DEFORMATIONS AND EMBEDDINGS OF THREE-DIMENSIONAL STRICTLY PSEUDOCONVEX CR MANIFOLDS

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Abstract. Abstract deformations of the CR structure of a compact strictly pseudoconvex hypersurface M in $\mathbb{C}^2$ are encoded by complex functions on M. In sharp contrast with the higher dimensional case, the natural integrability condition for 3-dimensional CR structures is vacuous, and generic deformations of a compact strictly pseudoconvex hypersurface $M \subseteq \mathbb{C}^2$ are not embeddable even in $\mathbb{C}^N$ for any N. A fundamental (and difficult) problem is to characterize when a complex function on $M \subseteq \mathbb{C}^2$ gives rise to an actual deformation of M inside $\mathbb{C}^2$. In this paper we study the embeddability of families of deformations of a given embedded CR 3-manifold, and the structure of the space of embeddable CR structures on S^3. We show that the space of embeddable deformations of the standard CR 3-sphere is a Frechet submanifold of $C^\infty(S^3, \mathbb{C})$ near the origin. We establish a modified version of the Cheng-Lee slice theorem in which we are able to characterize precisely the embeddable deformations in the slice (in terms of spherical harmonics). We also introduce a canonical family of embeddable deformations and corresponding embeddings starting with any infinitesimally embeddable deformation of the unit sphere in $\mathbb{C}^2$.

1. Introduction and Main Results

A fundamental problem in CR geometry is that of characterizing embeddability of abstract CR manifolds, where a CR manifold is said to be *embeddable* if it is CR embeddable in $\mathbb{C}^N$ for some N. By the work of Boutet de Monvel and Kohn [7, 24], embeddability of compact strictly pseudoconvex CR manifolds can be characterized in terms of a closed range property of $\bar\partial_b$. In particular, when the dimension of the CR manifold is at least 5 it is always embeddable [7]. On the other hand, compact strictly pseudoconvex CR 3-manifolds are generically not embeddable [8]. The first known examples of such nonembeddable CR 3-manifolds go back to Rossi [31] who showed that certain classical SU(2)-invariant structures on S^3 are not embeddable (though, being real analytic, they are locally embeddable); a locally nonembeddable example was given by Nirenberg [28, 29]. (Nirenberg's example can be compactified to give a CR structure on S^3, and his construction already indicated that nonembeddability was generic in the compact

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case.) The question of embeddability of compact strictly pseudoconvex CR 3-manifolds has continued to receive much attention, and many authors have sought to achieve a deeper understanding of the set of embeddable structures. Epstein [16, 17] has studied the set of embeddable deformations of a given compact embeddable CR structure in terms of index theory for the corresponding (relative) Szegő projectors, and shown that the set of embeddable structures is closed in the C^∞ topology [17]. Chanillo, Chiu and Yang [10, 11] have given a sufficient condition for embeddability in terms of CR Yamabe invariants; specifically they show that a compact CR structure is embeddable if it has positive Yamabe invariant and nonnegative CR Paneitz operator. A partial converse has recently been established by Takeuchi [32] who showed that the CR Paneitz operator of an embeddable compact CR 3-manifold is always nonnegative.

In this paper we study the embeddability of families of abstract deformations of a fixed compact strictly pseudoconvex CR 3-manifold embedded in $\mathbb{C}^2$, and the structure of the space of embeddable deformations (as a subset of the space of all abstract deformations) of the standard CR 3-sphere in $\mathbb{C}^2$. By the stability theorem of Lempert [27], a small abstract deformation of a compact strictly pseudoconvex hypersurface in $\mathbb{C}^2$ is embeddable (in $\mathbb{C}^N$ for some N) if and only if it is embeddable in $\mathbb{C}^2$. We therefore restrict our attention to embeddability in $\mathbb{C}^2$. We shall mainly consider CR structures on the 3-sphere S^3 near its standard CR structure, i.e. the strictly pseudoconvex CR structure that it inherits as the boundary of the unit ball in $\mathbb{C}^2$. Recall that a strictly pseudoconvex CR structure (M, H, J) on a smooth 3-manifold M is a contact distribution $H \subseteq TM$ equipped with a bundle endomorphism $J : H \to H$ satisfying $J^2 = -\mathrm{id}$. When $M = S^3$, by a result of Eliashberg [15], a CR structure can be embedded in $\mathbb{C}^2$ only if the underlying contact structure agrees with that of the standard CR sphere. Let $\Gamma(\mathcal{J})$ denote the space of smooth positively oriented CR structures on S^3 compatible with its standard contact distribution H. Let $\Gamma(\mathcal{J})_{emb} \subset \Gamma(\mathcal{J})$ denote the subset of CR structures that are embeddable in $\mathbb{C}^2$. In [13] it is shown that $\Gamma(\mathcal{J})$ is a smooth tame Frechet manifold in the sense of Hamilton [22]. Our first main result is the smoothness of the space of embeddable CR structures near the standard CR sphere:

Theorem 1.1. *$\Gamma(\mathcal{J})_{emb} \subset \Gamma(\mathcal{J})$ is a smooth tame Frechet submanifold near the standard CR sphere.*

To understand the embeddable CR structures on S^3 more concretely, we parametrize $\Gamma(\mathcal{J})$ by complex functions on S^3 in the following way. First, note that specifying a CR structure J compatible with H is the same as specifying its $\pm i$ eigenspaces $T^{1,0}$ and $T^{0,1} = \overline{T^{1,0}}$ as subbundles of $\mathbb{C} \otimes H$. Let (z, w) denote the coordinates on $\mathbb{C}^2$ and define the following vector fields on S^3,

$$Z_1 = \bar{w}\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial w}, \qquad Z_{\bar{1}} = \overline{Z_1} \tag{1}$$

spanning $T^{1,0}$ and $T^{0,1}$ respectively for the standard CR 3-sphere (S^3, H, J_0). A complex function $\varphi = \varphi_1{}^{\bar{1}}$ on S^3 with $\|\varphi\|_\infty < 1$ defines an oriented CR structure on (S^3, H) by defining its holomorphic tangent space ${}^{\varphi}T^{1,0}$ to be spanned by

$$Z_1^{\varphi} = Z_1 + \varphi_1{}^{\bar{1}} Z_{\bar{1}}.$$

(Up to complex conjugation, all CR structures compatible with H are realized this way.) Strictly speaking, φ should be interpreted as a section of $(T^{1,0})^* \otimes T^{0,1}$ and we refer to φ as the *deformation tensor*, though we usually trivialize $(T^{1,0})^* \otimes T^{0,1}$ using Z_1 and $Z_{\bar{1}}$ in order to think of φ as a function. We let $\mathfrak{D}$ denote the space of smooth deformation tensors, and let $\mathfrak{D}_{emb} \subset \mathfrak{D}$ be the subset of deformations that are embeddable in $\mathbb{C}^2$. The main goal of this paper is to better understand the space of embeddable deformation tensors $\mathfrak{D}_{emb}$ on S^3, thought of as a space of functions using the standard frame Z_1, $Z_{\bar{1}}$.

In [8] Burns and Epstein showed that there is an infinite dimensional linear space within the space of embeddable deformation tensors $\mathfrak{D}_{emb}$ near the origin (i.e. the trivial deformation corresponding to the standard structure on S^3), characterized by the vanishing of certain terms in the spherical harmonic decomposition. To make this more precise we introduce the spherical harmonic spaces $H_{p,q}$ of functions on S^3 that are the restrictions of harmonic homogeneous polynomials of bidegree (p, q) on $\mathbb{C}^2$ for each $p, q \geq 0$. We denote the component of φ in $H_{p,q}$ by $\varphi_{p,q}$, so that the L^2 orthogonal spherical harmonic decomposition of φ is given by $\varphi = \sum_{p,q} \varphi_{p,q}$. Define $\mathfrak{D}_{BE} \subset \mathfrak{D}$ to be the set of all deformation tensors φ such that $\varphi_{p,q} = 0$ if $q < p + 4$ (our deformation tensor is the conjugate of Burns and Epstein's). Burns and Epstein showed that if $\varphi \in \mathfrak{D}_{BE}$ is sufficiently small in C^4 then the deformation is embeddable. This has a clear conceptual explanation given by Bland [6] in terms of Lempert's theory of extremal discs for the Kobayashi metric, the corresponding circular representation, and nonnegativity of the Fourier coefficients of the conjugated deformation tensor $\overline{\varphi}$ (relative to an S^1-invariant frame); cf. [2, 26, 27, 30]. Examining the linearized action of the contact diffeomorphisms on the space of CR structures on S^3 suggests that the space of Burns-Epstein deformations (or more precisely a certain subspace of the Burns-Epstein deformations satisfying an additional condition along the critical diagonal $p = q + 4$) should give a slice for the action of the group of contact diffeomorphisms on the space of embeddable CR structures. But this has not been fully resolved in the literature; in particular, such a result has not been established in the C^∞ case. One of our main results is a slice theorem for the C^∞ embeddable CR structures on S^3 near its standard CR structure, see Theorem

1.3. To do this we first prove a modified version of the Cheng-Lee slice theorem [13] for the space of abstract deformations of the standard CR structure on S^3, and then show that restricting to a natural subspace of the modified slice gives a slice theorem for the embeddable CR structures.

Before stating our slice theorems we briefly discuss the corresponding linearized problem. Given any CR hypersurface $M \subseteq \mathbb{C}^2$, the infinitesimally embeddable abstract deformations may be understood concretely as follows. Let M_t be any smooth 1-parameter family of strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$ with $M_0 = M$, defined as the zero loci of a smooth family of defining functions ρ_t. It is always possible to find a family of contact diffeomorphisms $\psi_t : M \to M_t$ with $\psi_0 = \mathrm{id}$ parametrizing the family M_t. Using ψ_t one may pull back the CR structures of the M_t to M in order to obtain a family of CR structures on M whose holomorphic tangent spaces are spanned by $Z_1^t = Z_1^0 + \varphi_1{}^{\bar{1}}(t) Z_{\bar{1}}^0$ where Z_1^0 is a (unitary) frame for the holomorphic tangent space of $M = M_0$. For purely aesthetic reasons, we lower the index $\bar{1}$ on $\varphi_1{}^{\bar{1}}(t)$ using the Levi form of ρ_t to obtain $\varphi_{11}(t)$. A straightforward geometric calculation shows that if $\dot{\varphi} = \dot{\varphi}_{11} = \frac{d}{dt}\big|_{t=0} \varphi_{11}(t)$ then

$$\dot{\varphi}_{11} = (\nabla_1 \nabla_1 + i A_{11}) f \tag{2}$$

for some function f where $\operatorname{Re} f = -\dot{\rho} = -\frac{d}{dt}\big|_{t=0} \rho_t|_M$ is the normal velocity of the deformation at $t = 0$ (see, e.g, [5, 14, 23]); here ∇ is the Tanaka-Webster connection of the contact form $i\partial\rho_0|_M$ and A_{11} is the corresponding pseudohermitian torsion. In the case of the standard CR sphere, defined by $\rho_0 = 1 - |z|^2 - |w|^2$, (2) simply becomes

$$\dot{\varphi}_{11} = Z_1 Z_1 f. \tag{3}$$

The space $\mathfrak{D}_0$ of infinitesimally embeddable deformation tensors on S^3 is easily understood using spherical harmonics. The vector field Z_1 sends each $H_{p,q}$ isomorphically onto $H_{p-1,q+1}$ unless $p = 0$ in which case Z_1 is zero. It follows that $\dot{\varphi}$ is an embeddable infinitesimal deformation (i.e. $\dot{\varphi}$ is in the range of $Z_1 Z_1$) if and only if $\dot{\varphi}_{p,q} = 0$ for $q = 0, 1$.

Let $\mathcal{C}$ denote the space of contact diffeomorphisms on S^3. The Lie algebra of $\mathcal{C}$ is the space of contact (Hamiltonian) vector fields, which can be identified with $C^\infty(S^3, \mathbb{R})$ once a contact form on S^3 has been chosen (we always take the standard contact form $\theta = i(z d\bar{z} + w d\bar{w})$ on S^3 which normalizes the Levi form to be $h_{1\bar{1}} = 1$ in the frame Z_1). The linearization of the natural action $\mathcal{C} \times \Gamma(\mathcal{J}) \to \Gamma(\mathcal{J})$ at (id, J_0) is

$$(g, \dot{\varphi}_{11}) \mapsto \dot{\varphi}_{11} + i Z_1 Z_1 g, \tag{4}$$

where $g \in C^\infty$ is the potential for a contact (Hamiltonian) vector field and $\dot{\varphi}$ is a deformation tensor on S^3 (here we are identifying $\mathfrak{D}$ with the tangent space of $\Gamma(\mathcal{J})$ at

J_0). As an immediate consequence of (4), it was observed in Burns-Epstein [8] that an infinitesimal slice for the action of the contact diffeomorphisms on CR structures at J_0 is given by $\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp$ where $\mathfrak{D}_0^\perp \subseteq \mathfrak{D}$ is the L^2 orthogonal complement to $\mathfrak{D}_0$ and $\mathfrak{D}'_{BE} \subseteq \mathfrak{D}_{BE}$ is the subspace of all $\varphi \in \mathfrak{D}_{BE}$ that additionally satisfy the reality condition $\mathrm{Im}\,((Z_{\bar{1}})^2 \varphi_{p,p+4}) = 0$ along the critical diagonal. (The latter reality condition is equivalent to saying that φ must be L^2 orthogonal to the image of real S^1-invariant functions on S^3 under $i(Z_1)^2$, where the inner product is the real part of the complex inner product.)

In Cheng-Lee [13] it was shown that the space of marked CR structures on S^3 near the standard CR structure can be locally identified with $\mathcal{C} \times \mathcal{S}$ where $\mathcal{S}$ is the set of all deformation tensors φ such that $\mathrm{Im}\,(Z_{\bar{1}} Z_{\bar{1}} \varphi) = 0$. Marking here refers to the choice of a point in the CR Cartan frame bundle of the given CR structure on (S^3, H); the symmetry group of any marked CR structure is trivial, so working with marked structures eliminates the need to try to mod out by the noncompact symmetry group of the standard CR sphere. For our purposes, we need a modified version of the Cheng-Lee slice theorem which uses the linearly equivalent slice $\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp$. Let $\Gamma(\mathcal{J})^m$ denote the space of marked CR structures on (S^3, H), which we identify with the space $\mathfrak{D}^m$ of marked deformations of (S^3, H, J_0). The contact diffeomorphisms act naturally on $\Gamma(\mathcal{J})^m$ (see, e.g., [13]) and hence on $\mathfrak{D}^m$ by identification with $\Gamma(\mathcal{J})^m$.

Theorem 1.2. *Fix any marking y_0 of the standard CR sphere. Then*

(i) *The natural action $\mathcal{C} \times \mathfrak{D}^m \to \mathfrak{D}^m$ restricts to a local smooth tame diffeomorphism $P : \mathcal{C} \times (\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp) \times \{y_0\} \to \mathfrak{D}^m$ in a neighborhood of $(0, y_0) \in \mathfrak{D}^m$;*

(ii) *For $\Psi \in \mathcal{C}$ sufficiently near the identity, the image of $(\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp) \times \{y_0\}$ under Ψ is disjoint from itself unless $\Psi = \mathrm{Id}$.*

The proof of this modified Cheng-Lee slice theorem can be obtained by adapting the proof of Theorem B in [13]. For the reader's convenience we provide a slightly simplified proof of this theorem in Section 5. The advantage of this modified slice theorem is that a linear subspace of the slice gives a slice for the embeddable deformations. Let $\mathfrak{D}^m_{emb}$ denote the space of marked embeddable deformations of the standard CR sphere. We shall prove the following slice theorem for the set of embeddable deformations, which also immediately implies Theorem 1.1.

Theorem 1.3. *Fix any marking y_0 of the standard CR sphere. Then*

(i) *The natural action $\mathcal{C} \times \mathfrak{D}^m \to \mathfrak{D}^m$ restricts to a local smooth tame immersion $P_{emb} : \mathcal{C} \times \mathfrak{D}'_{BE} \times \{y_0\} \to \mathfrak{D}^m$ in a neighborhood of $(\mathrm{id}, 0) \in \mathcal{C} \times \mathfrak{D}'_{BE}$ whose image is a neighborhood of $(0, y_0)$ in $\mathfrak{D}^m_{emb}$;*

(ii) *For $\Psi \in \mathcal{C}$ sufficiently near the identity, the image of $\mathfrak{D}'_{BE} \times \{y_0\}$ under Ψ is disjoint from itself unless $\Psi = \mathrm{Id}$.*

We observe that by Theorem 1.2, Theorem 1.3 is equivalent to the statement that $(\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp) \setminus \mathfrak{D}'_{BE}$ consists solely of nonembeddable deformations (near the origin). This question was considered in [8] where it was shown that the nonembeddable deformations form a G_δ-set in $(\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp) \setminus \mathfrak{D}'_{BE}$; the results in [16] imply that this G_δ-set is open. Theorem 1.3 settles the question completely; a sufficiently small $\varphi \in \mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp$ is embeddable if and only if $\varphi \in \mathfrak{D}'_{BE}$.

Another consequence of Theorem 1.3 is a normal form for embeddable CR structures, unique up to an action of $\mathrm{Aut}(S^3)$ on $\mathfrak{D}'_{BE}$.

Corollary 1.4. *For sufficiently small deformations φ of the standard CR sphere, φ is an embeddable deformation if and only if there exists a smooth contact diffeomorphism such that the pulled back CR structure corresponds to a deformation $\tilde{\varphi} \in \mathfrak{D}'_{BE}$.*

Note that $\mathfrak{D}'_{BE}$ can be replaced by $\mathfrak{D}_{BE}$ in Corollary 1.4 at the expense of leaving also the freedom to act by an S^1-equivariant contact diffeomorphism on $\mathfrak{D}_{BE}$. Such a result was only previously known in finite regularity, with the notion of "sufficiently small" depending on the regularity; see the work of Bland and Bland-Duchamp [6, 3, 4].

The above characterization of embeddable deformations is satisfying, but it does not really give a practical means of checking for embeddability since one must first normalize the deformation tensor by an appropriate contact diffeomorphism. We would like to say something about the embeddability of a deformation without the need to first normalize it. At the linearized (i.e. infinitesimal) level this is clear, as explained above. To what extent does a similar characterization of embeddability hold beyond the linear level? By taking a completely different approach to the problem using geometric flows we provide the following result describing embeddable structures without the need to normalize by contact diffeomorphisms.

Theorem 1.5. *For $\dot{\varphi} \in \mathfrak{D}_0$ sufficiently small there exists a smooth family $\varphi(t) \in \mathfrak{D}_{emb}$ such that $\varphi(t) = t\dot{\varphi} + \psi(t)$ for $t \in [0, 1]$, where $\psi(t) = O(t^2)$ and $\psi(t) \in \mathfrak{D}_0^\perp$. Moreover, there exists a smooth family of embeddings $\Phi_t : S^3 \to \mathbb{C}^2$, with $\Phi_0 = \mathrm{Id}$, realizing the deformation $\varphi(t)$ for each $t \in [0, 1]$.*

Remark 1.6. We make two remarks.

(1) In fact, it follows from the more detailed version of the theorem, Theorem 4.12, that the family $\varphi(t)$ can be made canonical; the resulting time one map $\mathfrak{D}_0 \to \mathfrak{D}_{emb}$ taking $\dot{\varphi}$ to $\varphi(1)$ has a linearization at $\dot{\varphi} = 0$ which is the identity and hence can be thought of as an exponential map.

(2) Note that, in terms of spherical harmonics, the condition $\psi \in \mathfrak{D}_0^{\perp}$ means that $\psi_{p,q} = 0$ except possibly when $q = 0, 1$. Also, note that $\psi(t)$ will not be zero in general (as can be seen by an inspection of the proof of Proposition 4.3 for the special case where, say, $\dot{\varphi} \in H_{p,2}$).

In the special case where $\dot{\varphi} \in \mathfrak{D}_{BE}$ we naturally find that $\varphi(t) = t\dot{\varphi}$ (i.e. $\psi(t) = 0$) and we obtain analyticity of Φ_t in t. More precisely:

Theorem 1.7. *For $\dot{\varphi} \in \mathfrak{D}_{BE}$ sufficiently small there exists a family $\Phi_\tau : S^3 \to \mathbb{C}^2$ with complex parameter τ, such that for each τ, $|\tau| < 2$,*

(i) *Φ_τ is a smooth embedding which realizes the deformation $\varphi(\tau) = \tau\dot{\varphi}$;*

(ii) *Φ_τ is analytic in τ as a function with values in the Banach space of C^k maps $S^3 \to \mathbb{C}^2$ for any k.*

Note that this recovers the result of Burns-Epstein [8, Theorem 5.3] by setting $\tau = 1$.

As a by-product of our approach, we also establish the embeddability of a family of deformations of an embedded structure that satisfy a well known necessary condition (stated for $t = 0$ above), under a natural additional condition that forces the resulting family of embeddings to move outwards (or inwards). As mentioned above, given a CR 3-manifold (M, H, J) embedded in a complex surface, an infinitesimal deformation tensor $\dot{\varphi}$ will be infinitesimally embeddable if and only if it satisfies (2) for some complex function f on M. Given a family of CR hypersurfaces $M_t \subseteq \mathbb{C}^2$ with $M_0 = M$ contact parametrized by $\psi_t : M \to M_t$, with $\psi_0 = \mathrm{id}$, (2) applies at each time t on M_t. Pulling back using $\psi_t : M \to M_t$ we obtain a family of embeddable deformations $\varphi(t)$ with $\varphi(0) = 0$ and a family of complex functions f_t on M satisfying a certain second order equation at each time t, corresponding to (2); f_t can be interpreted as the complex normal component of the variational vector field $\dot{\psi}_t$ arising from the family of embeddings ψ_t (more precisely, as the corresponding function on M). Saying that a family of abstract deformations $\varphi(t)$ satisfies this condition (for some family f_t) in principle says that the deformation $\varphi(t)$ moves tangent to the space of embeddable deformations at each time t. Borrowing terminology from Jih-Hsin Cheng [12] we will refer to this condition on $\varphi(t)$ as the *(abstract) tangency condition*; we shall also refer to the family f_t as a *family of potentials* corresponding to $\varphi(t)$. The precise formulation of the (abstract) tangency condition is stated in Lemma 2.1. Given a family of deformations of an embeddable CR structure satisfying the abstract tangency condition, it is natural to ask whether this family is embeddable. Our result is the following:

Theorem 1.8. *Let M be a compact strictly pseudoconvex hypersurface in a complex surface X and let $\varphi(t)$ be a 1-parameter family of deformations of the induced CR structure* *on M with $\varphi(0) = 0$. Suppose $\varphi(t)$ satisfies the abstract tangency condition with a family of potentials f_t satisfying* $\operatorname{Re} f_t > 0$ *for all t. Then there exists $\epsilon > 0$ such that $\varphi(t)$ is an embeddable deformation for all $t \in [0, \epsilon)$.*

For a more precise statement see Theorem 3.1 below. This establishes for embeddable structures an analog of Cheng's theorem for fillable structures [12, Theorem A].

This paper is organized as follows. In Section 2 we give some preliminaries on deformations of 3-dimensional CR structures and introduce the tangency equation for families of embeddable deformations, which makes precise the tangency condition referred to in Theorem 1.8. In Section 3 we explain how one obtains embeddings from solutions to the tangency equation and establish Theorem 3.1, which implies Theorem 1.8. In Section 4 we study the solvability of the tangency equation for small deformations of the standard CR 3-sphere, and establish Theorems 1.5 and 1.7. Finally, in Section 5 we prove the slice theorems, Theorem 1.2 and Theorem 1.3. We remark that the main sections, Sections 3-5, are largely independent from each other. Section 3 is primarily geometric, and makes use of the Fefferman ambient metric construction in the framework of Hirachi-Marugame-Matsumoto [23]. In both Section 4 and Section 5 we make use of the Nash-Moser inverse function theorem as presented in Hamilton [22] (see also Cheng-Lee [13] for a brief introduction to this and Hamilton's tame Frechet category). These sections also make use of an elliptic regularity argument adapted from [8] which appears first in the proof of Proposition 4.8. The proof of Theorem 4.12 (a more precise version of Theorem 1.5) uses arguments from the theory of parabolic evolution equations.

2. Deformations of 3-Dimensional CR Structures

Let M be a smooth oriented 3-manifold. A *contact structure* on M is a rank 2 subbundle $H \subset TM$ which is nondegenerate in the sense that if H is locally given as the kernel of some 1-form θ, then $\theta \wedge d\theta$ is nowhere vanishing. A *CR structure* on (M, H) is given by a smooth endomorphism $J : H \to H$ such that $J^2 = -\mathrm{id}$. We refer to (M, H, J) as a *strictly pseudoconvex CR* 3-*manifold*. The partial complex structure J on $H \subset TM$ defines an orientation of H, and therefore defines an orientation on the annihilator subbundle $H^\perp := \operatorname{Ann}(H) \subset T^*M$. A nowhere vanishing section θ of $H^\perp$ is called a *contact form* for H. A contact form θ is positively oriented if $d\theta|_H$ is compatible with the orientation of H, equivalently, if $d\theta(\,\cdot\,, J\,\cdot\,)$ is positive definite on H. A CR structure (M, H, J) together with a choice of positively oriented contact form θ is referred to as a *pseudohermitian structure* [33, 34]. The *Reeb vector field* of a contact form θ is the vector field T uniquely determined by $\theta(T) = 1$ and $d\theta(T, \,\cdot\,) = 0$.

Given a CR manifold (M, H, J) we decompose the complexified contact distribution $\mathbb{C} \otimes H$ as $T^{1,0} \oplus T^{0,1}$, where J acts by i on $T^{1,0}$ and by $-i$ on $T^{0,1} = \overline{T^{1,0}}$. Let θ be a positively oriented contact form on M. Let Z_1 be a local frame for the *holomorphic tangent bundle* $T^{1,0}$ and $Z_{\bar{1}} = \overline{Z_1}$, so that $\{T, Z_1, Z_{\bar{1}}\}$ is a local frame for $\mathbb{C} \otimes TM$. Then the dual frame $\{\theta, \theta^1, \theta^{\bar{1}}\}$ is referred to as an *admissible coframe* and one has

$$d\theta = i h_{1\bar{1}} \theta^1 \wedge \theta^{\bar{1}} \tag{5}$$

for some positive smooth function $h_{1\bar{1}}$. The function $h_{1\bar{1}}$ is the component of the *Levi form* $\mathrm{L}_\theta(U, \overline{V}) = -i d\theta(U, \overline{V})$ on $T^{1,0}$, that is

$$\mathrm{L}_\theta(U^1 Z_1, V^{\bar{1}} Z_{\bar{1}}) = h_{1\bar{1}} U^1 V^{\bar{1}}.$$

It is convenient to scale Z_1 so that $h_{1\bar{1}} = 1$, and we will typically do so. In any case, we write $h^{1\bar{1}}$ for the multiplicative inverse of $h_{1\bar{1}}$. The Tanaka-Webster connection associated to θ is given in terms of such a local frame $\{T, Z_1, Z_{\bar{1}}\}$ by

$$\nabla Z_1 = \omega_1{}^1 \otimes Z_1, \quad \nabla Z_{\bar{1}} = \omega_{\bar{1}}{}^{\bar{1}} \otimes Z_{\bar{1}}, \quad \nabla T = 0$$

where the connection 1-forms $\omega_1{}^1$ and $\omega_{\bar{1}}{}^{\bar{1}}$ satisfy

$$d\theta^1 = \theta^1 \wedge \omega_1{}^1 + A^1{}_{\bar{1}}\, \theta \wedge \theta^{\bar{1}}, \text{ and} \tag{6}$$

$$\omega_1{}^1 + \omega_{\bar{1}}{}^{\bar{1}} = h^{1\bar{1}} d h_{1\bar{1}}, \tag{7}$$

for some function $A^1{}_{\bar{1}}$. The uniquely determined function $A^1{}_{\bar{1}}$ is known as the *pseudo-hermitian torsion*. Components of covariant derivatives will be denoted by adding ∇ with an appropriate subscript, so, e.g., if u is a function then $\nabla_1 u = Z_1 u$, $\nabla_1 \nabla_1 u = Z_1 Z_1 u - \omega_1{}^1(Z_1) Z_1 u$ and $\nabla_0 \nabla_1 u = T Z_1 u - \omega_1{}^1(T) Z_1 u$. We may also use $h_{1\bar{1}}$ and $h^{1\bar{1}}$ to raise and lower indices, so that $A_{\bar{1}\bar{1}} = h_{1\bar{1}} A^1{}_{\bar{1}}$ and $A_{11} = h_{1\bar{1}} A^{\bar{1}}{}_1$, with $A^{\bar{1}}{}_1 = \overline{A^1{}_{\bar{1}}}$. Note that when $h_{1\bar{1}} = 1$ raising and lowering indices is a trivial operation.

Let (M, H, J) be a compact, strictly pseudoconvex, three-dimensional CR manifold. Consider a smooth family of CR structures (M, H_t, J_t) on M with $(H_0, J_0) = (H, J)$. By Gray's theorem [21] this family may be pulled back by a smooth family of diffeomorphisms to a family of the form $(M, H, \tilde{J}_t)$. When considering families of CR structures on M we therefore always keep the contact distribution H fixed. If Z_1 is holomorphic tangent vector field on (M, H, J) then this amounts to requiring that the holomorphic tangent space of our deformed structure is spanned by a vector field of the form $Z_1 + \varphi_1{}^{\bar{1}} Z_{\bar{1}}$ for some complex function $\varphi = \varphi_1{}^{\bar{1}}$ with $|\varphi|^2 < 1$ on M. We shall fix a contact form θ on M such that Z_1 is unitary (i.e. $h_{1\bar{1}} = 1$ with respect to Z_1). Given a deformed CR structure spanned by $Z_1 + \varphi_1{}^{\bar 1} Z_{\bar 1}$ we will always work with the normalized frame

$$Z_1^{\varphi} = \frac{1}{\sqrt{1-|\varphi|^2}} \left(Z_1 + \varphi_1{}^{\bar 1} Z_{\bar 1} \right) \tag{8}$$

so that the Levi form of θ with respect to the deformed structure has component $h^{\varphi}_{1\bar 1} = 1$. Given a family (M, H, J_t) of CR structures on M, we may describe the deformation J_t by a deformation tensor $\varphi_1{}^{\bar 1}(t)$ via

$$Z_1^t := \frac{1}{\sqrt{1-|\varphi|^2}} \left(Z_1 + \varphi_1{}^{\bar 1}(t) Z_{\bar 1} \right), \tag{9}$$

where we use the shorthand notation

$$|\varphi|^2 = |\varphi_1{}^{\bar 1}(t)|^2. \tag{10}$$

The corresponding admissible coframe $(\theta, \theta^1_t, \theta^{\bar 1}_t)$ is obtained by choosing

$$\theta^1_t := \frac{1}{\sqrt{1-|\varphi|^2}} \left(\theta^1 - \varphi_{\bar 1}{}^1(t)\, \theta^{\bar 1} \right). \tag{11}$$

Note that J_t is easily recovered by writing $J_t = i Z_1^t \otimes \theta^1_t - i Z^t_{\bar 1} \otimes \theta^{\bar 1}_t$. It is useful to invert the transformations $(Z_1, Z_{\bar 1}) \mapsto (Z_1^t, Z_{\bar 1}^t)$ and $(\theta^1, \theta^{\bar 1}) \mapsto (\theta^1_t, \theta^{\bar 1}_t)$:

$$\begin{aligned} Z_1 &= \frac{1}{\sqrt{1-|\varphi|^2}} \left(Z_1^t - \varphi_1{}^{\bar 1}(t) Z^t_{\bar 1} \right), \\ \theta^1 &= \frac{1}{\sqrt{1-|\varphi|^2}} \left(\theta^1_t + \varphi_{\bar 1}{}^1(t) \theta^{\bar 1}_t \right). \end{aligned} \tag{12}$$

We denote by ∇^t the Tanaka-Webster connection of the pseudohermitian structure (M, H, J_t, θ), and by $A_{11}(t)$ its pseudohermitian torsion in the coframe $(\theta, \theta^1_t, \theta^{\bar 1}_t)$. For the connection form $\omega_1{}^1$ on M relative to the admissible coframe $(\theta, \theta^1, \theta^{\bar 1})$, we shall write

$$\omega_1{}^1 = \omega_1{}^1{}_1\, \theta^1 + \omega_1{}^1{}_{\bar 1}\, \theta^{\bar 1} + \omega_1{}^1{}_0\, \theta, \tag{13}$$

and similarly for the connection forms $\omega_1{}^1(t)$ of ∇^t,

$$\omega_1{}^1(t) = \omega_1{}^1{}_1(t)\, \theta^1_t + \omega_1{}^1{}_{\bar 1}(t)\, \theta^{\bar 1}_t + \omega_1{}^1{}_0(t)\, \theta. \tag{14}$$

Note that we then have, for a smooth function f,

$$\nabla^t_1 \nabla^t_1 f = (Z_1^t)^2 f - \omega_1{}^1{}_1(t) Z_1^t f. \tag{15}$$

The following lemma makes precise the *(abstract) tangency condition* referred to in the introduction.

Lemma 2.1. *Let (M, H, J_t) be a smooth 1-parameter family of CR structures on M corresponding to a family of admissible coframes $(\theta, \theta_t^1, \theta_t^{\bar{1}})$ with deformation tensor $\varphi_{11}(t)$ as above. If (M, H, J_t) is obtained from a smooth family M_t of strictly pseudoconvex hypersurfaces in a complex surface X by pulling the CR structures back to $M = M_0$ via a smooth family of contact diffeomorphisms $\psi_t : M \to M_t$ with $\psi_0 = \mathrm{id}$, then there exists a smooth family $f_t \in C^\infty(M, \mathbb{C})$ such that*

$$\nabla_1^t \nabla_1^t f_t + i A_{11}(t) f_t = \frac{\dot{\varphi}_{11}(t)}{1 - |\varphi(t)|^2}, \quad \dot{\varphi}_{11}(t) := \frac{d}{dt}\varphi_{11}(t), \tag{16}$$

expressed in terms of the coframe $(\theta, \theta_t^1, \theta_t^{\bar{1}})$ at each time t.

Remark 2.2. We refer to (16) as the *tangency equation.* Note that ∇^t and $A_{11}(t)$ depend only on $\varphi_{11}(t)$ and ∇, thus the equation may be written purely in terms of ∇, f_t and $\varphi_{11}(t)$; see (96).

Proof. By (2) at each time t the infinitesimal deformation tensor of the family (M, H, J_t) expressed in terms of the frame Z_1^t is in the image of the operator $\nabla_1^t \nabla_1^t + i A_{11}(t)$. (See, e.g., [14, Lemma 4.5] for a derivation of (2), noting that our convention in [14] for f_t differs by a factor of i.) Note that (except at $t = 0$) this is not $\frac{d}{dt}\varphi_{11}(t)$, since the latter refers to the frame Z_1 rather than Z_1^t. To find the infinitesimal deformation tensor relative to Z_1^t at a fixed time $t > 0$ we therefore compute (using the first equation in (12)):

$$\begin{aligned}
Z_1^{t+s} &= \frac{1}{\sqrt{1 - |\varphi(t+s)|^2}} \left(Z_1 + {\varphi_1}^{\bar{1}}(t+s) Z_{\bar{1}} \right) \\
&= \frac{1}{\sqrt{1 - |\varphi(t+s)|^2}} \frac{1}{\sqrt{1 - |\varphi(t)|^2}} \left(Z_1^t - {\varphi_1}^{\bar{1}}(t) Z_{\bar{1}}^t + {\varphi_1}^{\bar{1}}(t+s)(Z_{\bar{1}}^t - {\varphi_{\bar{1}}}^{1}(t) Z_1^t) \right) \\
&= \frac{1}{\sqrt{1 - |\varphi(t+s)|^2}} \frac{1}{\sqrt{1 - |\varphi(t)|^2}} \left((1 - {\varphi_1}^{\bar{1}}(t+s){\varphi_{\bar{1}}}^{1}(t)) Z_1^t + ({\varphi_1}^{\bar{1}}(t+s) - {\varphi_1}^{\bar{1}}(t)) Z_{\bar{1}}^t \right) \\
&= (1 + O(s)) Z_1^t + \left(\frac{{\dot{\varphi}_1}{}^{\bar{1}}(t)}{1 - |\varphi(t)|^2} s + O(s^2) \right) Z_{\bar{1}}^t.
\end{aligned} \tag{17}$$

The conclusion that there exists a family f_t satisfying (16) then follows from the discussion of (2) above (cf. [5, 14, 23]). By Lemma 4.5 of [14] the family f_t can be obtained by pulling back the complex Reeb component of $\frac{d}{dt}\psi_t|_{M_t}$ (with respect to the contact form $(\psi_t)_*\theta$ on M_t) to M via ψ_t and hence depends smoothly on t. □

3. Embeddings from Solutions to the Tangency Equation

In this section we shall show that the abstract tangency condition does in fact characterize embeddability, subject to a mild additional condition. More precisely, we shall show that it is possible to construct a smooth family of CR embeddings from a smooth family of potentials f_t that solve the tangency equation (16) for a given family of deformations $\varphi(t)$, provided $\operatorname{Re} f_t$ has a strict sign. The aim of this section is to prove the following result, from which Theorem 1.8 directly follows.

Theorem 3.1. *Let $M = M_0$ be a compact strictly pseudoconvex hypersurface in a complex surface X. Let (M, H, J_t) be a smooth family of CR structures on (M, H) with $J_0 = J$ where (M, H, J) is the CR structure induced on $M \subseteq X$. Let $\varphi(t)$ be the associated family of deformation tensors given by* (9)*. Assume that there is a smooth family of solutions $f_t \in C^\infty(M, \mathbb{C})$ to the tangency equation* (16) *such that $\operatorname{Re} f_t$ has a strict sign on M. Then, for a sufficiently small $\epsilon > 0$, there is a family of mappings $\psi\colon M \times [0, \epsilon) \to X$ such that:*

(i) *$\psi_t\colon M \to X$ is an embedding for each $t \in [0, \epsilon)$ with $\psi_0 = \mathrm{id}$, where $\psi_t := \psi(\cdot, t)$.*
(ii) *ψ_t is a CR diffeomorphism of (M, H, J_t) onto the image $M_t := \psi_t(M) \subset X$.*

The basic idea behind the proof of Theorem 3.1 is conceptually straightforward. Roughly speaking the functions $\operatorname{Re} f_t$ determine the normal velocity of the time evolution M_t of M in X; the functions $\operatorname{Im} f_t$ play a role in determining the contact parametrization of the family M_t. We can easily make this precise for $t = 0$, where we know that the contact parametrization ψ_t of the family M_t must be chosen such that the variational vector field at $t = 0$ is

$$\frac{d}{dt}\psi_t\bigg|_{t=0} = \operatorname{Re}\left(\frac{1}{2} f_0(JT + iT) - (\nabla^1 f_0) Z_1\right); \tag{18}$$

see, e.g., Lemma 4.5 of [14] (noting that the convention in [14] for the potential f_0 differs by a factor of i) or see the discussion below. (Note that the only term in (18) that is not tangent to M is $\frac{1}{2}\operatorname{Re} f_0\, JT$; note also that the contribution to the right hand side of (18) coming from $\operatorname{Im} f_0$ is a contact Hamiltonian vector field on M.) There is an obvious difficulty, however, in seeking to apply the corresponding result for $t > 0$ to determine the evolution of the map ψ_t, namely that we are given f_t as a function on M rather than on the hypersurface M_t (which we are trying to find). Moreover, f_t should really be treated as a weight $(1, 1)$-density, meaning that there is also an unknown family of conformal factors relating contact forms on M and M_t that must be determined in order to verify that we have the correct evolution.

A natural framework for resolving these problems is given by the construction of Hirachi-Marugame-Matsumoto in Sections 4.1 and 4.2 in [23], where they construct a family of contact diffeomorphisms $\underline{\Phi_t}$ parametrizing a family $M_t \subseteq X$ of CR hypersurfaces starting from a given strictly pseudoconvex hypersurface $M_0 \subseteq X$. Before we enter into the proof of Theorem 3.1, therefore, we shall examine the construction of Hirachi-Marugame-Matsumoto [23].

3.1. **The Hirachi-Marugame–Matsumoto construction.** We shall use the notation and setup in [23] restricted to the case of three-dimensional CR manifolds. For the reader's convenience we briefly review the essential definitions. As in [23] we start by assuming that M_t is a smooth family of strictly pseudoconvex real hypersurfaces in a complex manifold X (which we take to be a complex surface) with a smooth family of defining functions ρ_t such that $M_t = \{\rho_t = 0\}$. For $t = 0$, we may omit the subscript 0 and write $M_0 = M$, $\rho_0 = \rho$. We shall need the following quantities at each time t, but for simplicity we define them only for $t = 0$ and attach a sub or superscript t to the various objects as needed. Let

$$\vartheta = d^c\rho, \qquad \text{where } d^c = \frac{i}{2}(\partial - \overline{\partial}), \tag{19}$$

and endow each leaf $\{\rho = \epsilon\}$ with the contact form obtained by pulling back ϑ. Then there is a uniquely determined $(1, 0)$-vector field ξ on X near M such that

$$\xi\rho = 1 \qquad \text{and} \qquad \xi \lrcorner d\vartheta = i\kappa\overline{\partial}\rho \tag{20}$$

for some smooth function κ, called the *transverse curvature* of ρ; the vector field ξ has the form

$$\xi = \frac{1}{2}(JT + iT) = N + \frac{i}{2}T \tag{21}$$

where J is the complex structure of X and T is the Reeb vector field on each leaf $\{\rho = \epsilon\}$. The holomorphic tangent bundle of X thus decomposes as $T^{1,0}X = \ker \partial\rho \oplus \mathbb{C}\xi$ and we let L_1 be a local frame for $\ker \partial\rho$ (this is our one point of departure from the notation of [23], where the notation Z_1 is used; we will also use L_1^t rather than Z_1^t to denote a local frame for $\ker \partial\rho_t$). Moreover, we shall choose the tangent vector field L_1 in such a way that the Levi form of each leaf $\{\rho = \epsilon\}$ is $\ell_{1\bar{1}} = 1$.

Following Hirachi-Marugame-Matsumoto we shall assume that the defining function of each M_t is normalized to be a Fefferman defining function [19] and denote it by r_t rather than ρ_t. This normalization is not essential and is only done for consistency with the notation of [23]; in the proof of Theorem 3.1 we will therefore not require that r_t is a Fefferman defining function. Let K_X denote the canonical bundle of X. Let $\tilde{X}$ denote the $\mathbb{C}^*$ bundle $K_X^* = K_X \setminus \{0\}$ and let $\mathcal{N}_t = K_X^*|_{M_t} \subset \tilde{X}$. We take local coordinates

$z = (z^1, z^2)$ for X and trivialize K_X using $dz^1 \wedge dz^2$ to define a fiber coordinate λ. On $\tilde{X} = K_X^*$ we fix a branch $z^0 = \lambda^{1/3}$ and write

$$\boldsymbol{r}_t := |z^0|^2 r_t \tag{22}$$

which serves as a defining function for $\mathcal{N}_t$, and

$$\boldsymbol{\vartheta}_t := |z^0|^2 \vartheta_t \tag{23}$$

where $\vartheta_t = d^c r_t$. We define a time dependent frame (L_0^t, L_1^t, L_2^t) for $T^{1,0}\tilde{X}$ near $\mathcal{N}_t$ by taking $L_0^t = L_0 = z_0 \partial/\partial z_0$, L_1^t tangent to the leaves $\{r_t = \epsilon\}$ (as before), and $L_2^t = \xi^t$ (again we have departed from the notation of [23] by using L_A^t instead of Z_A^t). The *Fefferman ambient metric* of M_t is the Lorentz-Kähler metric $\tilde{g}_t$ defined by the Kähler form $i\partial\overline{\partial}\boldsymbol{r}_t$ on $\tilde{X}$ near $\mathcal{N}_t$ (if r_t is not assumed to be a Fefferman defining function, then $\tilde{g}_t$ is referred to as a *pre-ambient metric*; everything below remains valid in this case). With respect to the frame (L_0^t, L_1^t, L_2^t) the components of the ambient metric $\tilde{g}_t$ are given by

$$\tilde{g}^t_{A\bar{B}} = |z^0|^2 \begin{pmatrix} r_t & 0 & 1 \\ 0 & -\ell^t_{1\bar{1}} & 0 \\ 1 & 0 & -\kappa_t \end{pmatrix}, \tag{24}$$

where $\ell^t_{1\bar{1}} = 1$ is the Levi form on each leaf $\{r_t = \epsilon\}$ and κ_t is the transverse curvature of r_t. As in [23] we let $\tilde{\mathcal{E}}(w) = (K_X)^{-w/3} \otimes (\overline{K_X})^{-w/3}$, and let $\mathcal{E}(w) = \tilde{\mathcal{E}}(w)|_M$ denote the bundle of weight (w, w)-densities on M; locally, a section F of $\tilde{\mathcal{E}}(w)$ can be written as $|z^0|^{2w}\underline{F}$ where $\underline{F}$ is a function on X. Hence $\boldsymbol{r}_t$ is a section of $\tilde{\mathcal{E}}(1)$. As in [23] we abuse notation by using the same notation for $\tilde{\mathcal{E}}(w)$ and for its space of smooth sections, e.g., writing $\boldsymbol{r}_t \in \tilde{\mathcal{E}}(1)$.

We now choose a family of complex valued functions $F_t \in \tilde{\mathcal{E}}(1)$ such that

$$\operatorname{Re} F_t = -\dot{\boldsymbol{r}}_t, \tag{25}$$

and define, as in Section 4.1 in [23], the time dependent vector field

$$Y_t = \operatorname{Re}\left(\sum_{A=0}^{2} (F_t)^A L_A^t\right) = \operatorname{Re}\left(\sum_{A,B=0}^{2} \tilde{g}_t^{A\bar{B}} (L^t_{\bar{B}} F_t) L_A^t\right), \tag{26}$$

the gradient of F_t with respect to the ambient metric $\tilde{g}_t$. We define Φ_t to be the flow of Y_t, determined by $\frac{d}{dt}\Phi_t(p) = Y_t(\Phi_t(p))$ and $\Phi_0 = \mathrm{id}$. Denoting the projection $\tilde{X} \to X$ by π we also define $\underline{Y_t} = \pi_* Y_t$, whose flow is denoted $\underline{\Phi_t}$. For each t the map $\Phi_t : \tilde{X} \to \tilde{X}$ is a bundle map that covers the map $\underline{\Phi_t} : X \to X$. We recall from [23, Lemma 4.1] that (25) implies that

$$\boldsymbol{r}_t \circ \Phi_t = \boldsymbol{r}_0, \quad \Phi_t^* \boldsymbol{\vartheta}_t = \boldsymbol{\vartheta}_0. \tag{27}$$

Note that the imaginary part of F_t can be chosen freely in this construction, and different choices result in different families Φ_t all having the above properties.

We now develop some simple consequences of the construction outlined above. First, since Φ_t is a bundle map, in the local coordinates (z_0, z) on $\tilde{X}$ (which we locally identify with $\mathbb{C}^* \times X$) we may write

$$\Phi_t(z_0, z) = (z_0 \Phi_t^0(z), \underline{\Phi_t}(z)), \tag{28}$$

where $\Phi_t^0 \neq 0$. It follows immediately from (27) and (28) that

$$|\Phi_t^0|^2 \, r_t \circ \underline{\Phi_t} = r_0. \tag{29}$$

Lemma 3.2. *On M_0, the following identity holds:*

$$|\Phi_t^0|^2 \, \underline{\Phi_t^*} \vartheta_t = \vartheta_0. \tag{30}$$

Proof. We observe that the first identity in (27) implies that $\Phi_t^* d\boldsymbol{r}_t = d\boldsymbol{r}_0$. Combining this with the second identity in (27), we conclude, since $\boldsymbol{\vartheta}_t = d^c \boldsymbol{r}_t$, that $\Phi_t^* \partial \boldsymbol{r}_t = \partial \boldsymbol{r}_0$. We have

$$\partial \boldsymbol{r}_t = \partial(|z^0|^2 r_t) = |z^0|^2 \partial r_t + r_t \, \bar{z}^0 dz^0, \tag{31}$$

and therefore

$$\Phi_t^* \partial \boldsymbol{r}_t = |z^0|^2 |\Phi_t^0|^2 \underline{\Phi_t}^* \partial r_t + (r_t \circ \underline{\Phi_t}) \, \bar{z}^0 \bar{\Phi}_t^0 \, d(z^0 \Phi_t^0), \tag{32}$$

and similarly for $\bar{\partial} \boldsymbol{r}_t$. We conclude, by using (29), that

$$\Phi_t^* d^c \boldsymbol{r}_t = |z^0|^2 |\Phi_t^0|^2 \underline{\Phi_t}^* d^c r_t + O(r_0). \tag{33}$$

The conclusion of the lemma now follows from (27). ◻

We shall write

$$\Upsilon_t := -\log |\Phi_t^0|. \tag{34}$$

Thus, equation (30) can be written

$$\underline{\Phi_t^*} \vartheta_t = e^{2\Upsilon_t} \vartheta_0. \tag{35}$$

In particular, we have on M_0

$$\underline{\Phi_t^*} \theta_t = e^{2\Upsilon_t} \theta_0, \tag{36}$$

where $\theta_t = d^c r_t|_{M_t}$ and we are really pulling back by $\underline{\Phi_t}|_{M_0}$; understanding this function Υ_t will be important for us as it will enable us to correctly interpret the function f_t as a $(1,1)$-density when we use it to describe the evolution of M_t. It follows that $\underline{\Phi_t}$ restricts to a contact diffeomorphism $M_0 \to M_t$. Note, however, that $\underline{\Phi_t}$ will not be a CR diffeomorphism in general. We define the deformation tensor ${\varphi_1}^{\bar{1}}$ in the definition

(9) of Z_1^t on M_0 by requiring that $(\underline{\Phi_t})_* Z_1^t$ is a multiple of L_1^t, i.e. we define a new CR structure Z_1^t on M_0 by pulling back L_1^t via $\Phi_t|_{M_0} : M_0 \to M_t$. The requirement that the Levi forms remain one shows that we may assume (after possibly modifying L_1^t by a function of modulus one)

$$(\underline{\Phi_t})_*(e^{-\Upsilon_t} Z_1^t) = L_1^t. \tag{37}$$

For each t, we define the contact form $\hat{\theta}_t = \underline{\Phi_t}^* \theta_t = e^{2\Upsilon_t}\theta$ on M_0. By [23, Lemma 4.2] (and by inspecting the proof of Lemma 2.1) the functions $\hat{f}_t = \underline{F_t} \circ \underline{\Phi_t}$ on M_0 satisfy

$$\hat{\nabla}_1^t \hat{\nabla}_t^{\bar{1}} \hat{f}_t + i\hat{A}_1{}^{\bar{1}}(t)\hat{f}_t = \frac{\dot{\varphi}_1{}^{\bar{1}}(t)}{1 - |\varphi(t)|^2}, \tag{38}$$

expressed in terms of the frame $(\hat{\theta}_t = e^{2\Upsilon_t}\theta, \theta_t^1, \theta_t^{\bar{1}})$, where $\hat{\nabla}^t$ is the Tanaka-Webster connection of $\hat{\theta}_t$, $\hat{A}_1{}^{\bar{1}}(t)$ is its pseudohermitian torsion, and $\dot{\varphi}_1{}^{\bar{1}}(t) := \frac{d}{dt}\varphi_1{}^{\bar{1}}(t)$. The operator $\hat{\nabla}_1^t \hat{\nabla}_t^{\bar{1}} + i\hat{A}_1{}^{\bar{1}}(t)$ on the left hand side of (38) is CR invariant when acting on densities of weight $(1,1)$, and the right hand side of (38) does not depend on the contact form. Hence, the functions $f_t = e^{-2\Upsilon_t}\hat{f}_t = e^{-2\Upsilon_t}\underline{F_t} \circ \underline{\Phi_t}$ satisfy the tangency condition in Lemma 2.1,

$$\nabla_1^t \nabla_t^{\bar{1}} f_t + iA_1{}^{\bar{1}}(t) f_t = \frac{\dot{\varphi}_1{}^{\bar{1}}(t)}{1 - |\varphi(t)|^2}, \tag{39}$$

with respect to the frame the frame $(\theta, \theta_t^1, \theta_t^{\bar{1}})$.

Our aim now is to rewrite the differential equation determining $\underline{\Phi_t}|_{M_0}$ in terms of data on M_0 alone for all t (rather than on M_t for each t). In order to do this we need to understand how the normal direction JT^t on M_t relates to $(\underline{\Phi_t})_* T^0$. It turns out that these are nicely related, as can be seen from the following lemma.

Lemma 3.3. *On M_0, we have*

$$(\kappa_t \circ \underline{\Phi_t}) e^{2\Upsilon_t} = \kappa_0, \tag{40}$$

and along M_t, letting J denote the complex structure on X, we have

$$(\underline{\Phi_t})_* JT^0 = J(\underline{\Phi_t})_* T^0 \tag{41}$$

and

$$\xi^t = (\underline{\Phi_t})_* \left(e^{-2\Upsilon_t}(\xi^0 + 2(\nabla_t^1 \Upsilon_t) Z_1^t)\right) \tag{42}$$

where ∇^t is the Tanaka-Webster connection of the pseudohermitian structure corresponding to θ_0 and Z_1^t.

Proof. We know that along M_t, the vector field ξ^t has the form

$$\xi^t = \frac{1}{2}(JT^t + iT^t), \tag{43}$$

where T^t is the Reeb vector field of θ_t. Thus, if on M_0 we set

$$\hat{\theta}_t := e^{2\Upsilon_t}\theta_0, \tag{44}$$

then, by (36), we will have $T^t = (\underline{\Phi_t})_*\hat{T}^t$, where $\hat{T}^t$ is the Reeb vector field of $\hat{\theta}^t$. Moreover, from the definition of $\hat{T}^t$ and of T^0 we have

$$\hat{T}^t = e^{-2\Upsilon_t}\left(T^0 + 2i(\nabla_t^{\bar{1}}\Upsilon_t)Z^t_{\bar{1}} - 2i(\nabla_t^1\Upsilon_t)Z^t_1\right). \tag{45}$$

Thus, by (43), we have

$$\begin{aligned}\xi^t = J(\underline{\Phi_t})_*\left(\frac{1}{2}e^{-2\Upsilon_t}\left(T^0 + 2i(\nabla_t^{\bar{1}}\Upsilon_t)Z^t_{\bar{1}} - 2i(\nabla_t^1\Upsilon_t)Z^t_1\right)\right)\\ + i(\underline{\Phi_t})_*\left(\frac{1}{2}e^{-2\Upsilon_t}\left(T^0 + 2i(\nabla_t^{\bar{1}}\Upsilon_t)Z^t_{\bar{1}} - 2i(\nabla_t^1\Upsilon_t)Z^t_1\right)\right).\end{aligned} \tag{46}$$

By (37) and the fact that $J(\underline{\Phi_t})_*(e^{-\Upsilon_t}Z^t_{\bar{1}}) = JL^t_{\bar{1}} = -iL^t_{\bar{1}}$ and $i(\underline{\Phi_t})_*(e^{-\Upsilon_t}Z^t_{\bar{1}}) = iL^t_{\bar{1}}$, the terms involving $Z^t_{\bar{1}}$ in (46) cancel and we can write this identity as

$$\xi^t = \frac{1}{2}e^{-2\Upsilon_t\circ(\underline{\Phi_t})^{-1}}\left(J(\underline{\Phi_t})_*T^0 + i(\underline{\Phi_t})_*T^0\right) + 2(\underline{\Phi_t})_*\left(e^{-2\Upsilon_t}(\nabla_t^1\Upsilon_t)Z^t_1\right). \tag{47}$$

Next, we note that since $\underline{\Phi_t}$ is a contact diffeomorphism between M_0 and M_t, there is a (real) vector field S along M_0, transverse to M_0, such that

$$(\underline{\Phi_t})_*S = J(\underline{\Phi_t})_*T^0. \tag{48}$$

We may write

$$S = \alpha\xi^0 + \bar{\alpha}\bar{\xi}^0 + 2\beta Z^t_1 + 2\bar{\beta}Z^t_{\bar{1}}. \tag{49}$$

Our next observation, which follows readily by differentiating (29) and adding to (30), is that along M_0, we have in fact

$$\underline{\Phi_t}^*\partial r_t = e^{2\Upsilon_t}\partial r_0. \tag{50}$$

We compute

$$\alpha = \partial r_0(S) = e^{-2\Upsilon_t}\partial r_t((\underline{\Phi_t})_*S) = \partial r_t\left(J(\underline{\Phi_t})_*(e^{-2\Upsilon_t}T^0)\right), \tag{51}$$

and note that, by (47), $\xi^t = \frac{1}{2}J(\underline{\Phi_t})_*(e^{-2\Upsilon_t}T^0)$ modulo a vector field that is tangent to M_t. Since $\xi^t r_t = 1$ and $J(\underline{\Phi_t})_*(e^{-2\Upsilon_t}T^0)$ is real, we conclude that $\alpha = 1$. We may therefore write (47) as

$$\xi^t = (\underline{\Phi_t})_*\left(e^{-2\Upsilon_t}(\xi^0 + (2\nabla_t^1\Upsilon_t + \beta)Z^t_1 + \bar{\beta}Z^t_{\bar{1}})\right). \tag{52}$$

To determine β we use the other defining property of ξ^t:

$$i\kappa_t \bar{\partial} r_t = \xi^t \lrcorner d\vartheta_t = (\underline{\Phi_t})_* \left(e^{-2\Upsilon_t}(\xi^0 + (2\nabla^1_t \Upsilon_t + \beta)Z^t_1 + \bar{\beta} Z^t_{\bar{1}})\right)\lrcorner d\vartheta_t. \tag{53}$$

If we pullback (53) by $\underline{\Phi_t}$ and use (35) and (50), we obtain along M_0

$$i(\kappa_t \circ \underline{\Phi_t})e^{2\Upsilon_t}\bar{\partial} r_0 = \left(e^{-2\Upsilon_t}(\xi^0 + (2\nabla^1_t \Upsilon_t + \beta)Z^t_1 + \bar{\beta} Z^t_{\bar{1}})\right)\lrcorner e^{2\Upsilon_t} d\vartheta_0. \tag{54}$$

Hence

$$i(\kappa_t \circ \underline{\Phi_t})e^{2\Upsilon_t}\bar{\partial} r_0 = i\kappa_0 \bar{\partial} r_0 + \left((2\nabla^1_t \Upsilon_t + \beta)Z^t_1 + \bar{\beta} Z^t_{\bar{1}})\right)\lrcorner d\vartheta_0. \tag{55}$$

If we apply both sides to $\bar{\xi}^0$, we conclude that on M_0

$$(\kappa_t \circ \underline{\Phi_t})e^{2\Upsilon_t} = \kappa_0. \tag{56}$$

Thus, we also have, on M_0,

$$\left((2\nabla^1_t \Upsilon_t + \beta)Z^t_1 + \bar{\beta} Z^t_{\bar{1}})\right)\lrcorner d\vartheta_0 = 0. \tag{57}$$

If we apply both sides to Z^t_1 and recall that ϑ_0 equals θ_0 when applied to tangential vector fields, we conclude that $\bar{\beta} = 0$, so that $S = JT^0$ and hence

$$\xi^t = (\underline{\Phi_t})_* \left(e^{-2\Upsilon_t}(\xi^0 + 2\nabla^1_t \Upsilon_t Z^t_1)\right). \tag{58}$$

This completes the proof of the lemma. □

Remark 3.4. We remark that (41) will be critical in our construction used in the proof of Theorem 3.1, since it means that the normal derivatives $(\underline{\Phi_t})_* JT^0$ of $\underline{\Phi_t}$ are uniquely determined along M_0 by applying J to the tangential derivatives $(\underline{\Phi_t})_* T^0$, allowing us to write an equation for $\underline{\Phi_t}|_{M_0}$ in terms of data purely on M_0.

From Section 4.1 in [23] we know that, along M_t, $\underline{Y_t}$ is given by

$$\underline{Y_t} = \mathrm{Re}(\underline{F_t}\xi^t - (\ell^{1\bar{1}} L^t_{\bar{1}} \underline{F_t}) L^t_1), \tag{59}$$

where $\ell^{1\bar{1}} = (\ell_{1\bar{1}})^{-1} = 1$. It follows from (42) of Lemma 3.3, and the fact that $\underline{\Phi_t}$ is the flow of $\underline{Y_t}$, that

$$\underline{\dot{\Phi}_t} = \underline{\Phi_t}_* \mathrm{Re}\left(e^{-2\Upsilon_t}\left((\underline{F_t} \circ \underline{\Phi_t})\xi^0 + (2(\underline{F_t} \circ \underline{\Phi_t})\nabla^1_t \Upsilon_t - \nabla^1_t(\underline{F_t} \circ \underline{\Phi_t}))Z^t_1\right)\right), \tag{60}$$

where we have also used (37). Setting $f_t = e^{-2\Upsilon_t}\underline{F_t} \circ \underline{\Phi_t}$ as before we can rewrite (60) in the form

$$\underline{\dot{\Phi}_t} = \underline{\Phi_t}_* \mathrm{Re}\left(f_t \xi^0 - (\nabla^1_t f_t) Z^t_1\right). \tag{61}$$

In order to obtain an equation for $\underline{\dot{\Phi}_t}|_{M_0}$ in terms of $\underline{\Phi_t}|_{M_0}$ we recall that $\xi^0 = \frac{1}{2}(JT^0 + iT^0)$ so that by (41) we have

$$(\underline{\Phi_t})_*(\xi^0) = \frac{1}{2}(J(\underline{\Phi_t})_* T^0 + (\underline{\Phi_t})_* iT^0) \tag{62}$$

along M_0. It follows that if $\psi_t = \underline{\Phi_t}|_{M_0}$ then

$$\dot{\psi}_t = \frac{1}{2} J\psi_{t*} \operatorname{Re}\left(f_t T^0\right) + \psi_{t*} \operatorname{Re}\left(\frac{i}{2} f_t T^0 - (\nabla^1_t f_t) Z^t_1\right). \tag{63}$$

Theorem 3.1 will be proved by viewing (63) as an equation for the family of CR embeddings ψ_t in terms of the given functions f_t; to verify that ψ_t obtained in this way has the desired properties, however, we will need to reconstruct $\underline{F_t}$ (along M_t) from the functions f_t on M_0, which requires us to further understand how the conformal factor $e^{2\Upsilon_t}|_{M_0}$ evolves.

We shall derive an evolution equation for Υ_t along M_0 from the identity (cf. (29))

$$r_t \circ \underline{\Phi_t} = e^{2\Upsilon_t} r_0. \tag{64}$$

First, we need the precise formula for the inverse of the ambient metric $\tilde{g}_t$ in the frame $(L^t_0, L^t_1, L^t_2 := \xi^t)$, given by (24), namely:

$$\tilde{g}_t^{A\bar{B}} = |z^0|^{-2} \begin{pmatrix} \frac{\kappa_t}{1+\kappa_t r_t} & 0 & \frac{1}{1+\kappa_t r_t} \\ 0 & -\ell^{1\bar{1}} & 0 \\ \frac{1}{1+\kappa_t r_t} & 0 & -\frac{r_t}{1+\kappa_t r_t} \end{pmatrix}. \tag{65}$$

This means that the vector field $\underline{Y_t}$ is given by

$$\underline{Y_t} = \operatorname{Re}\left(\frac{\underline{F_t} - r_t \bar{\xi}^t \underline{F_t}}{1+\kappa_t r_t}\, \xi^t - (\ell^{1\bar{1}} L^t_{\bar{1}} \underline{F_t}) L^t_1\right). \tag{66}$$

Differentiating (64) with respect to t yields

$$\dot{r}_t + dr_t(\underline{\dot{\Phi}_t}) = 2\dot{\Upsilon}_t e^{2\Upsilon_t} r_0, \tag{67}$$

where r_t and dr_t are evaluated at $\underline{\Phi_t}$. Since $\underline{\dot{\Phi}_t} = \underline{Y_t}(\underline{\Phi_t})$, we obtain from (66) that

$$\dot{r}_t \circ \underline{\Phi_t} + \operatorname{Re}\left(\frac{\underline{F_t} - r_t \bar{\xi}^t \underline{F_t}}{1+\kappa_t r_t}\right) \circ \underline{\Phi_t} = 2\dot{\Upsilon}_t e^{2\Upsilon_t} r_0 \tag{68}$$

Using that $\operatorname{Re} \underline{F_t} = -\dot{r}_t$ yields

$$\operatorname{Re}\left(\frac{-\kappa_t r_t \underline{F_t} - r_t \bar{\xi}^t \underline{F_t}}{1+\kappa_t r_t}\right) \circ \underline{\Phi_t} = 2\dot{\Upsilon}_t e^{2\Upsilon_t} r_0. \tag{69}$$

Applying $N^0 = \frac{1}{2} J T^0$ to this and then restricting to M_0 we obtain

$$\left[\operatorname{Re}\left(-\kappa_t \underline{F_t} - \bar{\xi}^t \underline{F_t}\right) \circ \underline{\Phi_t}\right] N^0(r_t \circ \underline{\Phi_t}) = 2\dot{\Upsilon}_t e^{2\Upsilon_t} N^0 r_0. \tag{70}$$

By (64), we have along M_0,

$$N^0(r_t \circ \underline{\Phi_t}) = e^{2\Upsilon_t} N^0 r_0 = e^{2\Upsilon_t},$$

and, hence, (70) can be written

$$\operatorname{Re}\left(-\kappa_t \underline{F_t} - \bar{\xi}^t \underline{F_t}\right) \circ \underline{\Phi_t} = 2\dot{\Upsilon}_t. \tag{71}$$

By Lemma 3.3, we obtain

$$\dot{\Upsilon}_t = -\frac{1}{2} e^{-2\Upsilon_t} \operatorname{Re}\left((\underline{F_t} \circ \underline{\Phi_t})\kappa_0 + (\bar{\xi}^0 + 2(\nabla_t^{\bar{1}} \Upsilon_t) Z_{\bar{1}}^t)(\underline{F_t} \circ \underline{\Phi_t}) \right) \tag{72}$$

along M_0.

We now consider the 1-parameter family of smooth functions on M_0 given by $f_t = e^{-2\Upsilon_t} \underline{F_t} \circ \underline{\Phi_t}$. We seek to write (72) purely in terms of f_t and its tangential derivatives along M_0. But the term

$$e^{-2\Upsilon_t} \operatorname{Re}\left((\bar{\xi}^0 + 2\nabla_t^{\bar{1}} \Upsilon_t Z_{\bar{1}}^t)(\underline{F_t} \circ \underline{\Phi_t}) \right) = \operatorname{Re}(\bar{\xi}^t \underline{F_t}) \circ \underline{\Phi_t} \tag{73}$$

involves the normal derivative of $\underline{F_t} \circ \underline{\Phi_t}$. To remedy this we recall that in the construction of [23] the imaginary part of $\underline{F_t}$ can be chosen arbitrarily, and corresponds to contact reparametrizations of M_t. Since, however, the expression

$$\operatorname{Re} \bar{\xi}^t \underline{F_t} = \frac{1}{2}(JT^t \operatorname{Re} F_t + T^t \operatorname{Im} F_t) \tag{74}$$

involves the 'Reeb derivative' of $\operatorname{Im} \underline{F_t}$, it is difficult to use this freedom to eliminate the term $\operatorname{Re}(\bar{\xi}^t \underline{F_t}) \circ \underline{\Phi_t}$ (or to eliminate all of $\kappa_t \underline{F_t} + \bar{\xi}^t \underline{F_t}$, which would be ideal in that it leads to $\dot{\Upsilon}_t = 0$). It turns out that the most convenient normalization for our construction in the next section will be simply to impose that $N^t \operatorname{Re} \underline{F_t} = 0$ along M_t, where $N^t = \frac{1}{2} JT^t$; to achieve this we will also relax the requirement that r_t be a Fefferman defining function in Section 3.2. Recalling (63) and rewriting (72) assuming this normalization we obtain, in summary:

Proposition 3.5. *Let $f_t = e^{-2\Upsilon_t} \underline{F_t} \circ \underline{\Phi_t}|_{M_0}$, $\gamma_t = \Upsilon_t|_{M_0}$ and $\psi_t = \underline{\Phi_t}|_{M_0}$. If $N^t \operatorname{Re} \underline{F_t} = 0$ along M_t, then along M_0 we have*

$$\dot{\psi}_t = \frac{1}{2} J\psi_{t*} \operatorname{Re}\left(f_t T^0\right) + \psi_{t*} \operatorname{Re}\left(\frac{i}{2} f_t T^0 - (\nabla_t^1 f_t) Z_1^t \right) \tag{75}$$

$$\dot{\gamma}_t = -\operatorname{Re}\left(f_t \kappa_0 + \left(-\frac{i}{2} T^0 + (\nabla_t^{\bar{1}} \gamma_t) Z_{\bar{1}}^t - (\nabla_t^1 \gamma_t) Z_1^t - iT^0 \gamma_t \right) f_t \right). \tag{76}$$

Remark 3.6. Note that (75) does not require the normalization condition $N^t \operatorname{Re} \underline{F_t} = 0$ along M_t.

3.2. **Proof of Theorem 3.1.** We shall now identify a candidate for the embedding ψ_t in Theorem 3.1. Let ψ_t, γ_t, be the solution of

$$\dot{\psi}_t = \frac{1}{2} J\psi_{t*} \operatorname{Re}\left(f_t T^0\right) + \psi_{t*} \operatorname{Re}\left(\frac{i}{2} f_t T^0 - (\nabla_t^1 f_t) Z_1^t\right) \tag{77}$$

$$\dot{\gamma}_t = -\operatorname{Re}\left(f_t \kappa_0 + \left(-\frac{i}{2} T^0 + (\nabla_t^{\bar{1}} \gamma_t) Z_{\bar{1}}^t - (\nabla_t^1 \gamma_t) Z_1^t - i T^0 \gamma_t\right) f_t\right), \tag{78}$$

where $\psi_0 = \mathrm{id}$ and $\gamma_0 = 0$. We note that this is a (partially decoupled) PDE of transport type for ψ_t, γ_t, hence, the initial value problem with smooth data on M_0 at $t = 0$ can be uniquely solved in $(t, x) \in [0, \epsilon) \times M_0$ (by, e.g., the method of characteristics [18]); here, we have used that M_0 is compact and $\epsilon > 0$ is sufficiently small. By taking $\epsilon > 0$ small enough we may also ensure that $\psi_t : M_0 \to X$ is an embedding for each $t \in [0, \epsilon)$. For each t, let M_t denote the image of ψ_t.

We now seek to define r_t and $\underline{F_t}$ as in Section 3.1 satisfying the additional normalization condition $N^t \operatorname{Re} \underline{F_t} = 0$ in Proposition 3.5 (being able to satisfy this normalization condition is where the additional assumption that the functions $\operatorname{Re} f_t$ have strict sign comes in). Let ν^t denote $\frac{1}{2} J\psi_{t*}(e^{-2\gamma_t} T^0)$ and let x_t, $t \in [0, \epsilon)$, be a smooth family of defining functions with M_t being the zero locus of x_t and $\nu^t x_t = 1$ along M_t. Then $x_t \circ \psi_t = 0$ implies

$$\begin{aligned} 0 = dx_t(\dot{\psi}_t) + \dot{x}_t \circ \psi_t &= dx_t\left(\frac{1}{2} J\psi_{t*} \operatorname{Re}\left(f_t T^0\right)\right) + \dot{x}_t \circ \psi_t \\ &= e^{2\gamma_t} \operatorname{Re} f_t \, dx_t(\nu^t) + \dot{x}_t \circ \psi_t \\ &= e^{2\gamma_t} \operatorname{Re} f_t + \dot{x}_t \circ \psi_t \end{aligned}$$

along M_0, where dx_t is evaluated at ψ_t. Hence $e^{2\gamma_t} \operatorname{Re} f_t = -\dot{x}_t \circ \psi_t$ along M_0.

A general smooth family of defining functions for the smooth family of hypersurfaces M_t, $t \in [0, \epsilon)$, may be written as $r_t = e^{2\omega_t} x_t$ where ω_t depends smoothly on $t \in [0, \epsilon)$. Note that, since $\dot{r}_t = e^{2\omega_t} \dot{x}_t + 2\dot{\omega}_t e^{2\omega_t} x_t$ (so that $\dot{r}_t|_{M_t} = e^{2\omega_t} \dot{x}_t$), the family r_t will satisfy $e^{2\gamma_t} \operatorname{Re} f_t = -\dot{r}_t \circ \psi_t$ if and only if $\omega_t|_{M_t} = 0$. The family r_t also satisfies $\nu^t r_t = 1$ along M_t if and only if $\omega_t|_{M_t} = 0$.

Let $\omega_t = k_t x_t$ (so that $\omega_t|_{M_t} = 0$) where k_t is to be determined. We take $k_0 = 0$, so that $r_0 = x_0$. Let $\vartheta_t = d^c r_t$ and $\vartheta_t^0 = d^c x_t$. Then, along M_t, $\vartheta_t = d^c(e^{2\omega_t} x_t) = e^{2\omega_t} d^c x_t = d^c x_t = \vartheta_t^0$ (as ambient forms). Hence $\theta_t = \vartheta_t|_{TM_t}$ agrees with $\theta_t^0 = \vartheta_t^0|_{TM_t}$, so that the corresponding Reeb fields agree. It follows that $\xi^t = \xi_0^t$ along M_t, where ξ_0^t is defined in terms of x_t by (20). Let $N^t = \operatorname{Re} \xi^t$ and $N_0^t = \operatorname{Re} \xi_0^t$. Then $N^t = N_0^t$ along M_t, in particular, $N^t|_{M_t}$ is already determined (independent of the choice of k_t). We now compute along M_t,

$$N^t\,\dot r_t = N_0^t(e^{2\omega_t}\dot x_t + 2\dot\omega_t e^{2\omega_t}x_t) \tag{79}$$
$$= N_0^t\dot x_t + 2(N_0^t\,\omega_t)\dot x_t + 2\dot\omega_t,$$

since $e^{2\omega_t}|_{M_t} = 1$. Note that along M_t we have $N_0^t\,\omega_t = N_0^t\,(k_t x_t) = k_t$ and $\dot\omega_t = k_t\dot x_t$. Hence, along M_t,

$$N^t\,\dot r_t = N_0^t\,\dot x_t + 4k_t\dot x_t. \tag{80}$$

We shall now make use of the additional assumption on the family f_t that $\operatorname{Re} f_t$ has a strict sign for all $t \in [0,\epsilon)$. This is equivalent to saying that $\dot x_t|_{M_t}$ has a strict sign for all $t \in [0,\epsilon)$. (It is also equivalent to saying that the smooth family of hypersurfaces M_t form a foliation, i.e. they do not intersect each other.) Since $\dot x_t|_{M_t}$ has a strict sign for all $t \in [0,\epsilon)$ we may choose k_t such that $N_0^t\,\dot x_t + 4k_t\dot x_t = 0$ along M_t. It follows that $N^t\,\dot r_t = 0$ along M_t.

We now choose a smooth family of complex functions $\underline{F_t}$ on X such that $\operatorname{Re}\underline{F_t} = -\dot r_t$ and $\operatorname{Im}\underline{F_t}\circ\psi_t = e^{2\gamma_t}\operatorname{Im} f_t$. Note that one then has $\underline{F_t}\circ\psi_t = e^{2\gamma_t} f_t$. Let $\underline{Y_t}$ be defined as in (66) and let $\underline{\Phi_t}$ denote its flow. Let Υ_t be defined by $r_t\circ\underline{\Phi_t} = e^{2\Upsilon_t}r_0$. Since $N^t\,\dot r_t = 0$ along M_t we have $N^t\operatorname{Re}\underline{F_t} = 0$ along M_t and hence, by Proposition 3.5 and the uniqueness of solutions for the transport equation used to determine ψ_t, γ_t, we have

$$\underline{\Phi_t}|_{M_0} = \psi_t \quad\text{and}\quad \Upsilon_t|_{M_0} = \gamma_t \tag{81}$$

for $t \in [0,\epsilon)$. By (39) the embedding ψ_t realizes the deformed CR structure corresponding to $\varphi(t)$ for each t. This proves Theorem 3.1.

Remark 3.7. Note that if $\varphi(t)$ and f_t are analytic in t (with values in some Banach space of functions), then ψ_t and γ_t are also analytic in t (with values in the corresponding Banach spaces).

4. Solutions to the Tangency Equation on S^3

We our now going to consider deformations of the standard CR structure on S^3. Recall that a smooth linearized deformation tensor $\dot\varphi_{11}$ on S^3 is embeddable, i.e. there is a family of embeddable deformations $\varphi_{11}(t)$ of the unit sphere S^3 in $\mathbb{C}^2$ with $\varphi_{11}(0) = 0$ such that $\dot\varphi_{11} = \frac{d}{dt}\big|_{t=0}\varphi_{11}(t)$, if and only if

$$(Z_1)^2 f = \dot\varphi_{11}$$

for some $f \in C^\infty(S^3,\mathbb{C})$. In this section we shall construct, in a canonical way, such a deformation $\varphi_{11}(t)$ for a given embeddable linearized deformation tensor $\dot\varphi_{11}$. We do this by constructing a smooth family of complex functions f_t and a smooth family of deformations $\varphi_{11}(t)$ satisfying (16) and then appealing to Theorem 3.1. We start with some preliminary calculations.

4.1. **Deformations of S^3 with its standard CR structure.** We shall continue to use the notation of Section 2, where we now take (M, H, J, θ) to be the standard pseudohermitian structure on S^3. Recall that $\theta = i(zd\bar{z} + wd\bar{w})$ restricted to the unit sphere S^3, where (z, w) are the coordinates on $\mathbb{C}^2$. We also recall that we are using the frame Z_1 as in (1). We observe that in this case we have

$$\omega_1{}^1 = -iR\theta, \quad A_{11} = 0 \tag{82}$$

where $R = 2$ is the Tanaka-Webster scalar curvature of S^3 with its standard structure.

Let $\varphi_1{}^{\bar{1}}(t)$ be a smooth family of deformation tensors on S^3 with its standard pseudohermitian structure and frame. We shall consider the family of pseudohermitian structures defined by the admissible coframes $(\theta, \theta_t^1, \theta_t^{\bar{1}})$ where θ_t^1 is defined as in (11). We shall compute $\omega_1{}^1(t)$ and $A_{11}(t)$ in terms of the deformation tensor $\varphi_1{}^{\bar{1}}(t)$. We differentiate θ_t^1 and then substitute for θ^1 and $\theta^{\bar{1}}$ using (12):

(83)

$$\begin{aligned}
d\theta_t^1 &= \frac{1}{2}\frac{d|\varphi|^2}{(1-|\varphi|^2)^{3/2}} \wedge (\theta^1 - \varphi_{\bar{1}}{}^1\theta^{\bar{1}}) + \frac{1}{\sqrt{1-|\varphi|^2}}(d\theta^1 - d\varphi_{\bar{1}}{}^1 \wedge \theta^{\bar{1}} - \varphi_{\bar{1}}{}^1 d\theta^{\bar{1}}) \\
&= -\frac{1}{2}\theta_t^1 \wedge \frac{d|\varphi|^2}{1-|\varphi|^2} + \frac{1}{\sqrt{1-|\varphi|^2}}(-iR\theta^1 \wedge \theta - iR\varphi_{\bar{1}}{}^1\theta^{\bar{1}} \wedge \theta + \theta^{\bar{1}} \wedge d\varphi_{\bar{1}}{}^1) \\
&= \frac{1}{1-|\varphi|^2}\theta_t^1 \wedge (-\frac{d|\varphi|^2}{2} - iR\theta - iR|\varphi|^2\theta + \varphi_1{}^{\bar{1}} d\varphi_{\bar{1}}{}^1 - (Z_1^t \lrcorner d\varphi_{\bar{1}}{}^1)\theta_t^{\bar{1}}) - \frac{\varphi_{\bar{1}}{}^1{}_{,0}}{1-|\varphi|^2}\theta \wedge \theta_t^{\bar{1}}
\end{aligned}$$

where $\varphi_1{}^{\bar{1}} = \varphi_1{}^{\bar{1}}(t)$ and we have used that $T \lrcorner d\varphi_{\bar{1}}{}^1 - 2iR\varphi_{\bar{1}}{}^1 = \varphi_{\bar{1}}{}^1{}_{,0}$. We note that

$$d|\varphi|^2 = \varphi_1{}^{\bar{1}} d\varphi_{\bar{1}}{}^1 + \varphi_{\bar{1}}{}^1 d\varphi_1{}^{\bar{1}},$$

and hence

$$\begin{aligned}
\frac{d|\varphi|^2}{2} - \varphi_1{}^{\bar{1}} d\varphi_{\bar{1}}{}^1 &= \frac{1}{2}(-\varphi_1{}^{\bar{1}} d\varphi_{\bar{1}}{}^1 + \varphi_{\bar{1}}{}^1 d\varphi_1{}^{\bar{1}}) \\
&= \frac{1}{2}\Big(-\varphi_1{}^{\bar{1}}(\varphi_{\bar{1}}{}^1{}_{,1}\theta^1 + \varphi_{\bar{1}}{}^1{}_{,\bar{1}}\theta^{\bar{1}}) + \varphi_{\bar{1}}{}^1(\varphi_1{}^{\bar{1}}{}_{,1}\theta^1 + \varphi_1{}^{\bar{1}}{}_{,\bar{1}}\theta^{\bar{1}})\Big) \mod \theta \\
&= \frac{1}{2\sqrt{1-|\varphi|^2}}(-|\varphi|^2\varphi_{\bar{1}}{}^1{}_{,1} - \varphi_1{}^{\bar{1}}\varphi_{\bar{1}}{}^1{}_{,\bar{1}} + (\varphi_{\bar{1}}{}^1)^2\varphi_1{}^{\bar{1}}{}_{,1} + \varphi_{\bar{1}}{}^1\varphi_1{}^{\bar{1}}{}_{,\bar{1}})\theta_t^{\bar{1}}
\end{aligned} \tag{84}$$

where the last equality is modulo θ and θ_t^1. We also note that the θ-component is given by

$$\frac{d|\varphi|^2}{2} - \varphi_1{}^{\bar{1}} d\varphi_{\bar{1}}{}^1 = \left(\frac{1}{2}(-\varphi_1{}^{\bar{1}}\varphi_{\bar{1}}{}^1{}_{,0} + \varphi_{\bar{1}}{}^1\varphi_1{}^{\bar{1}}{}_{,0}) - 2iR|\varphi|^2\right)\theta \mod \theta_t^1,\, \theta_t^{\bar{1}}. \tag{85}$$

Next, we note that

$$Z_1^t \lrcorner d\varphi_{\bar{1}}{}^1 = \frac{1}{\sqrt{1-|\varphi|^2}}\,(\varphi_{\bar{1}}{}^1{}_{,1} + \varphi_1{}^{\bar{1}}\varphi_{\bar{1}}{}^1{}_{,\bar{1}}). \tag{86}$$

By using also the fact that $\omega_1{}^1(t)$ is purely imaginary (since $h_{1\bar{1}}^t = 1$), the following lemma follows from the calculations (83)–(86) and the structure equation (6):

Lemma 4.1. *Let $M = S^3$ with its standard pseudohermitian structure. Then,*

$$\begin{aligned}
\omega_1{}^1{}_1(t) &= \frac{1}{2(1-|\varphi|^2)^{3/2}}\,(2\varphi_1{}^{\bar{1}}{}_{,\bar{1}} + \varphi_1{}^{\bar{1}}\varphi_{\bar{1}}{}^1{}_{,1} + \varphi_{\bar{1}}{}^1\varphi_1{}^{\bar{1}}{}_{,1} + (\varphi_1{}^{\bar{1}})^2\varphi_{\bar{1}}{}^1{}_{,\bar{1}} - |\varphi|^2\varphi_1{}^{\bar{1}}{}_{,\bar{1}})\\
\omega_1{}^1{}_0(t) &= -2i + \frac{1}{2(1-|\varphi|^2)}\,(\varphi_1{}^{\bar{1}}\varphi_{\bar{1}}{}^1{}_{,0} - \varphi_{\bar{1}}{}^1\varphi_1{}^{\bar{1}}{}_{,0})\\
A_{11}(t) &= -\frac{\varphi_{11,0}}{1-|\varphi|^2}.
\end{aligned} \tag{87}$$

Remark 4.2. The expression for $\omega_1{}^1{}_1(t)$ can be simplified by noting that

$$\begin{aligned}
(Z_1 + \varphi_1{}^{\bar{1}}Z_{\bar{1}})\,\frac{1}{\sqrt{1-|\varphi|^2}} &= \frac{1}{2(1-|\varphi|^2)^{3/2}}(Z_1 + \varphi_1{}^{\bar{1}}Z_{\bar{1}})|\varphi|^2\\
&= \frac{1}{2(1-|\varphi|^2)^{3/2}}(\varphi_1{}^{\bar{1}}\varphi_{\bar{1}}{}^1{}_{,1} + \varphi_{\bar{1}}{}^1\varphi_1{}^{\bar{1}}{}_{,1} + (\varphi_1{}^{\bar{1}})^2\varphi_{\bar{1}}{}^1{}_{,\bar{1}} + |\varphi|^2\varphi_1{}^{\bar{1}}{}_{,\bar{1}}).
\end{aligned}$$

Consequently, we may write

$$\omega_1{}^1{}_1(t) = \frac{\varphi_1{}^{\bar{1}}{}_{,\bar{1}}}{\sqrt{1-|\varphi|^2}} + \tilde{Z}_1^t\,\frac{1}{\sqrt{1-|\varphi|^2}}, \tag{88}$$

where

$$\tilde{Z}_1^t = Z_1 + \varphi_1{}^{\bar{1}}Z_{\bar{1}}. \tag{89}$$

4.2. **Spherical harmonics.** We shall denote the space of spherical harmonic polynomials of bidegree (p,q) on $S^3 \subset \mathbb{C}^2$ by $H_{p,q}$. We recall that the spherical harmonic spaces $H_{p,q}$ are eigenspaces for T acting on functions,

$$Tu = i(p-q)u, \tag{90}$$

and that Z_1 maps $H_{r,s}$ isomorphically onto $H_{r-1,s+1}$ when $r \geq 1$ and $Z_1 = 0$ on $H_{0,s}$. An immediate consequence of this is that $\dot{\varphi}_{11}$ is in the image of $(Z_1)^2$ if and only if the spherical harmonic expansion of $\dot{\varphi}_{11}$ has vanishing components in $H_{p,q}$ for $q = 0,1$, and the kernel of $(Z_1)^2$ is given by those complex functions whose only nontrivial components are in $H_{p,q}$ for $p = 0,1$. It follows that if $\dot{\varphi}_{11}$ is in the image of $(Z_1)^2$ then there is a unique complex function f such that $(Z_1)^2 f = \dot{\varphi}_{11}$ and f has vanishing components in $H_{p,q}$ for $p = 0,1$, i.e. f is L^2-orthogonal to the kernel of $(Z_1)^2$. Note that ∇_1 always acts as Z_1 on tensors of any type, since the connection form is given by $\omega_1{}^1 = -iR\theta$.

4.3. **Formally embeddable deformations.** We now assume that $\varphi_{11}(0) = 0$. We shall expand a potential solution f_t and deformation tensor $\varphi_{11}(t)$ satisfying (16) in powers of t as follows:

$$f_t = \sum_{k=0}^{\infty} f^{(k)} t^k \tag{91}$$

$$\varphi_{11}(t) = \sum_{k=1}^{\infty} \varphi^{(k)} t^k, \tag{92}$$

where $\varphi^{(1)} = \dot{\varphi}_{11}$. We shall identify terms in (16) with equal powers of t. We obtain for t^0:

$$(\nabla_1)^2 f^{(0)} = \dot{\varphi}_{11}, \tag{93}$$

the solvability of which is equivalent to $\dot{\varphi}_{11}$ being an embeddable linearized deformation. Before we proceed we first rewrite equation (16) explicitly in terms of the deformation $\varphi_{11}(t)$. We note that $\omega_1{}^1(0) = -Ri\theta$ implies that $\nabla_1 = \nabla^t_1|_{t=0}$ and $\nabla_{\bar{1}} = \nabla^t_{\bar{1}}|_{t=0}$ act on any tensor simply as Z_1 and $Z_{\bar{1}}$. The left hand side of (16) can be written, by using the expression for $A_{11}(t)$ in Lemma 4.1,

$$\begin{aligned} &\frac{1}{1-|\varphi|^2}\left(\nabla_1 + \varphi_1{}^{\bar{1}}\nabla_{\bar{1}}\right)^2 f_t - \omega_1{}^1{}_1 \frac{1}{\sqrt{1-|\varphi|^2}}\left(\nabla_1 + \varphi_1{}^{\bar{1}}\nabla_{\bar{1}}\right) f_t \\ &\quad + \frac{1}{\sqrt{1-|\varphi|^2}}\left(\left(\nabla_1 + \varphi_1{}^{\bar{1}}\nabla_{\bar{1}}\right)\frac{1}{\sqrt{1-|\varphi|^2}}\right)\left(\nabla_1 + \varphi_1{}^{\bar{1}}\nabla_{\bar{1}}\right) f_t - \frac{i\varphi_{11,0}}{1-|\varphi|^2} f_t, \end{aligned} \tag{94}$$

where we have also abbreviated $\omega_1{}^1{}_1 = \omega_1{}^1{}_1(t)$ and $\varphi_{11} = \varphi_{11}(t)$. By using also (88), we find that this simplifies to

$$\frac{1}{1-|\varphi|^2}\left(\left(\nabla_1 + \varphi_1{}^{\bar{1}}\nabla_{\bar{1}}\right)^2 f_t - \varphi_1{}^{\bar{1}}{}_{,\bar{1}}\left(\nabla_1 + \varphi_1{}^{\bar{1}}\nabla_{\bar{1}}\right) f_t - i\varphi_{11,0} f_t\right). \tag{95}$$

By canceling a factor of $(1-|\varphi|^2)^{-1}$ in (16), we obtain the equation

$$\left(\nabla_1 + \varphi_1{}^{\bar{1}}\nabla_{\bar{1}}\right)^2 f_t - \varphi_1{}^{\bar{1}}{}_{,\bar{1}}\left(\nabla_1 + \varphi_1{}^{\bar{1}}\nabla_{\bar{1}}\right) f_t - i\varphi_{11,0} f_t = \frac{d}{dt}\varphi_{11}. \tag{96}$$

The operator acting on f_t on the left hand side of this equation can be expressed as $(\nabla_1)^2 + L_\varphi$ where

$$L_\varphi = \varphi_1{}^{\bar{1}}\nabla_1\nabla_{\bar{1}} + \varphi_1{}^{\bar{1}}\nabla_{\bar{1}}\nabla_1 + (\varphi_1{}^{\bar{1}})^2(\nabla_{\bar{1}})^2 + \varphi_1{}^{\bar{1}}{}_{,1}\nabla_{\bar{1}} - \varphi_1{}^{\bar{1}}{}_{,\bar{1}}\nabla_1 - i\varphi_{11,0}. \tag{97}$$

We note that L_φ has a Taylor expansion

$$L_\varphi = \sum_{k=1}^{\infty} t^k L^{(k)} \tag{98}$$

where the operators $L^{(k)}$ are given by

$$L^{(k)} = \varphi^{(k)}\nabla_1\nabla_{\bar 1} + \varphi^{(k)}\nabla_{\bar 1}\nabla_1 + \sum_{j=1}^{k-1}\varphi^{(j)}\varphi^{(k-j)}(\nabla_{\bar 1})^2 + (\nabla_1\varphi^{(k)})\nabla_{\bar 1} - (\nabla_{\bar 1}\varphi^{(k)})\nabla_1 - i\nabla_0\varphi^{(k)}, \tag{99}$$

where we have used the notation in (92) and recall that $h_{1\bar 1} = 1$, so $\varphi_1{}^{\bar 1} = \varphi_{11}$.

For the proof of the following proposition we introduce the orthogonal (in L^2) projections $\mathcal{P}_1, \mathcal{P}_2$ onto the image of $(\nabla_1)^2$ (i.e., the subspace of functions with vanishing components in $H_{p,q}$ for $q = 0, 1$) and its orthogonal complement, the kernel of $(\nabla_{\bar 1})^2$ (i.e., with non-vanishing components only in $H_{p,q}$ for $q = 0, 1$).

Proposition 4.3. *Given a smooth embeddable linearized deformation tensor $\dot\varphi_{11}$, there are unique formal power series $f_t = \sum_{k=0}^\infty f^{(k)}t^k$ and $\varphi_{11}(t) = t\dot\varphi_{11} + \sum_{k=2}^\infty \varphi^{(k)}t^k$, with $f^{(k)}$ and $\varphi^{(k)}$ smooth, satisfying (96) such that for each k, $f^{(k)}$ has vanishing components in $H_{p,q}$ for $p = 0, 1$, and for each $k \geq 2$, $\varphi^{(k)}$ has vanishing components in $H_{p,q}$ for $q \geq 2$.*

Remark 4.4. Note that $\varphi(t)$ is of the form $t\dot\varphi + \psi(t)$ where $\psi(t) = \sum_{k=2}^\infty \varphi^{(k)}t^k$ takes values in $\mathfrak{D}_0^\perp$.

Proof. By identifying coefficients of t^k in (96) we get for t^0, $(\nabla_1)^2 f^{(0)} = \dot\varphi_{11}$, and for t^k, $k \geq 1$,

$$(\nabla_1)^2 f^{(k)} + \sum_{j=1}^{k} L^{(j)} f^{(k-j)} = (k+1)\varphi^{(k+1)}. \tag{100}$$

We take $f^{(0)}$ to be the unique solution of $(\nabla_1)^2 f^{(0)} = \dot\varphi_{11}$ with vanishing components in $H_{p,q}$ for $p = 0, 1$. For $k \geq 1$ we define $f^{(k)}$ and $\varphi^{(k+1)}$ recursively by decomposing $\sum_{j=1}^k L^{(j)} f^{(k-j)} = A_k + B_k$, where

$$A_k = \mathcal{P}_1 \sum_{j=1}^{k} L^{(j)} f^{(k-j)}, \quad B_k = \mathcal{P}_2 \sum_{j=1}^{k} L^{(j)} f^{(k-j)}, \tag{101}$$

and then defining $f^{(k)}$ to be the unique solution to $(\nabla_1)^2 f^{(k)} = -A_k$ with vanishing components in $H_{p,q}$ for $p = 0, 1$, and $\varphi^{(k+1)}$ to be $B_k/(k+1)$. The solutions are easily seen to be smooth by standard properties of the solution operator to $(\nabla_1)^2$. This concludes the proof. □

Remark 4.5. Note that in Proposition 4.3 we could have instead allowed the components of the $f^{(k)}$ in $H_{p,q}$ for $p = 0, 1$ to be arbitrary, since $(\nabla_1)^2$ annihilates $H_{p,q}$ for $p = 0, 1$. Doing this we obtain the general formal solution to the tangency equation (96). Below we shall use this flexibility and allow $f^{(0)}$ to have a nontrivial component in $H_{0,0}$.

4.4. **Deformations in the Burns-Epstein region.** Recall from the introduction that the space of Burns-Epstein deformations $\mathfrak{D}_{BE}$ is the set of all deformation tensors φ such that $\varphi_{p,q} = 0$ if $q < p + 4$. The following lemma follows easily by inspection of the definition of L_φ given in (97).

Lemma 4.6. *Let $\varphi_{11} \in \mathfrak{D}_{BE}$ and $f \in (Z_{\bar{1}})^2 \mathfrak{D}_{BE}$. Then $L_\varphi f \in \mathfrak{D}_{BE}$.*

An immediate consequence of this is the following:

Lemma 4.7. *Let $\varphi_{11}(t) = \sum_{k=1}^{\infty} \varphi^{(k)} t^k$ and $f_t = \sum_{k=0}^{\infty} f^{(k)} t^k$ be formal power series with values in $C^\infty(S^3, \mathbb{C})$. If $\varphi^{(j)} \in \mathfrak{D}_{BE}$, $1 \le j \le k$, and $f^{(j)} \in (Z_{\bar{1}})^2 \mathfrak{D}_{BE}$, $0 \le j \le k-1$, then $\sum_{j=1}^{k} L^{(j)} f^{(k-j)} \in \mathfrak{D}_{BE}$.*

Proof. The lemma follows by applying Lemma 4.6 to $\tilde{\varphi}_{11} = \sum_{j=1}^{k} \varphi^{(j)} t^j$ and $\tilde{f}_t = \sum_{j=0}^{k-1} f^{(j)} t^j$ and then taking the t^k coefficient of $L_{\tilde{\varphi}} \tilde{f}$, which is the same as the t^k coefficient of $L_\varphi f$. □

We let H^s_{FS} denote the Folland-Stein Sobolev space [20] of complex valued functions on S^3 with s derivatives in L^2 in the directions tangent to the contact distribution H. (Note that on these spaces any Reeb vector field, being a commutator of vector fields tangent to H, behaves like a second order operator.) We denote the norm on H^s_{FS} by $\|\cdot\|_s$ (we will also occasionally use the standard Sobolev norm, which we denote by $\|\cdot\|_{H^s}$).

Proposition 4.8. *Given $\dot{\varphi}_{11} \in \mathfrak{D}_{BE}$, the unique formal power series $f_t = \sum_{k=0}^{\infty} f^{(k)} t^k$ and $\varphi_{11}(t) = \sum_{k=1}^{\infty} \varphi^{(k)} t^k$ given by Proposition 4.3 satisfy*

$$\varphi_{11}(t) = t\dot{\varphi}_{11} \qquad \text{and} \qquad f^{(k)} \in (Z_{\bar{1}})^2 \mathfrak{D}_{BE} \tag{102}$$

for all k. Moreover, for every $s \ge 10$ there is $C_s > 0$ such that the formal power series $f_t = \sum_{k=0}^{\infty} f^{(k)} t^k$ converges for $|t| < R_s = (C_s \|\dot{\varphi}_{11}\|_s)^{-1}$ to an analytic function taking values in H^s_{FS}, and for each fixed t, $|t| < R_s$, f_t is a C^∞ function on S^3.

Remark 4.9. In this paper we will not be concerned with optimal regularity in the finite regularity case. The choice $s \ge 10$ in the proposition is only for convenience and is not optimal.

In the proof we will make use of the standard solution operator for the Kohn Laplacian $\Box_b = -\nabla^{\bar{1}} \nabla_{\bar{1}}$ on S^3, denoted as in [8] by $\mathcal{Q}_0$. It is straightforward to check that we can use $(\overline{\mathcal{Q}_0} Z_{\bar{1}})^2$ to invert $(\nabla_1)^2$ in Folland-Stein spaces. More precisely, we have the following estimate.

Lemma 4.10. *If $g \in H^s_{FS}$ and $\mathcal{P}_2 g = 0$ then $u = (\overline{\mathcal{Q}_0} Z_{\bar 1})^2 g$ solves $(\nabla_1)^2 u = g$. Moreover, there is a constant C depending only on s such that*

$$\|u\|_{s+2} \le C\|g\|_s \tag{103}$$

Proof. This follows from the definition of $\mathcal{Q}_0$ and the fact that it gains two derivatives in Folland-Stein space [8]. □

Proof of Proposition 4.8. By the construction in the proof of Proposition 4.3 and an induction using Lemma 4.7 we obtain (102). It remains to be shown that f_t is analytic in t, when viewed as taking values in the Banach space H^s_{FS}, and that for each fixed t the function f_t is C^∞. Writing $\dot\varphi = \dot\varphi_{11}$, we then have

$$L^{(1)} = \dot\varphi \nabla_1 \nabla_{\bar 1} + \dot\varphi \nabla_{\bar 1}\nabla_1 + (\nabla_1\dot\varphi)\nabla_{\bar 1} - (\nabla_{\bar 1}\dot\varphi)\nabla_1 - i\nabla_0\dot\varphi,$$

$L^{(2)} = \dot\varphi^2(\nabla_{\bar 1})^2$, and $L^{(j)} = 0$ for $j \ge 3$. Thus, the $f^{(k)}$ are determined by the equations $(\nabla_1)^2 f^{(0)} = \dot\varphi$, $(\nabla_1)^2 f^{(1)} = -L^{(1)} f^{(0)}$,

$$(\nabla_1)^2 f^{(k)} = -L^{(1)} f^{(k-1)} - L^{(2)} f^{(k-2)}, \quad k \ge 2,$$

and the fact that they are orthogonal to the kernel of $(\nabla_1)^2$. By Lemma 4.10 we have $f^{(0)} = (\overline{\mathcal{Q}_0} Z_{\bar 1})^2\dot\varphi$, $f^{(1)} = -(\overline{\mathcal{Q}_0} Z_{\bar 1})^2 L^{(1)} f^{(0)}$ and

$$f^{(k)} = -(\overline{\mathcal{Q}_0} Z_{\bar 1})^2 \left(L^{(1)} f^{(k-1)} + L^{(2)} f^{(k-2)}\right), \quad k \ge 2.$$

It follows that $\|f^{(0)}\|_s \le C\|\dot\varphi\|_{s-2} \le C\|\dot\varphi\|_s$ and, using the expressions for $L^{(1)}$ and $L^{(2)}$ above,

$$\|f^{(1)}\|_s \le 5C\|\dot\varphi\|_s \|f^{(0)}\|_s \tag{104}$$

and

$$\|f^{(k)}\|_s \le C\left(5\|\dot\varphi\|_s\|f^{(k-1)}\|_s + \|\dot\varphi\|_s^2\|f^{(k-2)}\|_s\right), \quad k \ge 2. \tag{105}$$

Choose C_s such that $C_s^2 > C(5C_s + 1)$. By induction using the above display we then conclude $\|f^{(k)}\|_s \le (C_s\|\dot\varphi\|_s)^{k+1}$ for all $k \in \{0, 1, 2, \dots\}$. This proves that $f_t = \sum_{k=0}^\infty f^{(k)} t^k$ converges for $|t| < (C_s\|\dot\varphi_{11}\|_s)^{-1}$ to an analytic function valued in H^s_{FS} functions on S^3.

To complete the proof, we shall now show that for each fixed t, $|t| < (C_s\|\dot\varphi_{11}\|_s)^{-1}$, f_t is C^∞ smooth. This follows by an elliptic regularity argument parallel to that given in the proof of Theorem 5.3 in [8]. Fix $s_0 \ge 10$ and t with $|t| < (C_{s_0}\|\dot\varphi_{11}\|_{s_0})^{-1}$, and let $f = f_t$ and $\varphi = t\dot\varphi$. By construction f is orthogonal to the kernel of $(\nabla_1)^2$ and satisfies $(\nabla_1)^2 f = -L_\varphi f + \dot\varphi$. Applying $(\overline{\mathcal{Q}_0} Z_{\bar 1})^2$ to this last equation we get

$$f = -(\overline{\mathcal{Q}_0} Z_{\bar 1})^2 L_\varphi f + (\overline{\mathcal{Q}_0} Z_{\bar 1})^2 \dot\varphi.$$

Letting $A = -(\overline{\mathcal{Q}_0} Z_{\bar{1}})^2 L_\varphi$ we then have

$$(I - A)f = (\overline{\mathcal{Q}_0} Z_{\bar{1}})^2 \dot{\varphi}. \tag{106}$$

Since $\overline{\mathcal{Q}_0} \in Op\, S^{-2}_{\mathcal{V}}$ (in the notation of the Heisenberg pseudodifferential calculus of Beals and Greiner [1]), it is easy to see that $I - A \in Op\, S^0_{\mathcal{V}} \subset Op\, S^0_{\frac{1}{2},\frac{1}{2}}$; here we are using the notation $Op\, S^m_{\frac{1}{2},\frac{1}{2}}$ for the classical pseudodifferential operators of type $(\frac{1}{2}, \frac{1}{2})$ and order m. As in [8] (where A is taken to be $\mathcal{Q}_0 Z_1 \overline{\varphi} Z_1$) it is easy to see that if $\|\varphi\|_{L^\infty(S^3)}$ is sufficiently small, then the principal symbol of $I - A$ is positive. The argument on pages 832-833 of [8] then shows that there is a constant K_s such that

$$\|u\|_{H^s} \leq K_s \|(I - A)u\|_{H^s}.$$

Applying this to (106) we have

$$\|f\|_{H^s} \leq K_s \|(\overline{\mathcal{Q}_0} Z_{\bar{1}})^2 \dot{\varphi}\|_{H^s} \leq K'_s \|\dot{\varphi}\|_{H^s}$$

where in the last inequality we have used that $(\overline{\mathcal{Q}_0} Z_{\bar{1}})^2 \in Op\, S^{-2}_{\mathcal{V}} \subset Op\, S^{-1}_{\frac{1}{2},\frac{1}{2}}$. Since $\dot{\varphi} \in C^\infty(S^3, \mathbb{C})$ it follows that $f \in C^\infty(S^3, \mathbb{C})$. □

In order to apply Theorem 3.1 to produce a family of embeddings realizing a family of deformations $t\dot{\varphi}$ with $\dot{\varphi} \in \mathfrak{D}_{BE}$ we need to ensure that the family of solutions f_t to the tangency equation are such that $\operatorname{Re} f_t$ has strict sign. To do this we modify Proposition 4.8 making use of the freedom to add a constant to $f_0 = f^{(0)}$ due to the kernel of $(\nabla_1)^2$ and prove the following theorem.

Theorem 4.11. *For any $s \geq 10$, $\lambda \in \mathbb{R}$, $R > 0$ there exists $\epsilon > 0$ such that if $\dot{\varphi}_{11} \in \mathfrak{D}_{BE}$ satisfies $\|\dot{\varphi}\|_s < \epsilon$ then there is a unique formal power series $f_t = \lambda + \tilde{f}_t = \lambda + \sum_{k=0}^\infty \tilde{f}^{(k)} t^k$ such that*

- (i) *$\tilde{f}^{(k)} \in (Z_{\bar{1}})^2 \mathfrak{D}_{BE}$ for $k \geq 0$;*
- (ii) *f_t converges for $|t| < R$ to an analytic function taking values in H^s_{FS}, and for each fixed t, $|t| < R$, f_t is a C^∞ function on S^3;*
- (iii) *f_t solves*

$$\nabla^t_1 \nabla^t_1 f_t + i A_{11}(t) f_t = \frac{\dot{\varphi}_{11}}{1 - |\varphi(t)|^2} \tag{107}$$

  where $\varphi_{11}(t) = t\dot{\varphi}_{11}$;
- (iv) *if $|\lambda| \geq 1$ and $R > 2$ then $\operatorname{Re} f_t$ has a strict sign for $|t| \leq 1$.*

Proof. Recall that the equation (107) for f_t is equivalent to the equation

$$(\nabla_1)^2 f_t + L_\varphi f_t = \dot{\varphi}_{11}.$$

In terms of $\tilde{f}_t$ this equation takes the form

$$(\nabla_1)^2\tilde{f}_t + L_\varphi \tilde{f}_t = \dot{\varphi}_{11} + it\lambda\nabla_0\dot{\varphi}_{11}.$$

Formally this equation is equivalent to $(\nabla_1)^2\tilde{f}^{(0)} = \dot{\varphi}_{11}$, $(\nabla_1)^2\tilde{f}^{(1)} = -L^{(1)}\tilde{f}^{(0)} + i\lambda\nabla_0\dot{\varphi}_{11}$,

$$(\nabla_1)^2\tilde{f}^{(k)} = -L^{(1)}\tilde{f}^{(k-1)} - L^{(2)}\tilde{f}^{(k-2)}, \quad k \geq 2.$$

Since $\dot{\varphi}_{11} \in \mathfrak{D}_{BE}$ implies $\nabla_0\dot{\varphi}_{11} \in \mathfrak{D}_{BE}$, the unique formal solvability of this equation for $\tilde{f}_t = \sum_{k=0}^{\infty} \tilde{f}^{(k)}t^k$ satisfying (i) follows easily by induction using Lemma 4.7 as in the proof of Proposition 4.8. Arguing as in the proof of Proposition 4.8 we obtain $\|\tilde{f}^{(0)}\|_s \leq C\|\dot{\varphi}\|_{s-2} \leq C\|\dot{\varphi}\|_s$,

$$\|\tilde{f}^{(1)}\|_s \leq C\left(5\|\dot{\varphi}\|_s\|\tilde{f}^{(0)}\|_s + |\lambda|\|\dot{\varphi}\|_s\right) \tag{108}$$

and

$$\|\tilde{f}^{(k)}\|_s \leq C\left(5\|\dot{\varphi}\|_s\|\tilde{f}^{(k-1)}\|_s + \|\dot{\varphi}\|_s^2\|\tilde{f}^{(k-2)}\|_s\right), \quad k \geq 2. \tag{109}$$

It then follows by induction that $\|f^{(k)}\|_s \leq (C_s\|\dot{\varphi}\|_s)^{k+1}$ for all $k \in \{0, 1, 2, \ldots\}$ provided $C_s > 0$ satisfies $C_s^2 \geq C(5C_s + 1)$ and

$$C^2\left(5 + \frac{|\lambda|}{C\|\dot{\varphi}\|_s}\right) \leq C_s^2. \tag{110}$$

If such a C_s has been chosen, the radius of convergence of the power series is at least $\rho = (C_s\|\dot{\varphi}\|_s)^{-1}$. In order to find such a $C_s > 0$ and $\epsilon > 0$ with $\rho \geq R$ when $\|\dot{\varphi}\|_s < \epsilon$ we rewrite the last displayed equation as

$$C_s^2 \geq C^2\left(5 + \frac{C_s}{C}\frac{|\lambda|}{C_s\|\dot{\varphi}\|_s}\right) = C^2\left(5 + \frac{C_s}{C}|\lambda|\rho\right). \tag{111}$$

The above displayed inequality can be satisfied by choosing C_s such that

$$C_s^2 \geq C^2\left(5 + \frac{C_s}{C}|\lambda|R\right). \tag{112}$$

and then choosing $\epsilon = (C_sR)^{-1}$. Increasing C_s (and consequently shrinking ϵ) if necessary we may also ensure that $C_s^2 \geq C(5C_s + 1)$. Thus we have proved that for any $s \geq 10$, $\lambda \in \mathbb{R}$, $R > 0$ there exists $\epsilon > 0$ such that if $\dot{\varphi}_{11} \in \mathfrak{D}_{BE}$ satisfies $\|\dot{\varphi}\|_s < \epsilon$ then there is a unique formal power series $f_t = \lambda + \tilde{f}_t = \lambda + \sum_{k=0}^{\infty} \tilde{f}^{(k)}t^k$ satisfying $\tilde{f}^{(k)} \in (Z_{\bar{1}})^2\mathfrak{D}_{BE}$ for $k \geq 0$ such that f_t converges for $|t| < R$ to an analytic function taking values in H^s_{FS}; by construction f_t solves (107). The C^∞ smoothness of $\tilde{f}_t$, and hence of f_t, for fixed t follows as in the proof of Proposition 4.8.

We note that from the construction of f_t above we have

$$\|f_t - \lambda\|_\infty \le \|\tilde{f}_t\|_s \le \sum_{k=0}^{\infty} \|\tilde{f}^{(k)}\|_s |t|^k \le \sum_{k=0}^{\infty} (C_s\|\dot{\varphi}\|_s)^{k+1}|t|^k \le \frac{C_s\|\dot{\varphi}\|_s}{1-|t|C_s\|\dot{\varphi}\|_s} \le \frac{1}{R-|t|}. \tag{113}$$

From this it is easy to see that if $|\lambda| \ge 1$ and $R > 2$ then $\operatorname{Re} f_t$ has a strict sign for $|t| \le 1$. □

Theorem 1.7 now follows from Theorems 4.11 and 3.1.

4.5. **Families of embeddable deformations with general linearized term.** We now return to the case of general embeddable infinitesimal deformations $\dot{\varphi}_{11}$, for which analyticity in t of our formal solution may no longer hold. Our aim is to prove the following theorem.

Theorem 4.12. *For any $s \ge 10$, $\lambda < 0$, $T > 0$ there exists $\epsilon > 0$ such that if $\dot{\varphi}_{11} \in \mathfrak{D}_0$ satisfies $\|\dot{\varphi}\|_s < \epsilon$ then there are unique $f_t = \lambda + \tilde{f}_t \in C^\infty([0,T]\times S^3, \mathbb{C})$ and $\varphi_{11}(t) \in C^\infty([0,T]\times S^3,\mathbb{C})$ such that*

- (i) *$\tilde{f}_t \in (Z_{\bar{1}})^2\,(C^\infty(S^3,\mathbb{C}))$ for all $t \in [0,T]$;*
- (ii) *$\varphi_{11}(t) = t\dot{\varphi}_{11} + \psi_{11}(t)$ where $\psi_{11}(t) \in \mathfrak{D}_0^\perp$ for all $t \in [0,T]$ and $\psi_{11}(t) = O(t^2)$;*
- (iii) *f_t and $\varphi_{11}(t)$ solve the tangency equation (16).*

In the proof of Theorem 4.12 we will solve (16) by splitting it into two equations using the L^2 orthogonal projection $\mathcal{P}_1$ from $\mathfrak{D}$ onto $\mathfrak{D}_0$ and the complementary projection $\mathcal{P}_2$ onto $\mathfrak{D}_0^\perp$. When we apply $\mathcal{P}_1$ to (16) we get an equation with main term $(\nabla_1)^2 f_t$, and when we apply $\mathcal{P}_2$ we get an equation with main term $\left(\frac{d}{dt} + i\lambda\nabla_0\right)\mathcal{P}_2\varphi$ where $i\nabla_0$ acts on the deformation tensor $\mathcal{P}_2\varphi$ as $iT-4$ (since ${\omega_1}^1{}_0 = -iR = -2i$). A crucial point in the proof is that the operator $-iT$ when applied to elements of $\mathfrak{D}_0^\perp$ behaves like the sublaplacian $(-(Z_1Z_{\bar{1}} + Z_{\bar{1}}Z_1)$ on the standard CR sphere S^3, which acts by $2pq+p+q$ on $H_{p,q}$), cf. (90). For the proof of Theorem 4.12 we will need the following lemmas which are based on this observation. The first lemma is an immediate consequence of the action of the Reeb vector field and the sublaplacian on the spherical harmonics and the description of the Folland-Stein spaces in terms of spherical harmonic decompositions (see, e.g., [9]).

Lemma 4.13. *There exist constants $\beta, \gamma > 0$ such that the operator $-iT$ satisfies the following energy estimate*

$$\beta\|u\|_1^2 \le \int_{S^3} u(-iT)u + \gamma\|u\|_{L^2}^2 \tag{114}$$

for any $u \in H^1_{FS}$ such that $u = \mathcal{P}_2 u$.

The energy estimate in Lemma 4.13 plus a bootstrap argument imply the following lemma. In the following we denote by $H^k([0,T];\mathcal{B})$ the Sobolev space of functions on $[0,T]$ taking values in a given Banach space $\mathcal{B}$.

Lemma 4.14. *Let $\lambda < 0$, $s \geq 0$. If $g \in \bigcap_{k=0}^{s} H^k([0,T]; H^{2s-2k}_{FS})$ satisfies $g = \mathcal{P}_2 g$ then there is a unique solution $u \in \bigcap_{k=0}^{s+1} H^k([0,T]; H^{2s+2-2k}_{FS})$ to*

$$\begin{cases} \left(\frac{d}{dt} + \lambda(iT-4)\right) u = g \\ \qquad\qquad u(0) = 0. \end{cases} \tag{115}$$

Moreover, there is a constant C depending only on λ and s such that

$$\sum_{k=0}^{s+1} \left\| \frac{d^k u}{dt^k} \right\|_{L^2([0,T];H^{2s+2-2k}_{FS})} \leq C \sum_{k=0}^{s} \left\| \frac{d^k g}{dt^k} \right\|_{L^2([0,T];H^{2s-2k}_{FS})}. \tag{116}$$

Remark 4.15. Note that there is of course nothing particularly special about the number -4 in the lemma, which could be replaced by any nonpositive real number. As explained above, this is the result that we will need in the proof of Theorem 4.12.

We will need the following refinement of Lemma 4.14, whose proof is similar.

Lemma 4.16. *Let $\lambda < 0$, $s \geq 0$. Assume P_t is a smooth 1-parameter family of operators in $Op\, S^2_{\mathcal{V}}$, $t \in [0,T]$, satisfying $\mathcal{P}_1 \circ P_t = 0$ and*

$$\int_{S^3} u P_t u \leq \frac{1}{8} \|u\|_1^2 \tag{117}$$

for all $u \in H^1_{FS}$ such that $u = \mathcal{P}_2 u$. If $g \in \bigcap_{k=0}^{s} H^k([0,T]; H^{2s-2k}_{FS})$ satisfies $g = \mathcal{P}_2 g$ then there is a unique solution $u \in \bigcap_{k=0}^{s+1} H^k([0,T]; H^{2s+2-2k}_{FS})$ to

$$\begin{cases} \left(\frac{d}{dt} + \lambda(iT-4) + P_t\right) u = g \\ \qquad\qquad u(0) = 0. \end{cases} \tag{118}$$

Moreover, there is a constant C depending only on λ and s such that

$$\sum_{k=0}^{s+1} \left\| \frac{d^k u}{dt^k} \right\|_{L^2([0,T];H^{2s+2-2k}_{FS})} \leq C \sum_{k=0}^{s} \left\| \frac{d^k g}{dt^k} \right\|_{L^2([0,T];H^{2s-2k}_{FS})}. \tag{119}$$

Proof of Theorem 4.12. We start by recalling that (16) is equivalent to

$$(\nabla_1)^2 f + L_\varphi f = \frac{d}{dt}\varphi \tag{120}$$

where $f = f_t$, $\varphi = \varphi_{11}(t)$ and L_φ is as in (97). As before let $\mathcal{P}_1$ denote the L^2 orthogonal projection onto the image of $(\nabla_1)^2$ and let $\mathcal{P}_2$ denote the complementary orthogonal projection onto the kernel of $(\nabla_{\bar{1}})^2$. It is easy to see that the above displayed equation will hold with $\mathcal{P}_1\varphi(t) = t\dot{\varphi}$, if and only if

$$\begin{aligned}(\nabla_1)^2 f + \mathcal{P}_1(L_\varphi f) &= \dot{\varphi}\\ \mathcal{P}_1\varphi' &= \dot{\varphi}\\ \mathcal{P}_2\varphi' - \mathcal{P}_2(L_\varphi f) &= 0\end{aligned}$$

where $\varphi' = \frac{d}{dt}\varphi$. Writing $f = \lambda + \tilde{f}$ and noting that $L_\varphi f = L_\varphi \tilde{f} - i\lambda\nabla_0\varphi$ we may write this system as

$$\begin{aligned}(\nabla_1)^2\tilde{f} + \mathcal{P}_1(L_\varphi\tilde{f}) - i\lambda\nabla_0(\mathcal{P}_1\varphi) &= \dot{\varphi}\\ \mathcal{P}_1\varphi' &= \dot{\varphi}\\ (\tfrac{d}{dt} + i\lambda\nabla_0)\mathcal{P}_2\varphi - \mathcal{P}_2(L_\varphi\tilde{f}) &= 0.\end{aligned}$$

This leads us to consider the operator $\mathcal{F}$ taking pairs $(\tilde{f}, \varphi)$ where $\tilde{f} \in L^2([0,T]; H^s_{FS})$, $\mathcal{P}_1\varphi \in H^1([0,T]; H^s_{FS})$, $\mathcal{P}_2\varphi \in L^2([0,T]; H^s_{FS}) \cap H^1([0,T]; H^{s-2}_{FS})$ and $\varphi(0) = 0$ to triples (a, b_1, b_2) with $a = \mathcal{P}_1 a \in L^2([0,T]; H^{s-2}_{FS})$, $b_1 = \mathcal{P}_1 b \in L^2([0,T]; H^s_{FS})$, and $b_2 = \mathcal{P}_2 b_2 \in L^2([0,T]; H^{s-2}_{FS})$ given by

(121)
$$\mathcal{F}(\tilde{f}, \varphi) = \Big((\nabla_1)^2\tilde{f} + \mathcal{P}_1(L_\varphi\tilde{f}) - i\lambda\nabla_0(\mathcal{P}_1\varphi),\ \tfrac{d}{dt}\mathcal{P}_1\varphi,\ (\tfrac{d}{dt} + i\lambda\nabla_0)\mathcal{P}_2\varphi - \mathcal{P}_2(L_\varphi\tilde{f})\Big).$$

We indicate the domain and range of $\mathcal{F}$ (that have just been specified) by $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Linearizing $\mathcal{F}$ around $(0,0)$ we obtain

$$D\mathcal{F}_{(0,0)}(\tilde{f}, \varphi) = \Big((\nabla_1)^2\tilde{f} - i\lambda\nabla_0(\mathcal{P}_1\varphi),\ \tfrac{d}{dt}\mathcal{P}_1\varphi,\ (\tfrac{d}{dt} + i\lambda\nabla_0)\mathcal{P}_2\varphi\Big). \tag{122}$$

By Lemmas 4.10 and 4.14 the map $D\mathcal{F}_{(0,0)}$ is a bijection from $\mathcal{B}_1$ to $\mathcal{B}_2$. Hence by the Banach space inverse function theorem there is a neighborhood of the origin in $\mathcal{B}_1$ which is mapped bijectively under $\mathcal{F}$ to a neighborhood of the origin in $\mathcal{B}_2$ (in particular, there is a neighborhood of the origin in $\mathcal{B}_1$ on which the linearization of $\mathcal{F}$ is invertible).

The preceding observations allow us to find a weak solution of our original equation provided $\|\dot{\varphi}\|_s$ is sufficiently small (we could obtain a finite regularity classical solution in this way if we liked). The main difficulty in establishing the smooth regularity of our solution lies in the fact that if φ is not (known to be) smooth, then we cannot apply the previously used regularity theory for the solution operator to $(\nabla_1)^2 + \mathcal{P}_1 \circ L_\varphi$ for φ small in L^∞. We therefore seek to apply the Nash-Moser inverse function theorem (in the form presented in [22]). Let $\mathcal{A}_1$ denote the subspace of $\mathcal{B}_1$ consisting of those (f, φ) that are C^∞ in space and time, viewed as a tame Frechet space with respect to the $H^{\frac{s}{2}}([0,T]; H^s_{FS}) \times H^{\frac{s}{2}}([0,T]; H^s_{FS})$-norms (the precise choice of norms here is not particularly important). Similarly, let $\mathcal{A}_2$ denote the subspace of $\mathcal{B}_2$ consisting of those

(a, b_1, b_2) that are C^∞ in space and time, viewed as a tame Frechet space in analogous fashion. We now need to show that for all $(\tilde{f}_0, \varphi_0)$ in a neighborhood of the identity in $\mathcal{A}_1$ the linearization $D\mathcal{F}_{(\tilde{f}_0,\varphi_0)}$ of $\mathcal{F}$ is an invertible tame Frechet map from $\mathcal{A}_1$ to $\mathcal{A}_2$ and that the inverse $D\mathcal{F}^{-1}_{(\tilde{f}_0,\varphi_0)} : \mathcal{A}_2 \to \mathcal{A}_1$ depends smoothly on $(\tilde{f}_0, \varphi_0)$; the only hard part will be showing that $D\mathcal{F}_{(\tilde{f}_0,\varphi_0)} : \mathcal{A}_1 \to \mathcal{A}_2$ is surjective, for $(\tilde{f}_0, \varphi_0)$ sufficiently small.

From the first part of the proof we know that for all $(\tilde{f}_0, \varphi_0)$ in some neighborhood of the origin in $\mathcal{A}_1$ we have that $D\mathcal{F}_{(\tilde{f}_0,\varphi_0)}(\tilde{f}, \varphi) = (a, b_1, b_2)$ has a unique solution $(\tilde{f}, \varphi) \in \mathcal{B}_1$ for all $(a, b_1, b_2) \in \mathcal{A}_2$. We need to show that (provided $\tilde{f}_0$ and φ_0 are sufficiently small) if $(\tilde{f}, \varphi) \in \mathcal{B}_1$ solves $D\mathcal{F}_{(\tilde{f}_0,\varphi_0)}(\tilde{f}, \varphi) = (a, b_1, b_2)$ with $(a, b_1, b_2) \in \mathcal{A}_2$ then $\tilde{f}$ and φ are C^∞ (in space and time). We first observe that $D\mathcal{F}_{(\tilde{f}_0,\varphi_0)}(\tilde{f}, \varphi) = (a, b_1, b_2)$ may be written as

$$(\nabla_1)^2 \tilde{f} + \mathcal{P}_1(L_{\varphi_0}\tilde{f} + A_{\tilde{f}_0,\varphi_0}\varphi) - i\lambda\nabla_0(\mathcal{P}_1\varphi) = a \tag{123}$$

$$\tfrac{d}{dt}\mathcal{P}_1\varphi = b_1 \tag{124}$$

$$(\tfrac{d}{dt} + i\lambda\nabla_0)\mathcal{P}_2\varphi - \mathcal{P}_2(L_{\varphi_0}\tilde{f} + A_{\tilde{f}_0,\varphi_0}\varphi) = b_2 \tag{125}$$

where $A_{\tilde{f}_0,\varphi_0}$ is a first order operator involving only spatial derivatives, but second order acting between Folland-Stein spaces (since it involves $-i\tilde{f}_0\nabla_0$). The only fact we need about $A_{\tilde{f}_0,\varphi_0}$ is that it has leading term $-i\tilde{f}_0\nabla_0$ as an operator in $Op\, S^2_{\mathcal{V}}$. Note that $\mathcal{P}_1\varphi(t) = \int_0^t b_1(\tau)d\tau$ is clearly C^∞ in space and time (since b_1 is). We therefore only need to show that $\tilde{f}$ and $\mathcal{P}_2\varphi$ are smooth. Eliminating $\mathcal{P}_1\varphi$ from the above displayed equations we obtain a system of equations of the form

$$((\nabla_1)^2 + \mathcal{P}_1 L_{\varphi_0})\tilde{f} + \mathcal{P}_1 A_{\tilde{f}_0,\varphi_0}\mathcal{P}_2\varphi = \tilde{a} \tag{126}$$

$$(\tfrac{d}{dt} + i\lambda\nabla_0 - \mathcal{P}_2 A_{\tilde{f}_0,\varphi_0})\mathcal{P}_2\varphi - \mathcal{P}_2 L_{\varphi_0}\tilde{f} = \tilde{b}_2 \tag{127}$$

where $\tilde{a} = \mathcal{P}_1\tilde{a}$ and $\tilde{b}_2 = \mathcal{P}_2\tilde{b}_2$ are C^∞ in space and time. Recall that $\tilde{f} \in L^2([0,T]; H^s_{FS})$ and $\mathcal{P}_1\varphi \in L^2([0,T]; H^s_{FS}) \cap H^1([0,T]; H^{s-2}_{FS})$. Our approach to the regularity theory for the above system is based on the observation that for $\tilde{f}_0, \varphi_0$ small (in an appropriate sense) the operator $(\nabla_1)^2 + \mathcal{P}_1 L_{\varphi_0}$ can be seen as small perturbation of $(\nabla_1)^2$, for which we have a good solution operator gaining two derivatives in Folland-Stein spaces, and the operator $\frac{d}{dt} + i\lambda\nabla_0 - \mathcal{P}_2 A_{\tilde{f}_0,\varphi_0}$ can be seen as a small perturbation of the operator $\frac{d}{dt} + i\lambda\nabla_0$, which behaves like a heat operator when acting on the image of $\mathcal{P}_2$ (provided we use Folland-Stein norms to measure spatial regularity). The first step is to use the subellicity of the equation (126) for $\tilde{f}$ to show that $\tilde{f}$ is as regular as $\mathcal{P}_2\varphi$ is. Applying the operator $(\mathcal{Q}_0 Z_{\bar{1}})^2$ (the solution operator for $(\nabla_1)^2$) to (126) we obtain

$$(1 + (\mathcal{Q}_0 Z_{\bar{1}})^2 \mathcal{P}_1 L_{\varphi_0})\tilde{f} = (\mathcal{Q}_0 Z_{\bar{1}})^2(\tilde{a} - \mathcal{P}_1 A_{\tilde{f}_0,\varphi_0}\mathcal{P}_2\varphi). \tag{128}$$

Arguing as previously, the pseudodifferential operator $(\overline{\mathcal{Q}_0} Z_{\bar{1}})^2 \mathcal{P}_1 \circ L_{\varphi_0}$ is of order zero and its principal symbol will be small provided φ_0 is sufficiently small in L^∞ (in space and time). Hence, for φ_0 sufficiently small in L^∞, $(1 + (\mathcal{Q}_0 Z_{\bar{1}})^2 \mathcal{P}_1 L_{\varphi_0})$ is an invertible operator of order zero and hence

$$\tilde{f} = (1 + (\mathcal{Q}_0 Z_{\bar{1}})^2 \mathcal{P}_1 L_{\varphi_0})^{-1} (\mathcal{Q}_0 Z_{\bar{1}})^2 (\tilde{a} - \mathcal{P}_1 A_{\tilde{f}_0, \varphi_0} \mathcal{P}_2 \varphi). \tag{129}$$

So, modulo the addition of a term which is smooth in space and time, $\tilde{f}$ is given by a zero order pseudodifferential operator applied to $\mathcal{P}_2\varphi$. It follows that $\tilde{f}$ has precisely the same smoothness properties as $\mathcal{P}_2\varphi$. In particular, $\tilde{f} \in H^1([0,T]; H^{s-2}_{FS})$. Moreover, we may eliminate $\tilde{f}$ from (127) to obtain an equation of the form

$$(\tfrac{d}{dt} + i\lambda \nabla_0 + P_t)\mathcal{P}_2 \varphi = \tilde{b}_2' \tag{130}$$

for $\mathcal{P}_2\varphi$, where $\tilde{b}_2'$ is C^∞ and P_t is a smooth 1-parameter family of operators in $Op\, S^2_{\mathcal{V}}$, $t \in [0,T]$, satisfying $\mathcal{P}_1 \circ P_t = 0$ and

$$\int u P_t u \leq \frac{1}{8} \|u\|_1^2 \tag{131}$$

for all $u \in H^1_{FS}$ provided $\tilde{f}_0$, φ_0 are sufficiently small in the H^2_{FS}-norm. Recalling that $i\nabla_0$ acts on $\mathcal{P}_2\varphi$ as $iT - 4$, by Lemma 4.16 it follows that $\mathcal{P}_2\varphi$ is C^∞ in space and time, and hence so is $\tilde{f}$. This proves that $D\mathcal{F}_{(\tilde{f}_0, \varphi_0)}$ is an invertible map from $\mathcal{A}_1$ to $\mathcal{A}_2$ for all $\tilde{f}_0, \varphi_0$ sufficiently small. That $D\mathcal{F}_{(\tilde{f}_0, \varphi_0)}$ is a tame Frechet map follows immediately from the formula for this map; that $D\mathcal{F}^{-1}_{(\tilde{f}_0, \varphi_0)}$ is tame and that it depends smoothly on $\tilde{f}_0, \varphi_0$ can be seen from the above proof of invertibility. Theorem 4.12 now follows by the Nash-Moser inverse function theorem. □

We conclude this section by observing that Theorem 1.5 now follows from Theorems 4.12 and 3.1 (cf. also Remark 3.7 for Theorem 1.5 (ii)).

5. Proof of Theorem 1.2 and Theorem 1.3

In this section we shall prove the slice theorems, Theorem 1.2 and Theorem 1.3, described in the introduction. The proof of Theorem 1.2 is an application of the Nash-Moser inverse function theorem (along the lines of [13], Theorem B).

Proof of Theorem 1.2. In a slight abuse of notation we identify $\mathcal{C} \times (\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp) \times \{y_0\}$ with $\mathcal{C} \times (\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp)$. One can define a natural action of $\mathcal{C}$ on $\mathcal{C} \times (\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp)$ so that the map P in the statement of the theorem is equivariant (cf. [13], pp. 1284-1285). In order to check the conditions of the Nash-Moser inverse function theorem we need to consider the linearization of the map P in Theorem 1.2 at all points in a neighborhood of $(\mathrm{id}, 0)$ in $\mathcal{C} \times (\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^\perp)$; by the $\mathcal{C}$-equivariance of P it will suffice to consider only points of the form (id,φ), as in the proof of Theorems A and B in [13]. Recall that $\mathfrak{D}^m \cong \mathfrak{D}\times Y$, where Y is the CR Cartan bundle of S^3 (which may be identified with $\mathrm{SU}(2,1)$ modulo its finite center). As in Cheng-Lee [13] we write $P=(P_1,P_2)$ where P_1 takes values in $\mathfrak{D}$ and P_2 takes values in Y. In order to compute the linearization of $P=(P_1,P_2)$ we will make use of the local smooth tame parametrization $\Psi_e : C^\infty(S^3,\mathbb{R})\to\mathcal{C}$ of the contact diffeomorphism group in a neighborhood of the identity given in Theorem C of [13] (we identify these two spaces in the calculation below, and refer to points in $C^\infty(S^3,\mathbb{R})$ rather than $\mathcal{C}$; so, e.g., we write 0 instead of id). Using this parametrization, the linearization of P_1 at $(0,\varphi)$ is given by

$$DP_1(0,\varphi)(\dot f,\dot\varphi) = ((\nabla_1)^2 + L_\varphi)\dot f + \dot\varphi \tag{132}$$

for $\dot f\in C^\infty(S^3,\mathbb{R})$, $\dot\varphi\in\mathfrak{D}'_{BE}\oplus\mathfrak{D}_0^\perp$ where L_φ is as in (97) (cf. [13], equation (5.1); note that we are calculating exclusively in terms of the frame Z_1). We decompose $C^\infty(S^3,\mathbb{R})$ as a direct sum $C^\infty_{CR}(S^3,\mathbb{R})\oplus C^\infty_\perp(S^3,\mathbb{R})$ where $C^\infty_{CR}(S^3,\mathbb{R}) = \ker(\nabla_1)^2\cap C^\infty(S^3,\mathbb{R}) = \bigoplus_{p,q\in\{0,1\}} H_{p,q}\cap C^\infty(S^3,\mathbb{R})$ is the 8-dimensional space of potential functions for the infinitesimal CR automorphisms of the standard S^3, and $C^\infty_\perp(S^3,\mathbb{R}) = \overline{\bigoplus_{p,q\neq 0,1} H_{p,q}}\cap C^\infty(S^3,\mathbb{R})$. We also decompose $\mathfrak{D}=\mathfrak{D}'_{BE}\oplus(Z_1)^2(C^\infty(S^3,\mathbb{R}))\oplus\mathfrak{D}_0^\perp$ and let $\Pi:\mathfrak{D}\to(Z_1)^2(C^\infty(S^3,\mathbb{R})) = (Z_1)^2(C^\infty_\perp(S^3,\mathbb{R}))$ denote the corresponding projection (the projection is oblique, but is bounded in H^s_{FS} for every s, cf. [8, page 833]). Note that if $\dot\varphi\in\mathfrak{D}'_{BE}\oplus\mathfrak{D}_0^\perp$ then $\Pi\dot\varphi=0$. We construct a family of inverse maps $VP(0,\varphi)$ to the family of linearized maps $DP(0,\varphi)$ as follows. Given $(K,X)\in\mathfrak{D}\times\mathfrak{su}(2,1)\cong T_{(0,\varphi)}\mathfrak{D}^m$ we need to solve uniquely the following linear equations

$$\Pi DP_1(0,\varphi)(\dot f,\dot\varphi) = (\nabla_1)^2\dot g + \Pi L_\varphi \dot g + \Pi L_\varphi \dot h = \Pi K \tag{133}$$

$$(\mathrm{id}-\Pi)DP_1(0,\varphi)(\dot f,\dot\varphi) = \dot\varphi + (\mathrm{id}-\Pi)L_\varphi\dot f = (\mathrm{id}-\Pi)K \tag{134}$$

$$DP_2(0,\varphi)(\dot f,\dot\varphi) = X \tag{135}$$

where $\dot f = \dot g+\dot h$ with $\dot h\in C^\infty_{CR}(S^3,\mathbb{R})$ and $\dot g\in C^\infty_\perp(S^3,\mathbb{R})$. As in the proof of Proposition 4.8, by an elliptic regularity argument the map

$$(\nabla_1)^2 + \Pi L_\varphi : C^\infty_\perp(S^3,\mathbb{R}) \to (Z_1)^2(C^\infty_\perp(S^3,\mathbb{R})) \tag{136}$$

has a smooth tame solution operator $((\nabla_1)^2+\Pi L_\varphi)^{-1}$ for φ sufficiently small (in the L^∞ sense) with smooth tame dependence on φ. Using this solution operator we may solve (133) for $\dot g$, viewing $\dot h\in C^\infty_{CR}(S^3,\mathbb{R})$ as a free 8-dimensional parameter for now. One may then simply choose $\dot\varphi$ to satisfy (134), again viewing $\dot h$ as a parameter. Plugging the solutions for $\dot g$ and $\dot\varphi$ into (135) yields a finite dimensional equation to be solved for $\dot h$ in terms of X; solvability for small φ follows easily by the standard finite dimensional inverse function theorem after checking that this map is injective at $(0,\varphi)=(0,0)$, where the map becomes an identification between potentials for infinitesimal CR automorphisms and the corresponding elements of $\mathfrak{su}(2,1)$. This establishes the existence of a smooth tame family $VP(0,\varphi)$ of inverses to the family $DP(0,\varphi)$ of linearized maps, for sufficiently small φ. Part (i) of the theorem now follows by the Nash-Moser inverse function theorem.

Part (ii) follows easily from inspecting the linearized action of the contact diffeomorphisms on the slice. □

We claim that the restriction of the map $P : \mathcal{C} \times (\mathfrak{D}'_{BE} \oplus \mathfrak{D}_0^{\perp}) \times \{y_0\} \to \mathfrak{D}^m$ from Theorem 1.2 to $P_{emb} : \mathcal{C} \times \mathfrak{D}'_{BE} \times \{y_0\} \to \mathfrak{D}^m$ locally parametrizes the set of marked embeddable deformations of the standard CR sphere. This will be proved in Proposition 5.2 below. An argument from [6] (which will be fleshed out in the proof of Proposition 5.2 below) shows that the natural map $\mathcal{C} \times \mathfrak{D}_{BE} \times \{y_0\} \to \mathfrak{D}^m$ is surjective; but this map only becomes injective after we further restrict to the map $P_{emb} : \mathcal{C} \times \mathfrak{D}'_{BE} \times \{y_0\} \to \mathfrak{D}^m$. In order to show that the restricted map P_{emb} is surjective we need the following lemma. Let $\mathfrak{D}_{\mathrm{cd}}$ denote the set of smooth deformation tensors on the standard CR 3-sphere whose spherical harmonic decomposition is supported on the critical diagonal, i.e. the deformation tensors $\varphi = \sum_p \varphi_{p,p+4}$. Note that $\mathfrak{D}_{\mathrm{cd}}$ is precisely the space of deformation tensors corresponding to S^1-invariant CR structures. Let $\mathfrak{D}'_{\mathrm{cd}} = \{\, \varphi \in \mathfrak{D}_{\mathrm{cd}} \mid \mathrm{Im}(\nabla^1)^2\varphi = 0\,\}$.

Lemma 5.1. *Let $\varphi_0 \in \mathfrak{D}_{\mathrm{cd}}$ be sufficiently small. Then there exists an S^1-equivariant contact diffeomorphism of S^3 (unique modulo S^1-equivariant automorphisms of the CR sphere) pulling the CR structure corresponding to φ_0 back to one with deformation tensor $\tilde{\varphi}_0 \in \mathfrak{D}'_{\mathrm{cd}}$. Moreover, the contact diffeomorphism can be chosen to smoothly depend on φ_0.*

Proof. The argument is to establish a slice theorem essentially as in the proof of Theorem 1.2, but can be made slightly simpler due to the relevant automorphism group now being compact so that markings are not needed (this is analogous to the situation of Theorem A in [13]). Let $\mathcal{C}^{S^1}$ denote the space of S^1-equivariant contact diffeomorphisms of S^3 and let $H \subseteq \mathrm{Aut}(S^3)$ denote the subgroup of group of S^1-equivariant automorphism of the CR sphere S^3. By restricting the local parametrization $\Psi_e : C^\infty(S^3,\mathbb{R}) \to \mathcal{C}$ of the contact diffeomorphism group (in a neighborhood of the identity) given in Theorem C of [13] to S^1-invariant functions we obtain a smooth tame parametrization $C^\infty(S^3,\mathbb{R})^{S^1} \to \mathcal{C}^{S^1}$ of $\mathcal{C}^{S^1}$ in a neighborhood of the identity. As in Theorem D of [13], by restricting this map to the space $\mathfrak{W}$ of functions f in $C^\infty(S^3,\mathbb{R})$ with spherical harmonic decomposition of the form $f = \sum_{p\geq 2} f_{p,p}$ (i.e. functions $f \in C^\infty(S^3,\mathbb{R})^{S^1}$ with $f_{0,0} = f_{1,1} = 0$) we obtain, as the image of the restricted map, a local slice $\mathcal{W} \subseteq \mathcal{C}^{S^1}$ for the coset space $\mathcal{C}^{S^1}/H$.

Let $P_0 : \mathcal{W} \times \mathfrak{D}'_{\mathrm{cd}} \to \mathfrak{D}_{\mathrm{cd}}$ denote the natural map (where the contact diffeomorphism acts by pullback on the CR structure corresponding to the deformation tensor). One can define a natural action of $\mathcal{C}^{S^1}$ on $\mathcal{W} \times \mathfrak{D}'_{\mathrm{cd}}$ so that the map P_0 is equivariant (cf. [13], pp. 1284-1285). In order to check the conditions of the Nash-Moser inverse function theorem we need to consider the linearization of P_0 at all points in a neighborhood of $(\mathrm{id}, 0)$ in $\mathcal{C}^{S^1} \times \mathfrak{D}_{\mathrm{cd}}$; by the equivariance of P_0 it will suffice to consider only points of the form (id, φ), as in the proof of Theorem A in [13]. In order to compute the linearization we will make use of the local smooth tame parametrization of $\mathcal{W}$ by the space of functions $\mathfrak{W}$ (we identify these two spaces in the calculation below, and refer to points in $\mathfrak{W}$ rather than $\mathcal{W}$). Using this parametrization, the linearization of P_0 at $(0, \varphi)$ is given by

$$DP_0(0,\varphi)(\dot{f},\dot{\varphi}) = ((\nabla_1)^2 + L_\varphi)\dot{f} + \dot{\varphi} \tag{137}$$

for $\dot{f} \in \mathfrak{W}$, $\dot{\varphi} \in \mathfrak{D}'_{cd}$ where L_φ is as in (97) (cf. [13], equation (5.1)). We construct a family of inverse maps $VP_0(0,\varphi)$ to the family of linearized maps $DP_0(0,\varphi)$ as follows. Let $\Pi_0 : \mathfrak{D}_{cd} \to {\mathfrak{D}'_{cd}}^{\perp} \subseteq \mathfrak{D}_{cd}$ denote the L^2 orthogonal projection. (Note that ${\mathfrak{D}'_{cd}}^{\perp}$ is the image of $\mathfrak{W}$, or equivalently of $C^\infty(S^3,\mathbb{R})^{S^1}$, under $(Z_1)^2$.) For $\chi \in \mathfrak{D}_{cd}$ we write $\chi = \chi_1 + \chi_2$ where $\chi_1 = \Pi_0\chi$ and $\chi_2 = (\mathrm{id} - \Pi_0)\chi \in \mathfrak{D}'_{cd}$. Given $\chi \in \mathfrak{D}_{cd}$ we first solve

$$((\nabla_1)^2 + \Pi_0 L_\varphi)\dot{f} = \chi_1$$

using the same argument as in the proof of Proposition 4.8. As in the proof of Proposition 4.8, by an elliptic regularity argument the map $((\nabla_1)^2 + \Pi_0 L_\varphi) : \mathfrak{W} \to {\mathfrak{D}'_{cd}}^{\perp}$ has a smooth tame inverse for sufficiently small φ. Since we are free to choose $\dot{\varphi} \in \mathfrak{D}'_{cd}$ to solve the $(\mathrm{id} - \Pi_0)$ projection of

$$((\nabla_1)^2 + L_\varphi)\dot{f} + \dot{\varphi} = \chi$$

we obtain a smooth family of inverses $VP_0(0,\varphi)$. Thus, by the Nash-Moser inverse function theorem, given any sufficiently small deformation tensor φ_0 there exists a (unique small) S^1-equivariant contact diffeomorphism (in $\mathcal{W}$) pulling the corresponding CR structure back to one with deformation tensor $\tilde{\varphi}_0 \in \mathfrak{D}'_{cd}$.

□

Proposition 5.2. *Fix any marking y_0 of the standard CR sphere. The natural map $P_{emb} : \mathcal{C} \times \mathfrak{D}'_{BE} \times \{y_0\} \to \mathfrak{D}^m$ is a local bijection from an open neighborhood of $(\mathrm{id}, 0, y_0)$ to an open neighborhood of $(0, y_0)$ in the subset $\mathfrak{D}^m_{emb}$ of marked embeddable deformations of the standard CR 3-sphere.*

Proof. That P_{emb} maps $\mathcal{C} \times \mathfrak{D}'_{BE} \times \{y_0\}$ into $\mathfrak{D}^m_{emb}$ follows from [8, Theorem 5.3], cf. also Theorem 1.7 in this paper. Injectivity then follows from Theorem 1.2 above. To see that the map is surjective, we first let φ be an embeddable deformation, with φ

sufficiently small such that there is an embedding $\Phi : S^3 \to \mathbb{C}^2$ with image a strictly convex hypersurface near the standard sphere realizing φ (i.e. such that $\Phi_*(Z_1 + {\varphi_1}^{\bar{1}} Z_{\bar{1}})$ is a $(1,0)$-vector field along the image of Φ). The hypersurface $M = \Phi(S^3)$ bounds a convex domain Ω which has Kobayashi indicatrix $B \subseteq T_0\mathbb{C}^2 \cong \mathbb{C}^2$ based at 0. Let $\Psi : B \to \Omega$ denote the circular representation of Ω [25, 2], which is smooth up to the boundary (and away from the origin). By [2], equation (3.5), $\Psi|_{\partial B}$ is a contact diffeomorphism from ∂B to $M = \partial\Omega$ and so the CR structure on M pulls back to a deformation $\varphi_{M,\partial B}$ of the CR structure on ∂B; moreover, $\overline{\varphi_{M,\partial B}}$ (when expressed in terms of an S^1-invariant framing) has only positive Fourier coefficients with respect to the natural S^1 action on ∂B [2].

The radial projection from ∂B to S^3 is clearly S^1-equivariant, but is not (in general) a contact diffeomorphism (so it can be thought of as endowing S^3 with a second S^1-invariant contact distribution). We may correct for this possible discrepancy by using the S^1-invariant version of Gray's classical theorem (since our two contact distributions on S^3 are isotopic through S^1-invariant contact distributions), which tells us that there exists an S^1-invariant contact diffeomorphism from ∂B to S^3. This contact diffeomorphism allows us to push forward the intrinsic CR structure on ∂B to an S^1-invariant CR structure on S^3 compatible with the standard contact distribution H; with respect to the standard frame Z_1 on S^3 this CR structure has deformation tensor $\tilde{\varphi}_{\partial B,S^3} \in \mathfrak{D}_{cd}$ (by S^1-invariance). Using Lemma 5.1 there exists an S^1-equivariant contact diffeomorphism which pulls the CR structure corresponding to $\tilde{\varphi}_{\partial B,S^3}$ back to one with deformation tensor $\varphi_{\partial B,S^3} \in \mathfrak{D}'_{cd}$. We shall denote by ψ the contact diffeomorphism $\partial B \to S^3$ that pushes forward the CR structure on ∂B to the one on S^3 with deformation tensor $\varphi_{\partial B,S^3} \in \mathfrak{D}'_{cd}$. Using this contact diffeomorphism we may also push forward the CR structure with deformation tensor $\varphi_{M,\partial B}$ from ∂B to one on S^3 with deformation tensor φ_{M,S^3}. Note that φ_{M,S^3} (which describes the CR structure of M relative to S^3) differs from $\psi_*\varphi_{M,\partial B}$, since the latter is the deformation tensor for the CR structure of M relative to the CR structure of ∂B after we have identified ∂B and M with S^3 using the circular representation and the map ψ. Knowing that the deformation tensor of ∂B relative to S^3 is $\varphi_{\partial B,S^3}$ and the deformation tensor of M relative to ∂B is $\psi_*\varphi_{M,\partial B}$, it is easy to show that

$$\varphi_{M,S^3} = \frac{\psi_*\varphi_{M,\partial B} + \varphi_{\partial B,S^3}}{1 + (\psi_*\varphi_{M,\partial B}) \cdot \overline{\varphi_{\partial B,S^3}}}. \tag{138}$$

We now claim that $\varphi_{M,S^3} \in \mathfrak{D}'_{BE}$. Choose a unitary S^1-invariant framing $Z_1^{\partial B}$ for the CR structure on ∂B and similarly a unitary S^1-invariant framing Z_1^0 on the standard CR sphere S^3. Working in these frames the identity (138) becomes an identity of functions, and $\psi_*\varphi_{M,\partial B}$ is just $\varphi_{M,\partial B} \circ \psi^{-1}$. Since ψ is S^1-equivariant, it follows from (138) that φ_{M,S^3} has only non-positive Fourier coefficients, and moreover, the zeroth Fourier component of

φ_{M,S^3} is simply $\varphi_{\partial B,S^3}$. When expressed in the standard framing $Z_1, Z_{\bar{1}}$ of S^3 (which are not S^1 invariant since $\mathcal{L}_T Z_1 = -2iZ_1$) then Fourier coefficients are shifted by -4 $(= -2-2)$ and hence, viewed in this frame, the deformation φ_{M,S^3} lies in $\mathfrak{D}_{BE}$. Moreover, since the spherical harmonic coefficients of φ_{M,S^3} agree with $\varphi_{\partial B,S^3}$ along the critical diagonal (which corresponded to the zeroth Fourier mode when using S^1-invariant framings) and $\varphi_{\partial B,S^3} \in \mathfrak{D}'_{cd}$ we have that $\varphi_{M,S^3} \in \mathfrak{D}'_{BE}$ when expressed in terms of the standard framing for S^3. This establishes that (φ, y_1) is in the image of P_{emb} for some marking y_1.

It remains to show that (φ, y) is in the image of P_{emb} for all markings y in a uniform neighborhood of y_0 (for φ sufficiently small). Note that in the preceding argument we could have chosen a different base point p for the Kobayashi indicatrix. Note also that $\mathrm{Aut}(S^3) = \mathrm{Aut}(\mathbb{B}^2)$ acts simply and transitively on the set of pointed frames in $\mathbb{B}^2$. Using this we can act on the marking y_1 of (φ, y_1) while keeping φ fixed as follows. Given a point $p \in \mathbb{B}^2$ and a unitary frame (e_1, e_2) for $T_p\mathbb{C}^2$ we repeat the above construction of $\varphi_{M,S^3} \in \mathfrak{D}'_{BE}$ but now use the Kobayashi indicatrix B_p centered at p and identify $\partial B_p \subseteq T_p\mathbb{C}^2$ with $\partial B = \partial B_0 \subseteq T_0\mathbb{C}^2$ using the linearization of the automorphism of $\mathbb{B}^2$ that takes $(p, (e_1, e_2))$ to the point 0 with the standard frame. In this way we obtain a family (ψ_s, φ_s) of points in $\mathcal{C} \times \mathfrak{D}'_{BE}$ parametrized by $s \in \mathrm{Aut}(S^3)$ whose images under P_{emb} are all of the form (φ, y_s). Since $\mathrm{Aut}(S^3) \cong Y$ is finite dimensional and the map $s \mapsto y_s$ in the special case $\varphi = 0$ is just the natural identification, the map $s \mapsto y_s$ is a local diffeomorphism for sufficiently small φ. This proves the result. □

Theorem 1.3 now follows from Proposition 5.2 and Theorem 1.2.

Remark 5.3. For comparison, we note that in [6, Theorem 14.2 and Theorem 15.1] Bland gives a normal form for CR structures and for embeddable CR structures on S^3 near the standard structure with respect to the action of contact diffeomorphisms and S^1-equivariant diffeomorphisms (which do not preserve the contact distribution). In the notation of the proof of Proposition 5.2 Bland's normal form for the embeddable deformation φ is obtained by pushing $\varphi_{M,\partial B}$ forward to S^3 using the radial projection from ∂B to S^3 and viewing this as a deformation of the CR structure of ∂B pushed forward to S^3 (recall that via this identification the contact distribution of ∂B is S^1-invariant but does not in general match the standard contact distribution of S^3, which is why S^1-equivariant diffeomorphisms are needed for this normalization). Our approach has been to keep the underlying contact structure fixed, which allows us to view the deformation in normal form as a deformation of the standard CR structure.

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