma = \mathcal{Z}(\sigma)$ (assumed non-empty), and there the operator is first order. In particular if the structure is asymptotically ASC in the sense that $I^2 = \pm 1 + \sigma f$, for some smooth (weight -1) density f , then along Σ

$$I \cdot D = (d + 2w - 2)\delta_n ,$$

on $\mathcal{E}^\Phi[w]$, where δ_n is the conformal Robin operator,

$$\delta_n \stackrel{g}{=} n^a \nabla_a^g - w H^g ,$$

of [13, 5] (twisted with the tractor connection); here n^a is a length ± 1 normal and H^g the mean curvature, as measured in the metric g .

8.2. A boundary calculus for the degenerate Laplacian. Let (M, \mathbf{c}) be a conformal structure of dimension $d \geq 3$ and of any signature. Given σ a section of $\mathcal{E}[1]$, write I_A for the corresponding scale tractor as usual. That is $I_A = \frac{1}{d} D_A \sigma$. Then $\sigma = X^A I_A$.

8.2.1. The $\mathfrak{sl}(2)$ -algebra. Suppose that $f \in \mathcal{E}^\Phi[w]$, where \mathcal{E}^Φ denotes any tractor bundle. Select $g \in \mathbf{c}$ for the purpose of calculation, and write $I_A \stackrel{g}{=} (\sigma, \nu_a, \rho)$ to simplify the notation. Then using $\nu_a = \nabla_a \sigma$, we have

$$\begin{aligned} I \cdot D(\sigma f) &= (d + 2w)((w + 1)\sigma \rho f + \sigma \nu_a \nabla^a f + f \nu_a \nu^a) \\ &\quad - \sigma(\sigma \Delta f + 2\nu_a \nabla^a f + f \Delta \sigma + (w + 1)\mathcal{J}\sigma f) , \end{aligned}$$

while

$$-\sigma I \cdot D f = -\sigma(d + 2w - 2)(w \rho f + \nu_a \nabla^a f) + \sigma^2(\Delta f + w \mathcal{J}f) .$$

So, by virtue of the fact that $\rho = -\frac{1}{d}(\Delta \sigma + \mathcal{J}\sigma)$, we have

$$[I \cdot D, \sigma]f = (d + 2w)(2\sigma \rho + \nu_a \nu^a)f .$$

Now $I^A I_A = I^2 \stackrel{g}{=} 2\sigma \rho + \nu_a \nu^a$, whence the last display simplifies to

$$[I \cdot D, \sigma]f = (d + 2w)I^2 f .$$

Denoting by \mathbf{w} the weight operator on tractors, we have the following.

Lemma 8.1. *Acting on any section of a weighted tractor bundle we have*

$$[I \cdot D, \sigma] = I^2(d + 2\mathbf{w}) ,$$

where \mathbf{w} is the weight operator.

Remark 8.2. A similar computation to above shows that, more generally,

$$I \cdot D(\sigma^\alpha f) - \sigma^\alpha I \cdot D f = \sigma^{\alpha-1} \alpha I^2(d + 2\mathbf{w} + \alpha - 1)f ,$$

for any constant α . ■

The operator $I \cdot D$ lowers conformal weight by 1. On the other hand, as an operator (by tensor product) σ raises conformal weight by 1. We can record this by the commutator relations

$$[\mathbf{w}, I \cdot D] = -I \cdot D \quad \text{and} \quad [\mathbf{w}, \sigma] = \sigma,$$

so with the Lemma we see that the operators σ , $I \cdot D$, and \mathbf{w} , acting on weighted scalar or tractor fields, generate an $\mathfrak{sl}(2)$ Lie algebra, provided I^2 is nowhere vanishing. It is convenient to fix a normalisation of the generators; we record this and our observations as follows.

Proposition 8.3. *Suppose that (M, c, σ) is such that I^2 is nowhere vanishing. Setting $x := \sigma$, $y := -\frac{1}{I^2} I \cdot D$, and $h := d + 2\mathbf{w}$ we obtain the commutation relations*

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h,$$

of standard $\mathfrak{sl}(2)$ -algebra generators.

In the case of $I^2 = 0$ the result is an Inönü-Wigner contraction of the $\mathfrak{sl}(2)$ -algebra:

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = 0,$$

where h and x are as before, but now $y = -I \cdot D$.

Subsequently \mathfrak{g} will be used to denote this ($\mathfrak{sl}(2)$) Lie algebra of operators. From Proposition 8.3 (and in concordance with remark 8.2) follow some useful identities in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Corollary 8.4.

$$[x^k, y] = x^{k-1}k(d + 2\mathbf{w} + k - 1) = x^{k-1}k(h + k - 1)$$

(58)

and

$$[x, y^k] = y^{k-1}k(d + 2\mathbf{w} - k + 1) = y^{k-1}k(h - k + 1).$$

8.3. Tangential operators and holographic formulæ. Suppose that $\sigma \in \Gamma(\mathcal{E}[1])$ is such that $I_A = \frac{1}{d} D_A \sigma$ satisfies that $I^A I_A = I^2$ is nowhere zero. As explained in Section 5.3, the zero locus $\mathcal{Z}(\sigma)$ of σ is then either empty or forms a smooth hypersurface.

Conversely if Σ is any smooth oriented hypersurface then, at least in a neighbourhood of Σ , there is a smooth defining function s . Now take $\sigma \in \Gamma(\mathcal{E}[1])$ to be the unique density which gives s in the trivialisation of $\mathcal{E}[1]$ determined by some $g \in \mathfrak{c}$. It follows then that $\Sigma = \mathcal{Z}(\sigma)$ and $\nabla^g \sigma$ is non-zero at all points of Σ . If $\nabla^g \sigma$ is nowhere null along Σ then Σ is nondegenerate and I^2 is nowhere vanishing in a neighbourhood of Σ , and we are in the situation of the previous paragraph. We call such a σ a *defining density* for Σ (recall our definition from Lecture 6), and to simplify the discussion we shall take M to agree with this neighbourhood of Σ . Until further notice σ will mean such a section of $\mathcal{E}[1]$ with $\Sigma = \mathcal{Z}(\sigma)$ non-empty and nondegenerate. Note that Σ has a conformal structure \bar{c} induced in the obvious way from (M, \mathfrak{c}) and is a conformal infinity for the metric $g_+ := \sigma^{-2} \mathfrak{g}$ on $M \setminus \Sigma$.

8.3.1. *Tangential operators.* Suppose that σ is a defining density for a hypersurface Σ . Let $P : \mathcal{E}^\Phi[w_1] \rightarrow \mathcal{E}^\Phi[w_2]$ be some linear differential operator defined in a neighbourhood of Σ . We shall say that P acts tangentially (along Σ) if $P \circ \sigma = \sigma \circ \tilde{P}$, where $\tilde{P} : \mathcal{E}^\Phi[w_1 - 1] \rightarrow \mathcal{E}^\Phi[w_2 - 1]$ is some other linear operator on the same neighbourhood. The point is that for a tangential operator P we have

$$P(f + \sigma h) = Pf + \sigma \tilde{P}h.$$

Thus along Σ the operator P is insensitive to how f is extended off Σ . It is easily seen that if P is a tangential differential operator then there is a formula for P , along Σ , involving only derivatives tangential to Σ , and the converse also holds.

Using Corollary 8.4 we see at once that certain powers of $I \cdot D$ act tangentially on appropriately weighted tractor bundles. We state this precisely. Suppose that Σ is a (nondegenerate) hypersurface in a conformal manifold (M^{n+1}, \mathbf{c}) , and σ a defining density for Σ . Then recall $\Sigma = \mathcal{Z}(\sigma)$ and I^2 is nowhere zero in a neighbourhood of Σ , where $I_A := \frac{1}{n+1} D_A \sigma$ is the scale tractor. The following holds.

Theorem 8.5. *Let \mathcal{E}^Φ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along Σ*

$$(59) \quad P_k : \mathcal{E}^\Phi\left[\frac{k-n}{2}\right] \rightarrow \mathcal{E}^\Phi\left[\frac{-k-n}{2}\right] \quad \text{given by} \quad P_k := \left(-\frac{1}{I^2} I \cdot D\right)^k$$

is a tangential differential operator, and so determines a canonical differential operator $P_k : \mathcal{E}^\Phi\left[\frac{k-n}{2}\right]|_\Sigma \rightarrow \mathcal{E}^\Phi\left[\frac{-k-n}{2}\right]|_\Sigma$.

Proof. The P_k are differential by construction. Thus the result is immediate from Corollary 8.4 with $\tilde{P} = \left(-\frac{1}{I^2} I \cdot D\right)^k$. \square

8.4. The extension problems and their asymptotics. Henceforth we consider an almost pseudo-Riemannian structure (M, c, σ) with σ a defining density for a hypersurface Σ and I^2 nowhere zero. We consider the problem of solving, off Σ asymptotically,

$$I \cdot Df = 0,$$

for $f \in \Gamma(\mathcal{E}^\Phi[w_0])$ and some given weight w_0 . For simplicity we henceforth calculate on the side of Σ where σ is non-negative, so effectively this amounts to working locally along the boundary of a conformally compact manifold.

8.4.1. *Solutions of the first kind.* Here we treat an obvious Dirichlet-like problem where we view $f|_\Sigma$ as the initial data. Suppose that f_0 is an arbitrary smooth extension of $f|_\Sigma$ to a section of $\mathcal{E}^\Phi[w_0]$ over M . We seek to solve the following problem:

Problem 8.6. Given $f|_\Sigma$, and an arbitrary extension f_0 of this to $\mathcal{E}^\Phi[w_0]$ over M , find $f_i \in \mathcal{E}^\Phi[w_0 - i]$ (over M), $i = 1, 2, \dots$, so that

$$f^{(\ell)} := f_0 + \sigma f_1 + \sigma^2 f_2 + \dots + O(\sigma^{\ell+1})$$

solves $I \cdot Df = O(\sigma^\ell)$, off Σ , for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

Remark 8.7. For $i \geq 1$ we do not assume that the f_i are necessarily non-vanishing along Σ . Also, note that the stipulation “given $f|_{\Sigma}$, and an arbitrary extension f_0 of this” can be rephrased as “given f_0 in the space of sections with a fixed restriction to Σ (denoted $f|_{\Sigma}$)”. \blacksquare

We write $h_0 = d + 2w_0$ so that $hf_0 = h_0f_0$, for example. The existence or not of a solution at generic weights is governed by the following result.

Lemma 8.8. *Let $f^{(\ell)}$ be a solution of Problem 8.6 to order $\ell \in \mathbb{Z}_{\geq 0}$. Then provided $\ell \neq h_0 - 2 = n + 2w_0 - 1$ there is an extension*

$$f^{(\ell+1)} = f^{(\ell)} + \sigma^{\ell+1}f_{\ell+1},$$

unique modulo $\sigma^{\ell+2}$, which solves

$$I \cdot Df^{(\ell+1)} = 0 \quad \text{modulo} \quad O(\sigma^{\ell+1}).$$

If $\ell = h_0 - 2$ then the extension is obstructed by $P_{\ell+1}f_0|_{\Sigma}$, where $P_{\ell+1} = (-\frac{1}{I^2}I \cdot Df)^{\ell+1}$ is the tangential operator on densities of weight w_0 given by Theorem 8.5.

Proof. Note that $I \cdot Df = 0$ is equivalent to $-\frac{1}{I^2}I \cdot Df = 0$ and so we can recast this as a formal problem using the Lie algebra $\mathfrak{g} = \langle x, y, h \rangle$ from Proposition 8.3. Using the notation from there

$$yf^{(\ell+1)} = yf^{(\ell)} - x^{\ell}(\ell + 1)(h + \ell)f_{\ell+1} + O(x^{\ell+1}).$$

Now $hf_{\ell+1} = (h_0 - 2(\ell + 1))f_{\ell+1}$, thus

$$(60) \quad yf^{(\ell+1)} = yf^{(\ell)} - x^{\ell}(\ell + 1)(h_0 - \ell - 2)f_{\ell+1} + O(x^{\ell+1}).$$

By assumption $yf^{(\ell)} = O(x^{\ell})$, thus if $\ell \neq h_0 - 2$ we can solve $yf^{(\ell+1)} = O(x^{\ell+1})$ and this uniquely determines $f_{\ell+1}|_{\Sigma}$.

On the other hand if $\ell = h_0 - 2$ then (60) shows that, modulo $O(x^{\ell+1})$,

$$yf^{(\ell)} = y(f^{(\ell)} + x^{\ell+1}f_{\ell+1}),$$

regardless of $f_{\ell+1}$. It follows that the map $f_0 \mapsto x^{-\ell}yf^{(\ell)}$ is tangential and $x^{-\ell}yf^{(\ell)}|_{\Sigma}$ is the obstruction to solving $yf^{(\ell+1)} = O(x^{\ell+1})$. By a simple induction this is seen to be a non-zero multiple of $y^{\ell+1}f_0|_{\Sigma}$. \square

Thus by induction we conclude the following.

Proposition 8.9. *For $h_0 \notin \mathbb{Z}_{\geq 2}$ Problem (8.6) can be solved to order $\ell = \infty$. For $h_0 \in \mathbb{Z}_{\geq 2}$ the solution is obstructed by $[P_{h_0-1}f]|_{\Sigma}$; if, for a particular f , $[P_{h_0-1}f]|_{\Sigma} = 0$ then there is a family of solutions to order $\ell = \infty$ parametrised by sections $f_{h_0-1} \in \Gamma\mathcal{E}^{\Phi}[-d - w_0 + 1]|_{\Sigma}$.*

APPENDIX A. CONFORMAL KILLING VECTOR FIELDS AND ADJOINT TRACTORS

Here we discuss conformal Killing vector fields and the prolongation of the conformal Killing equation which is given by a connection on a conformal tractor bundle called the adjoint tractor bundle. Although the adjoint bundle is not a new tractor bundle *per se* (being the bundle of skew endomorphisms of \mathcal{E}^A , naturally isomorphic to $\mathcal{E}_{[AB]}$), it is a very important tractor bundle and is worthy of our further attention; we discuss briefly the connection of the conformal Cartan bundle with the adjoint tractor bundle and its calculus. We finish this section by applying the adjoint tractor calculus and the prolongation of the conformal Killing equation to Lie derivatives of tractors with respect to conformal Killing vector fields.

A.1. The conformal Cartan bundle and the adjoint tractor bundle. In the lectures we have been discussing tractor calculus on the so called standard tractor bundle on a conformal manifold (M, c) , which can be seen as the associated bundle $\mathcal{G} \times_P \mathbb{R}^{p+1, q+1}$ to the P -principal conformal Cartan bundle $\mathcal{G} \rightarrow M$. We have not discussed the construction of the conformal Cartan bundle in the lectures because it is easier to construct the standard tractor bundle directly from the underlying conformal structure. Indeed, having understood the conformal standard tractor bundle one could simply define the conformal Cartan bundle as an adapted frame bundle for the standard tractor bundle \mathcal{T} ; from this point of view the conformal Cartan connection simply encodes the tractor parallel transport of the frames, cf. (40). Although for calculational purposes the standard tractor approach presented in the lectures is optimal, we have seen (e.g., in section 5.3.1) that the Cartan geometric point of view can afford additional insights. This is also the case when it comes to discussing the adjoint tractor bundle.

If $P \subset G = O(p, q)$ is the stabiliser of a null ray in $\mathbb{R}^{p+1, q+1}$ then P contains a subgroup which may be identified with $CO(p, q)$. The conformal Cartan bundle of (M, c) is then easily defined as the extension of the conformal orthogonal frame bundle $\mathcal{G}_0 \rightarrow M$ to a principal P -bundle $\mathcal{G} \rightarrow M$ corresponding to the inclusion $CO(p, q) \subset P$. The total space of this bundle is then $\mathcal{G}_0 \times_{CO(p, q)} P$.

The conformal Cartan connection ω is a 1-form on \mathcal{G} taking values in the Lie algebra of G , which can be written as a (vector space) direct sum

$$\mathbb{R}^d \oplus \mathfrak{co}(p, q) \oplus (\mathbb{R}^d)^*$$

where $d = p + q$. The \mathbb{R}^d component of ω is simply the trivial extension of the *soldering form* of \mathcal{G}_0 (which is tautologically defined on any reduction of the frame bundle of a manifold) to \mathcal{G} ; this component corresponds to the terms $-\mu_a$ and $+\mathbf{g}_{ab}\rho$ in the formula (22) for the standard tractor connection. The next component arises from noting that sections of the bundle $\mathcal{G} \rightarrow \mathcal{G}_0$ are in 1-1 correspondence with affine connections on M which preserve the conformal metric \mathbf{g} (*Weyl connections*); the $\mathfrak{co}(p, q)$ component of ω is the 1-form γ on \mathcal{G} whose pullback to \mathcal{G}_0 by any section is the connection 1-form for the corresponding Weyl connection. In order to obtain the formula for the tractor connection from that

of the Cartan connection one must first pull everything down from \mathcal{G} to \mathcal{G}_0 using a Weyl connection, allowing you to break tractors (in associated bundles to \mathcal{G}) up into weighted tensors (in associated bundles to \mathcal{G}_0). If one does this using the Levi-Civita connection ∇ of a metric g then one can see that the component γ of ω gives rise to the three terms involving ∇ in (22). The final component of ω gives rise to the two terms involving the Schouten tensor in (22) and is determined by the other two components of ω and an algebraic condition on the curvature

$$d\omega + \omega \wedge \omega$$

of ω (which can be identified with the tractor curvature). For more details see the first chapter of [12].

The conformal Cartan connection has the following three properties:

- $\omega_u : T_u\mathcal{G} \rightarrow \mathcal{G} \times \mathfrak{g}$ is a linear isomorphism for each $u \in \mathcal{G}$;
- ω is P -equivariant, i.e. $(r^p)^*\omega = Ad(p^{-1}) \circ \omega$ for all $p \in P$ (where r^p denotes the right action of p on \mathcal{G});
- ω returns the generators of fundamental vector fields, i.e.

$$\omega \left(\left. \frac{d}{dt} \right|_0 u \cdot \exp(tX) \right) = X$$

for all $u \in \mathcal{G}$ and $X \in \mathfrak{p} = \text{Lie}(P)$.

These properties more generally define what is called a *Cartan connection* of type (G, P) on a P -principal bundle \mathcal{G} . In the model case the Lie group G can be seen as a P -principal bundle over the model space G/P and the Maurer-Cartan 1-form ω_{MC} , which evaluates left invariant vector fields at the identity, is a Cartan connection which has vanishing curvature by the Maurer-Cartan structure equations

$$d\omega_{MC} + \omega_{MC} \wedge \omega_{MC} = 0.$$

The *adjoint tractor bundle* $\mathcal{A} \rightarrow M$ of a conformal manifold (M, \mathbf{c}) is the associated bundle to the conformal Cartan bundle \mathcal{G} corresponding to the adjoint representation of G (restricted to P), i.e. $\mathcal{A} = \mathcal{G} \times_P \mathfrak{g}$. Since $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$ can be identified with the skew-symmetric endomorphisms of $\mathbb{R}^{p+1, q+1}$ the adjoint tractor bundle can similarly be identified with the bundle of skew-symmetric endomorphisms of the standard tractor bundle \mathcal{T} (with respect to the tractor metric). Clearly then we may identify \mathcal{A} with $\mathcal{E}_{[AB]}$ by lowering a tractor index. An adjoint tractor \mathbb{L}^A_B can be written in terms of the direct sum decomposition of \mathcal{T} (and hence \mathcal{A}) as

$$(61) \quad \begin{pmatrix} -\nu & -l_b & 0 \\ -\rho^a & \mu^a_b & l^a \\ 0 & \rho_b & \nu \end{pmatrix}$$

where $\mu_{ab} = \mu_{[ab]}$ and the matrix acts on standard tractors from the left, as the tractor curvature does in (32) (the tractor curvature is in fact an adjoint tractor valued 2-form). It is easy to see that the two appearances of l^a make up the ‘‘top slot’’ of \mathbb{L}^A_B , so that there is an invariant projection Π from \mathcal{A} to TM that takes \mathbb{L}^A_B to l^a .

By writing an adjoint tractor \mathbb{L}^A_B in terms of the splitting tractors (X^A, Z^A, Y^A) corresponding to a choice of metric g one can easily obtain the formula for the tractor connection acting on \mathbb{L}^A_B using (40). If \mathbb{L}^A_B is given in matrix form (w.r.t. g) by (61) then we have

$$(62) \quad \nabla_a \mathbb{L}^B_C \stackrel{g}{=} \begin{pmatrix} * & * & 0 \\ * & \mathbf{g}^{bb'} (\nabla_a \mu_{b'c} - 2\mathbf{P}_{a[b'l'c]} + 2\mathbf{g}_{a[b'l'c]}) & \nabla_a l^b + \mu_a^b + \nu \delta_a^b \\ 0 & \nabla_a \rho_c - \mathbf{P}_{ab}^b \mu_{bc} + \nu \mathbf{P}_{ac} & \nabla_a \nu - \mathbf{P}_{ab}^b l_b - \rho_a \end{pmatrix}$$

where the entries marked with a $*$ are determined by skew-symmetry.

A.2. Prolonging the conformal Killing equation. Note that if \mathbb{L} is a parallel adjoint tractor, and $l = \Pi(\mathbb{L})$, then from the above display we must have

$$\nabla_a l_b + \mu_{ab} + \nu g_{ab} = 0$$

where μ_{ab} is skew. This implies that $\nabla_{(a} l_{b)_o} = 0$, in other words l^a is a *conformal Killing vector field*. A conformal Killing vector field k^a is a nonzero solution of the conformally invariant equation

$$\nabla_{(a} k_{b)_o} = 0,$$

where $k_b = g_{bc} k^c$. Geometrically this equation says that the local flow of k preserves any metric $g \in \mathbf{c}$ up to a conformal factor, equivalently, the Lie derivative of any metric $g \in \mathbf{c}$ with respect to k is proportional to g . It is natural to ask whether there is a 1-1 correspondence between conformal Killing vector fields and (nonzero) parallel adjoint tractor fields – the answer to this question turns out to be no, except for on the flat model (where one can use this correspondence to easily write the $d(d-1)$ -dimensional space of Killing vector fields explicitly).

One can however construct a different conformally invariant connection on the adjoint tractor bundle which does prolong the conformal Killing equation, i.e. for which parallel sections are in 1-1 correspondence with solutions. One can obtain this system directly by (fixing g and $\nabla = \nabla^g$ and then) writing the equation $\nabla_{(a} k_{b)_o} = 0$ as

$$\nabla_a k_b = \mu_{ab} + \nu g_{ab},$$

introducing the new variables $\mu_{ab} \in \Gamma(\mathcal{E}_{[ab]}[2])$ and $\nu \in \Gamma(\mathcal{E})$. Beyond this the key step is to introduce the fourth variable

$$\rho_a = \nabla_a \nu + \mathbf{P}_{ab} k^b \in \Gamma(\mathcal{E}_a)$$

(rather than $\rho_a = \nabla_a \nu$), cf. (62). The remainder of the prolongation process simply involves taking covariant derivatives of the above two displays and then skew-symmetrising over certain pairs of indices in them in order to bring out curvature terms (as well as using Bianchi identities to simplify expressions); from these “differential consequences” of the above two displays one may derive expressions for $\nabla_a \mu_{bc}$ and $\nabla_a \rho_b$ which are linear in the other three variables (respectively). (The way to accomplish this process efficiently is to suppose you have a flat Levi-Civita connection first and go through the process to obtain both expressions, then

go back through the same steps and take into account the non-vanishing curvature for the general case.) The result is the following system of differential equations

$$\begin{aligned}\tilde{\nabla}_a k_b &= \nu g_{ab} + \mu_{ab} \\ \tilde{\nabla}_a \mu_{bc} &= -2P_{a[b} k_{c]} - 2g_{a[b} \rho_{c]} + W_{dabc} k^d \\ \tilde{\nabla}_a \nu &= \rho_a - P_{ab} k^b \\ \tilde{\nabla}_a \rho_b &= -P_{ab}^c \mu_{bc} - P_{ab} \nu - C_{cab} k^c\end{aligned}$$

where $C_{abc} = 2\nabla_{[a} P_{bc]}$ is the Cotton tensor.

From the above system we can see that if we define the connection $\tilde{\nabla}$ on \mathcal{A} by

$$(63) \quad \tilde{\nabla} \mathbb{L} = \nabla \mathbb{L} + i_{\Pi(\mathbb{L})} \kappa \quad \text{for all } \mathbb{L} \in \Gamma(\mathcal{A})$$

then there is a one to one correspondence between (nonzero) $\tilde{\nabla}$ -parallel sections of \mathcal{A} and conformal Killing vector fields on (M, \mathbf{c}) . To check this we simply calculate (in a scale g) that

$$\begin{aligned}\tilde{\nabla}_a \begin{pmatrix} * & * & 0 \\ * & \mu^b_c & l^b \\ 0 & \rho_c & \nu \end{pmatrix} &= \begin{pmatrix} * & * & 0 \\ * & g^{bb'} (\nabla_a \mu_{b'c} - 2P_{a[b'} l_{c]} + 2g_{a[b'} \rho_{c]}) & \nabla_a l^b + \mu_a^b + \nu \delta_a^b \\ 0 & \nabla_a \rho_c - P_{ab}^c \mu_{bc} + \nu P_{ac} & \nabla_a \nu - P_a^b l_b - \rho_a \end{pmatrix} \\ &+ \begin{pmatrix} * & * & 0 \\ * & W_{da}^b c^d & 0 \\ 0 & -C_{dac} l^d & 0 \end{pmatrix}\end{aligned}$$

and observe that by setting the right hand side equal to zero (and substituting $k^a = -l^a$) we recover our prolonged system for the conformal Killing equation. Note that it is possible to take a much more abstract and theoretical approach to obtaining this prolonged system and the corresponding connection $\tilde{\nabla}$ on \mathcal{A} (see, e.g., [6, 12]). From the general theory (or direct observation) we also have the invariant linear differential operator $L : TM \rightarrow \mathcal{A}$ which takes a vector field l^a on M to the adjoint tractor \mathbb{L}^A_B given in a scale g by (61) with $\mu_{ab} = -\nabla_{[a} l_{b]}$, $\nu = -\frac{1}{d} \nabla_a l^a$, and $\rho_a = \nabla_a \nu - P_{ab} l^b$. Clearly $\Pi \circ L = \text{id}_{TM}$ and consequently L is referred to as a *differential splitting operator*.

A.3. The fundamental derivative and Lie derivatives of tractors. Let \mathbb{V} be a representation of P and let $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$. If a function $\tilde{v} : \mathcal{G} \rightarrow \mathbb{V}$ satisfies

$$\tilde{v}(u \cdot p) = p^{-1} \cdot \tilde{v}(u) \quad \text{for all } u \in \mathcal{G}, p \in P$$

then we say that \tilde{v} is *P-equivariant*. Observe that if $\tilde{v} : \mathcal{G} \rightarrow \mathbb{V}$ is a smooth *P*-equivariant map then the map $v : M \rightarrow \mathcal{V}$ which takes x to $[(u, v(u))]$ for some $u \in \mathcal{G}_x$ defines a smooth section of $\mathcal{V} \rightarrow M$ (being independent of the choice of $u \in \mathcal{G}_x$ for each x). It is easy to see that sections of $\mathcal{V} \rightarrow M$ are in 1-1 correspondence with such functions. (We have used this already in discussing conformal densities where \mathbb{R}_+ replaces P and \mathcal{Q} replaces \mathcal{G} , see section 2.4.) Now observe that the conformal Cartan connection ω allows us to view smooth *P*-equivariant functions $\tilde{\mathbb{L}} : \mathcal{G} \rightarrow \mathfrak{g}$ as smooth vector fields $V_{\tilde{\mathbb{L}}}$ on \mathcal{G} (since $\omega_u : T_u \mathcal{G} \rightarrow \mathfrak{g}$ is an isomorphism

for each $u \in \mathcal{G}$ and ω is smooth). From the properties of the Cartan connection it is not hard to see that any such vector field $V_{\mathbb{L}}$ must be P -invariant, that is,

$$(r^p)^*V_{\mathbb{L}} = V_{\mathbb{L}} \quad \text{for all } p \in P$$

where $r^p : \mathcal{G} \rightarrow \mathcal{G}$ denotes the right action of $p \in P$. Moreover, the Cartan connection gives a 1-1 correspondence between P -equivariant functions $\tilde{\mathbb{L}} : \mathcal{G} \rightarrow \mathfrak{g}$ and P -invariant vector fields on \mathcal{G} . Combining this with the previous observation we see that one may naturally identify $\Gamma(\mathcal{A})$ with the space $\mathfrak{X}(\mathcal{G})^P$ of P -invariant vector fields on \mathcal{G} .

The observations of the preceding paragraph allow us to define a new canonical differential operator acting on sections of any vector bundle \mathcal{V} associated to the conformal Cartan bundle \mathcal{G} . Fix $\mathbb{L} \in \Gamma(\mathcal{A})$ and let $v \in \Gamma(\mathcal{V}) = \Gamma(\mathcal{G} \times_P \mathbb{V})$, then $V_{\mathbb{L}}\tilde{v} = d\tilde{v}(V_{\mathbb{L}})$ is again a P -equivariant function from \mathcal{G} to \mathbb{V} and thus defines a section $D_{\mathbb{L}}v$ of \mathcal{V} . Thus for each section $\mathbb{L} \in \Gamma(\mathcal{A})$ we have a first order differential operator $D_{\mathbb{L}} : \mathcal{V} \rightarrow \mathcal{V}$. It is easy to see that $D_{f\mathbb{L}} = fD_{\mathbb{L}}$ for any $f \in C^\infty(M)$ so that we really have a (first order) differential operator taking sections of \mathcal{V} to sections of $\mathcal{A}^* \otimes \mathcal{V}$. The operator

$$D : \mathcal{V} \rightarrow \mathcal{A}^* \otimes \mathcal{V}$$

defined in this way is referred to as the *fundamental derivative* (or *fundamental D operator*). This operator was introduced by Čap and Gover in [9].

Now if k is a conformal Killing vector field on (M, \mathbf{c}) and v is a section of $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$ then we may talk about the Lie derivative of v with respect to k ; since $\mathcal{V} \rightarrow M$ is a natural bundle in the category of conformal manifolds (with diffeomorphisms as maps) one may pull the section v of \mathcal{V} back by the (local) flow of k and define the Lie derivative by

$$\mathcal{L}_k v = \left. \frac{d}{dt} \right|_{t=0} (Fl_t^k)^*(v).$$

Notice that if \mathbb{L} is an section of the adjoint tractor bundle then $V_{\mathbb{L}}\tilde{v}$ (which gives the P -equivariant function corresponding to $D_{\mathbb{L}}v$) may be written as the Lie derivative $\mathcal{L}_{V_{\mathbb{L}}}\tilde{v}$, it should not come as a total surprise then that the Lie derivative and the fundamental derivative are connected. In fact, one has that

$$\mathcal{L}_k v = D_{L(k)}v$$

for any conformal Killing vector field k on (M, \mathbf{c}) and any section v of a natural bundle $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$ (this is proven in [14]).

Using the results of [9] (and carefully comparing sign conventions) we have that

$$(64) \quad D_{\mathbb{L}}\tau \stackrel{g}{=} \nabla_l \tau + w\nu\tau$$

for sections τ of the density bundle $\mathcal{E}[w]$ (which can be thought of as an associated bundle to \mathcal{G}_0 and hence to \mathcal{G}), where \mathbb{L} is given in the scale g by (61). We also have that

$$D_{\mathbb{L}}V = \nabla_{\Pi(\mathbb{L})}^T V - \mathbb{L}(V)$$

for all $\mathbb{L} \in \Gamma(\mathcal{A})$ and $V \in \Gamma(\mathcal{T})$. The operator $D_{\mathbb{L}}$ is easily seen to satisfy the Leibniz property and hence for any tractor field $T^{A\dots B}_{C\dots D}$ we have

$$\begin{aligned} D_{\mathbb{L}}T^{A\dots B}_{C\dots D} &= \nabla_{\Pi(\mathbb{L})}^T T^{A\dots B}_{C\dots D} - \mathbb{L}^A_{A'} T^{A'\dots B}_{C\dots D} - \dots - \mathbb{L}^B_{B'} T^{A\dots B'}_{C\dots D} \\ &\quad + \mathbb{L}^{C'}_C T^{A\dots B}_{C'\dots D} + \dots + \mathbb{L}^{D'}_D T^{A\dots B}_{C\dots D'}. \end{aligned}$$

From all of this we can finally write down an explicit formula for the Lie derivative of a standard tractor field V^A in terms of ‘‘slots’’: if k is a conformal Killing vector field on (M, \mathbf{c}) and $V^A \stackrel{g}{=} (\sigma, \mu^a, \rho)$ then

$$\begin{aligned} \mathcal{L}_k V^A &= k^b \nabla_b V^A + \mathbb{K}^A_B V^B \\ &\stackrel{g}{=} k^b \begin{pmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu^a + \rho \delta_b^a + \sigma P_b^a \\ \nabla_b \rho - P_{ab} \mu^a \end{pmatrix} + \begin{pmatrix} -\nu & k_b & 0 \\ -\rho^a & \mu^a_b & -k^a \\ 0 & \rho_b & \nu \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^b \\ \rho \end{pmatrix} \\ &\stackrel{g}{=} \begin{pmatrix} k^b \nabla_b \sigma - \nu \sigma \\ k^b \nabla_b \mu^a + \mu^a_b \mu^b - \sigma \nabla^a \nu \\ k^b \nabla_b \rho + \nu \rho + \mu^a \nabla_a \nu \end{pmatrix} = \begin{pmatrix} \mathcal{L}_k \sigma \\ \mathcal{L}_k \mu^a - \sigma \nabla^a \nu \\ \mathcal{L}_k \rho + \mu^a \nabla_a \nu \end{pmatrix} \end{aligned}$$

where $\mathbb{K} = L(-k)$ and we have used that $\rho_a = \nabla_a \nu + P_{ab} k^b$ and that $\mathcal{L}_k = D_{-\mathbb{K}}$ on densities so that by (64) we have $\mathcal{L}_k \sigma = \nabla_k \sigma - \nu \sigma$, $\mathcal{L}_k \rho = \nabla_k \rho + \nu \rho$, and

$$\begin{aligned} \mathcal{L}_k \mu^a &= (k^b \nabla_b \mu^a - \mu^b \nabla_b k^a) + \nu \mu^a \\ &= k^b \nabla_b \mu^a - \mu^b (\mu_b^a + \nu \delta_b^a) + \nu \mu^a \\ &= k^b \nabla_b \mu^a + \mu^a_b \mu^b \end{aligned}$$

since μ^a has conformal weight -1 .

Note that one can also, of course, calculate the expression for \mathcal{L}_k on densities and on standard tractor fields directly from the definition (which was done in [14]) by looking at what you get when you pull back densities and standard tractors by the local flow of k . As a check of our above formula for $\mathcal{L}_k V^A$ we observe that in the case where k is a Killing vector field for g then the flow of k preserves g and hence also preserves the splitting tractors X^A , Z_a^A and Y^A so that $\mathcal{L}_k X^A = 0$, $\mathcal{L}_k Z_a^A = 0$, and $\mathcal{L}_k Y^A = 0$; thus by the Leibniz property and linearity one immediately has that if $V^A = \sigma X^A + \mu^a Z_a^A + \rho Y^A$ then $\mathcal{L}_k V^A = (\mathcal{L}_k \sigma) X^A + (\mathcal{L}_k \mu^a) Z_a^A + (\mathcal{L}_k \rho) Y^A$ which is consistent with our above formula for $\mathcal{L}_k V^A$ since $\mathcal{L}_k g = 0$ forces $\nu = \frac{1}{d} \nabla_a^g k^a$ to be zero.

Remark A.1. On vector fields the fundamental derivative acts according to

$$D_{\mathbb{L}} v^b = l^a \nabla_a v^b + (\mu_a^b + \nu \delta_a^b) v^a$$

where \mathbb{L} is given in the scale g by 61 and $\nabla = \nabla^g$. Thus if k is a conformal Killing vector field then applying $\mathcal{L}_k = D_{L(k)}$ on vector fields simply returns the usual formula for the Lie derivative in terms of a torsion free (in this case Levi-Civita) connection:

$$\mathcal{L}_k v^b = k^a \nabla_a v^b - (\nabla_a k^b) v^a.$$

Similarly, using the Leibniz property of the fundamental derivative, we have that

$$D_{\mathbb{L}}T^{b\dots c}_{d\dots e} = l^a \nabla_a T^{b\dots c}_{d\dots e} - (\mu_a^b + \nu \delta_a^b) T^{a\dots c}_{d\dots e} - \dots - (\mu_a^c + \nu \delta_a^c) T^{b\dots a}_{d\dots e} \\ + (\mu_d^a + \nu \delta_d^a) T^{b\dots c}_{a\dots e} + \dots + (\mu_e^a + \nu \delta_e^a) T^{b\dots c}_{d\dots a},$$

and again applying $\mathcal{L}_k = D_{L(k)}$ for a conformal killing vector field k simply yields the standard formula for the Lie derivative. These observations further demonstrate the consistency of our claims. What's more, we may now calculate the fundamental derivative of any weighted tensor-tractor field $T^{b\dots c}_{d\dots e}{}^{B\dots C}_{D\dots E}$. For instance, by writing \mathbf{g}_{ab} as $\sigma^2 g_{ab}$ and using the Leibniz property one may easily show that

$$D_{\mathbb{L}}\mathbf{g}_{ab} = 0$$

for all $\mathbb{L} \in \Gamma(\mathcal{A})$. ■

A.3.1. Static and stationary spacetimes. As an application of the above, we observe that if $g \in \mathbf{c}$ is an Einstein metric with corresponding scale tractor I and k is a Killing vector field for g with $\mathbb{K} = L(-k)$ then $\mathbb{K}(I) = 0$. This follows from the fact $\mathcal{L}_k I = 0$ since the flow of k preserves g (and hence also I), and from the above formula for the Lie derivative of I ,

$$\mathcal{L}_k I^A = \nabla_k I^A + \mathbb{K}^A_B I^B = \mathbb{K}^A_B I^B,$$

since I is parallel. Moreover, if k is *hypersurface orthogonal* (i.e. its orthogonal distribution is integrable) then it is easy to see that $\mathbb{K}_{[AB}\mathbb{K}_{C]D}X^D = 0$ so that K_{AB} is simple and can be written as $2v_{[A}K_{B]}$ where $K_B = X^A \mathbb{K}_{AB}$ and $v_A K^A = 0$; on top of this, from $\mathbb{K}_{AB}I^B = 0$ we obtain $K_A I^A = 0$ and hence also $v_A I^A = 0$. These observations form the starting point for the development of conformal tractor calculus adapted to static and stationary spacetimes. In particular, we note that in the case where (M, g, k) is a static spacetime then $K_A = uN_A$ where N_A is the normal tractor to each of the spacelike hypersurfaces in the foliation given by $k^\perp \subset TM$ and u is the so called *static potential* (if we trivialise the density bundles using g); thus $K_A I^A = 0$ and $K_A v^A = 0$ imply that both I^A and v^A lie in the intrinsic tractor bundle of the foliating spacelike hypersurfaces and one can carry out dimensional reduction using tractors. One can in fact still carry out dimensional reduction in the stationary case by identifying tractors which are Lie dragged by k and orthogonal to $K_B = X^A \mathbb{K}_{AB}$ with conformal tractors on the manifold of integral curves of k (see [14] for further details in both cases).

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SNC & ARG: DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF AUCKLAND, PRIVATE BAG 92019, AUCKLAND 1142, NEW ZEALAND

E-mail address: `r.gover@auckland.ac.nz`

E-mail address: `sean.curry@auckland.ac.nz`