BOUNDED STRICTLY PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$ WITH OBSTRUCTION FLAT BOUNDARY II

SEAN N. CURRY AND PETER EBENFELT

Abstract. On a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $n > 1$, the smoothness of the Cheng-Yau solution to Fefferman's complex Monge-Ampere equation up to the boundary is obstructed by a local CR invariant of the boundary. For a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^2$ diffeomorphic to the ball, we prove that the global vanishing of this obstruction implies biholomorphic equivalence to the unit ball, subject to the existence of a holomorphic vector field satisfying a mild approximate tangency condition along the boundary. In particular, by considering the Euler vector field multiplied by i the result applies to all domains in a large C^1 open neighborhood of the unit ball in $\mathbb{C}^2$. The proof rests on establishing an integral identity involving the CR curvature of $\partial\Omega$ for any holomorphic vector field defined in a neighborhood of the boundary. The notion of ambient holomorphic vector field along the CR boundary generalizes naturally to the abstract setting, and the corresponding integral identity still holds in the case of abstract CR 3-manifolds.

1. Introduction

In this paper we continue our study of compact obstruction flat CR 3-manifolds, begun in [11]. We recall that if $\Omega \subset \mathbb{C}^n$, $n > 1$, is a bounded strictly pseudoconvex domain with smooth boundary $M = \partial\Omega$, then by [9] there is a unique solution $u > 0$ to the Dirichlet problem

$$\begin{cases} \mathcal{J}(u) := (-1)^n \det \begin{pmatrix} u & u_{z^{\bar{k}}} \\ u_{z^j} & u_{z^j z^{\bar{k}}} \end{pmatrix} = 1 \ \text{in} \ \Omega, \\ u = 0 \ \text{on} \ \partial\Omega \end{cases} \tag{1.1}$$

governing the existence of a complete Kähler-Einstein metric on Ω with Kähler potential $-\log u$. By [9, 24] the solution u is smooth in Ω and $C^{n+2-\epsilon}$ up to the boundary, for any $\epsilon > 0$. Moreover, u has asymptotic expansion

$$u \sim \rho \sum_{k=0}^{\infty} \eta_k (\rho^{n+1} \log \rho)^k, \quad \eta_k \in C^\infty(\overline{\Omega}) \tag{1.2}$$

where ρ is a Fefferman defining function for Ω, meaning ρ is normalized by $\mathcal{J}(\rho) = 1 + O(\rho^{n+1})$; see [16]. Such a defining function is unique modulo $O(\rho^{n+2})$ and Graham [18, 19] showed that the coefficients η_k are local CR invariants modulo $O(\rho^{n+1})$. Graham also showed that if $b\eta_1 = \eta_1|_M$ vanishes, then $\eta_k = 0$ for all $k \geq 1$, and u is smooth up to M. Thus $b\eta_1$ is precisely the obstruction to C^∞ boundary regularity of the Cheng-Yau solution u. We say that a strictly pseudoconvex CR manifold is *obstruction flat* if $b\eta_1 = 0$. Graham [18, 19] showed that there is a large family of local (noncompact) strictly pseudoconvex obstruction flat hypersurfaces in $\mathbb{C}^n$, not locally CR equivalent to the unit sphere (not *locally spherical*). In this paper we shall consider the problem of classifying the *compact* strictly

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pseudoconvex obstruction flat hypersurfaces in $\mathbb{C}^2$. In this case the problem coincides with that of classifying the smooth bounded strictly pseudoconvex domains Ω for which the trace of the log term in the asymptotic expansion of the Bergman kernel vanishes on $M = \partial\Omega$ [19]. In this formulation the problem is subject to a strong form of the classical Ramadanov conjecture [26]; recall that the Ramadanov conjecture states that the full vanishing of the coefficient of the log term (not just of the trace on the boundary) characterizes the unit ball in $\mathbb{C}^n$. The reader is referred to [11, Section 2] for more detail. The problem is also equivalent to classifying the compact strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$ that are critical points for the Burns-Epstein invariant, or equivalently for the total Q'-curvature, among deformations in $\mathbb{C}^2$ (Kuranishi wiggles); see [6, 8, 11, 20, 21].

In [11] the authors proved that if a compact strictly pseudoconvex CR 3-manifold with infinitesimal symmetry is obstruction flat then it must be locally spherical, generalizing previous results in [2, 25, 13]. For the case of boundaries in $\mathbb{C}^2$, having an infinitesimal symmetry means the existence of a holomorphic vector field whose real part is tangent to the boundary. In this paper we generalize the aforementioned result of [11] in the embedded case, by substantially relaxing the condition on the tangency of the real part of the holomorphic vector field. The main technical idea is Proposition 4.2, which takes an ambient holomorphic vector field and gives a solution of a prolonged system of differential equations involving the Cartan/tractor curvature. This is a partial generalization of Lemma 7.3 from [11] which was used in the proof of Theorem 1.3 of that paper; the proof of our main result Theorem 1.7 below imitates the proof of Theorem 1.3 of [11] with Proposition 4.2 replacing the role of Lemma 7.3.

The restriction of a holomorphic vector field X to a strictly pseudoconvex hypersurface M in $\mathbb{C}^2$ is determined by a single complex function (or density) u given by $u = \overline{iX\rho}$ where ρ is a defining function for M; the function $\bar{u} = iX\rho$ is a kind of Hamiltonian potential for X (see Lemma 3.1). The holomorphic vector field X gives rise to an infinitesimal symmetry of M precisely when u is real, which means that the real part of X is tangent to M. For our main result, we shall need a much weaker condition on u; we merely require that the imaginary part of u is not too large relative to the real part. We make the following definition.

Definition 1.1. Let M be a smooth hypersurface in $\mathbb{C}^2$, and let ρ be any defining for M in a neighborhood of $p \in M$. Let $\epsilon \geq 0$. We say that the real part of a $(1,0)$-vector field X is *ϵ-approximately tangent* to M at p if $u = \overline{iX\rho}$ satisfies

$$(\operatorname{Im} u)^2 \leq \epsilon \, (\operatorname{Re} u)^2$$

at p. We say that $\operatorname{Re} X$ is *strictly ϵ-approximately tangent* at p if the above inequality is strict.

Remark 1.2. (i) It is easy to see that the notion of ϵ-approximate tangency does not depend on the choice of defining function ρ.
(ii) Note that if X is a $(1,0)$-vector field, then $\operatorname{Re} X$ is tangent to M if and only if $\operatorname{Re} X$ is 0-approximately tangent to M.
(iii) If $u(p) \neq 0$, then strict ϵ-approximate tangency means $|\arg u| < \alpha$ or $|\arg(-u)| < \alpha$ at p with $\alpha = \arctan\sqrt{\epsilon}$; a similar condition was introduced by Barrett in his study of Sobolev regularity for the Bergman projection on bounded domains [1].
(iv) In our main results we will only need the notion of strict 1-approximate tangency. Note that this is equivalent to $\operatorname{Re} u^2 > 0$.

Theorem 1.3. *Let $\Omega \subset \mathbb{C}^2$ be a smooth bounded strictly pseudoconvex domain for which there exists a holomorphic vector field X on $\overline{\Omega}$ whose real part is strictly* 1*-approximately tangent to $\partial\Omega$ almost everywhere. If $\partial\Omega$ is obstruction flat, then $\partial\Omega$ is locally spherical.*

Taking X to be i times the $(1,0)$-Euler vector field, Theorem 1.3 applies to all domains in a large C^1 open neighborhood of the unit ball. More precisely, we define $\mathscr{D}$ to be the space of smooth embedded strictly pseudoconvex 3-spheres in $\mathbb{C}^2$ (with the C^∞ topology). By identifying an element $M \in \mathscr{D}$ with the domain $\Omega \subset \mathbb{C}^2$ that it bounds, we think of $\mathscr{D}$ as the space of strictly pseudoconvex domains $\Omega \subset \mathbb{C}^2$ with smooth boundary $\partial\Omega$ diffeomorphic to S^3. Let $\mathscr{U} \subset \mathscr{D}$ denote the open neighborhood of the unit ball $\mathbb{B}^2$ consisting of domains for which X is strictly 1-approximately tangent to the boundary; this holds if the real part of the Euler field is closer to being normal than tangent to the boundary of the domain.

Corollary 1.4. *For all domains Ω in the neighborhood $\mathscr{U}$ of the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$, if $\partial\Omega$ is obstruction flat, then Ω is biholomorphic to the unit ball.*

Remark 1.5. While $\mathscr{D}$ is naturally equipped with the C^∞ topology, by construction the set $\mathscr{U}$ is clearly also open in the (relative) C^1 topology.

The conjugate u of the Hamiltonian potential for a holomorphic vector field X satisfies the differential equation

$$\nabla_1\nabla_1 u + iA_{11}u = 0 \tag{1.3}$$

on the CR manifold $M \subset \mathbb{C}^2$, where ∇ denotes the Tanaka-Webster connection of some contact form and A_{11} is its pseudohermitian torsion; the operator $\nabla_1\nabla_1 + iA_{11}$ is in fact CR invariant when acting on CR densities of weight $(1,1)$, cf. Section 2.1, and the corresponding equation makes sense on an abstract CR manifold. We have the following more general result, which implies Theorem 1.3.

Theorem 1.6. *Let (M,H,J) be a compact strictly pseudoconvex CR* 3*-manifold for which there exists a weight $(1,1)$ complex solution u of* (1.3) *such that $(\operatorname{Im} u)^2 < (\operatorname{Re} u)^2$ almost everywhere. If (M,H,J) is obstruction flat, then it is locally spherical.*

This theorem will follow easily from:

Theorem 1.7. *Let (M,H,J) be a compact strictly pseudoconvex CR* 3*-manifold which is obstruction flat. For any weight $(1,1)$ complex solution u of* (1.3) *we have*

$$\int_M u^2|Q|^2 = 0$$

where Q is Cartan's umbilical tensor.

Remark 1.8. (i) We remark that Cartan's umbilical tensor Q vanishes on an open set if and only if the CR 3-manifold is locally spherical on that open set (see Section 2.2).
(ii) Note that since u has weight $(1,1)$ and $|Q|^2 = \boldsymbol{h}^{1\bar{1}}\boldsymbol{h}^{1\bar{1}}Q_{11}Q_{\bar{1}\bar{1}}$ has weight $(-4,-4)$, the integrand $u^2|Q|^2$ has weight $(-2,-2)$ and therefore may be naturally identified with a complex volume form on M. See Section 2.1 for a discussion of weights and densities.

The main technical ingredient in the proof of this result is Proposition 4.2, which constructs from a solution u of (1.3) a solution of a prolonged system of $\bar{\partial}_b$-equations involving the CR curvature.

1.1. **Outline of the Paper.** In Section 2 we recall some background on CR geometry and weighted pseudohermitian calculus, and make some observations concerning complex solutions of the infinitesimal automorphism equation (1.3). In Section 3 we establish the connection between weight $(1,1)$ complex solutions of the infinitesimal automorphism equation on an embedded CR manifold and ambient holomorphic vector fields. In Section 4 we introduce the CR tractor calculus and prove Proposition 4.2. In Section 5 we prove our main results, Theorems 1.6 and 1.7.

2. Strictly pseudoconvex CR 3-manifolds

For the reader's convenience, we here recall the general setup from [11]. A *strictly pseudoconvex CR 3-manifold* is a triple (M,H,J) where M is a smooth oriented 3-manifold, $H\subset TM$ is a contact distribution, and $J:H\to H$ is a smooth bundle endomorphism such that $J^2=-\mathrm{id}$. The partial complex structure J on $H\subset TM$ defines an orientation of H, and therefore defines an orientation on the annihilator subbundle $H^\perp := \mathrm{Ann}(H)\subset T^*M$. Given any contact form θ for H, $\mathrm{d}\theta|_H$ is a nondegenerate bilinear form. A contact form θ for H is positively oriented if $\mathrm{d}\theta(\,\cdot\,,J\,\cdot\,)$ is positive definite on H. A strictly pseudoconvex CR structure (M,H,J) together with a choice of positively oriented contact form θ is referred to as a *pseudohermitian structure*. The *Reeb vector field* of a contact form θ is the vector field T uniquely determined by $\theta(T)=1$ and $T\lrcorner\,\mathrm{d}\theta=0$.

Given a CR manifold (M,H,J) we decompose the complexified contact distribution $\mathbb{C}\otimes H$ as $T^{1,0}\oplus T^{0,1}$, where J acts by i on $T^{1,0}$ and by $-i$ on $T^{0,1}=\overline{T^{1,0}}$. Let θ be a positively oriented contact form on M. Let Z_1 be a local frame for the *holomorphic tangent bundle* $T^{1,0}$ and $Z_{\bar 1}=\overline{Z_1}$, so that $\{T,Z_1,Z_{\bar 1}\}$ is a local frame for $\mathbb{C}\otimes TM$. Then the dual frame $\{\theta,\theta^1,\theta^{\bar 1}\}$ is referred to as an *admissible coframe* and one has

$$\mathrm{d}\theta = ih_{1\bar 1}\theta^1\wedge\theta^{\bar 1} \tag{2.1}$$

for some positive smooth function $h_{1\bar 1}$. The function $h_{1\bar 1}$ is the component of the Levi form $\mathrm{L}_\theta(U,\overline{V})=-2i\mathrm{d}\theta(U,\overline{V})$ on $T^{1,0}$, that is

$$\mathrm{L}_\theta(U^1Z_1,V^{\bar 1}Z_{\bar 1}) = h_{1\bar 1}U^1V^{\bar 1}.$$

It is sometimes convenient to scale Z_1 so that $h_{1\bar 1}=1$, but we will not assume this. We write $h^{1\bar 1}$ for the multiplicative inverse of $h_{1\bar 1}$. The Tanaka-Webster connection associated to θ is given in terms of such a local frame $\{T,Z_1,Z_{\bar 1}\}$ by

$$\nabla Z_1=\omega_1{}^1\otimes Z_1,\quad \nabla Z_{\bar 1}=\omega_{\bar 1}{}^{\bar 1}\otimes Z_{\bar 1},\quad \nabla T=0$$

where the connection 1-forms $\omega_1{}^1$ and $\omega_{\bar 1}{}^{\bar 1}$ satisfy

$$\mathrm{d}\theta^1=\theta^1\wedge\omega_1{}^1 + A^1{}_{\bar 1}\,\theta\wedge\theta^{\bar 1},\ \text{and} \tag{2.2}$$

$$\omega_1{}^1+\omega_{\bar 1}{}^{\bar 1} = h^{1\bar 1}\mathrm{d}h_{1\bar 1}, \tag{2.3}$$

for some function $A^1{}_{\bar 1}$. The uniquely determined function $A^1{}_{\bar 1}$ is known as the *pseudohermitian torsion*. Components of covariant derivatives will be denoted by adding ∇ with an appropriate subscript, so, e.g., if u is a function then $\nabla_1 u = Z_1 u$, $\nabla_0 u = Tu$ and $\nabla_0\nabla_1 u = TZ_1u - \omega_1{}^1(T)Z_1u$. We may also use $h_{1\bar 1}$ and $h^{1\bar 1}$ to raise and lower indices, so that $A_{\bar 1\bar 1}=h_{1\bar 1}A^1{}_{\bar 1}$ and $A_{11}=h_{1\bar 1}A^{\bar 1}{}_1$, with $A^{\bar 1}{}_1=\overline{A^1{}_{\bar 1}}$.

We recall some useful formulae, which can be found in [12, 23]. The *pseudohermitian (scalar) curvature* R is defined by the structure equation

$$\mathrm{d}\omega_1{}^1 = Rh_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + (\nabla^1 A_{11})\,\theta^1 \wedge \theta - (\nabla^{\bar{1}} A_{\bar{1}\bar{1}})\,\theta^{\bar{1}} \wedge \theta.$$

The torsion of the Tanaka-Webster connection (as an affine connection) is captured by the following formulae, for a smooth function f,

$$\nabla_1\nabla_{\bar{1}} f - \nabla_{\bar{1}}\nabla_1 f = -ih_{1\bar{1}}\nabla_0 f, \quad \text{and} \quad \nabla_1\nabla_0 f - \nabla_0\nabla_1 f = A^{\bar{1}}{}_1\nabla_{\bar{1}} f.$$

The pseudohermitian curvature R may therefore equivalently be defined by the Ricci identity

$$\nabla_1\nabla_{\bar{1}}V^1 - \nabla_{\bar{1}}\nabla_1 V^1 + ih_{1\bar{1}}\nabla_0 V^1 = Rh_{1\bar{1}}V^1 \tag{2.4}$$

for any local section $V^1 Z_1$ of $T^{1,0}$. Commuting 0 and 1 (or $\bar{1}$) derivatives on $V^1 Z_1$ gives torsion according to the following formulae

$$\nabla_1\nabla_0 V^1 - \nabla_0\nabla_1 V^1 - A^{\bar{1}}{}_1\nabla_{\bar{1}} V^1 = (\nabla^1 A_{11})V^1, \quad \text{and} \tag{2.5}$$

$$\nabla_{\bar{1}}\nabla_0 V^1 - \nabla_0\nabla_{\bar{1}} V^1 - A^1{}_{\bar{1}}\nabla_1 V^1 = (\nabla^{\bar{1}} A_{\bar{1}\bar{1}})V^1.$$

In dimension 3, the Bianchi identities of [23, Lemma 2.2] reduce to

$$\nabla_0 R = 2\mathrm{Re}\,(\nabla^1\nabla^1 A_{11}). \tag{2.6}$$

2.1. **Weighted pseudohermitian calculus.** Let (M, H, J) be a CR 3-manifold, and let $\Lambda^{1,0}$ denote the complex rank 2 bundle of $(1,0)$-forms on M. The bundle $\Lambda^{2,0} = \Lambda^2(\Lambda^{1,0})$ of $(2,0)$-forms is referred to as the *canonical line bundle* of M, and denoted by $\mathscr{K}$. We assume throughout that its dual $\mathscr{K}^*$ admits a (global) cube root, which we fix and denote by $\mathcal{E}(1,0)$. Note that this is equivalent to assuming that the integral first Chern class of $\mathscr{K}$ is divisible by 3; in particular this holds for hypersurfaces in $\mathbb{C}^2$. We then define the *CR density line bundle* of weight (w, w') to be $\mathcal{E}(w,w') = \mathcal{E}(1,0)^w \otimes \overline{\mathcal{E}(1,0)}^{w'}$, where $w, w' \in \mathbb{C}$ with $w - w' \in \mathbb{Z}$. Note that for w real the bundle $\mathcal{E}(w,w)$ is invariant under conjugation, and hence contains a real subbundle $\mathcal{E}_{\mathbb{R}}(w,w)$. Note also that by definition $\mathcal{E}(3,0) = \mathscr{K}^*$, so $\mathcal{E}(-3,0) = \mathscr{K}$.

Trivializing the bundle TM/H determines a contact form on M via the natural map $TM \to TM/H$. Similarly, a choice of non-vanishing section ζ (i.e. a trivialization) of $\mathscr{K}$ determines canonically a contact form θ on M by the requirement [14] (see also [23]) that

$$\theta \wedge \mathrm{d}\theta = i\theta \wedge (T \lrcorner\, \zeta) \wedge (T \lrcorner\, \bar{\zeta}). \tag{2.7}$$

In this case we say that θ is *volume normalized* with respect to ζ. Combining these observations, we may realize TM/H as a real CR density line bundle as follows. A contact form θ determines canonically a section $|\zeta|^2 = \zeta \otimes \bar{\zeta}$ of $\mathscr{K} \otimes \overline{\mathscr{K}} = \mathcal{E}(-3,-3)$ by the condition that ζ satisfy (2.7) (ζ is only determined up to phase at each point). If we rescale θ to $\hat{\theta} = e^{\Upsilon}\theta$, with $\Upsilon \in C^\infty(M,\mathbb{R})$, then the corresponding section $|\hat{\zeta}|^2$ equals $e^{3\Upsilon}|\zeta|^2$. Thus, the map which assigns to a contact form θ the section $|\zeta|^{2/3}$ of $\mathcal{E}_{\mathbb{R}}(-1,-1)$ extends to a canonical isomorphism of $H^\perp$ with $\mathcal{E}_{\mathbb{R}}(-1,-1)$. Dually TM/H is canonically isomorphic to $\mathcal{E}_{\mathbb{R}}(1,1)$. This identification gives us a tautological 1-form $\boldsymbol{\theta}$ of weight $(1,1)$, corresponding to the map $TM \to TM/H = \mathcal{E}_{\mathbb{R}}(1,1)$.

We define the *CR Levi form* $\boldsymbol{L} : T^{1,0} \otimes T^{0,1} \to \mathbb{C}TM/\mathbb{C}H$ by

$$\boldsymbol{L}(U, \overline{V}) = 2i[U, \overline{V}] \mod \mathbb{C}H.$$

On a strictly pseudoconvex CR 3-manifold the CR Levi form is a bundle isomorphism, so we have $T^{1,0}\otimes T^{0,1}\cong \mathbb{C}TM/\mathbb{C}H=\mathcal{E}(1,1)$. The CR Levi form may be interpreted as a Hermitian bundle metric on $T^{1,0}\otimes\mathcal{E}(-1,0)$, and we would like to have a more concise notation for bundles like this one. We use the symbol $\mathcal{E}$ decorated with appropriate indices to denote the tensor bundles constructed from $T^{1,0}$ and $T^{0,1}$ (this is Penrose's abstract index notation). For example, $\mathcal{E}^1=T^{1,0}$, $\mathcal{E}_{\bar 1}=(T^{0,1})^*$, and $\mathcal{E}_{1\bar 1}=(T^{1,0})^*\otimes(T^{0,1})^*$. We will now generally use abstract index notation for sections of these bundles. So, for example, V^1 may denote a global section of $\mathcal{E}^1=T^{1,0}$ (previously written locally as V^1Z_1). This keeps the notation from getting too heavy, and allows us to globalize our previous local formulas. Note that a choice of contact form allows us to decompose the complexified tangent bundle $\mathbb{C}TM$ as $\mathcal{E}^1\oplus\mathcal{E}^{\bar 1}\oplus\mathcal{E}(1,1)$. Using abstract index notation we may therefore decompose V globally as $V\overset{\theta}{=}(V^1,V^{\bar 1},V^0)$. Generally we denote the tensor product of a complex vector bundle $\mathcal{V}$ on M with $\mathcal{E}(w,w')$ by appending (w,w'), as in $\mathcal{V}(w,w')$. The CR Levi form will be thought of as a section $\boldsymbol{h}_{1\bar 1}$ of $\mathcal{E}_{1\bar 1}(1,1)$, with inverse $\boldsymbol{h}^{1\bar 1}$. The Levi form will be used to identify $\mathcal{E}^{1\bar 1}$ with $\mathcal{E}(1,1)$, and $\mathcal{E}_{1\bar 1}$ with $\mathcal{E}(-1,-1)$, and to raise and lower indices without comment.

Observe that the Tanaka-Webster connection ∇ of a pseudohermitian structure θ extends naturally to act on the CR density bundles, since ∇ acts on the canonical bundle $\mathscr{K}$. Since the Tanaka-Webster connection of θ preserves θ, and also preserves the section $|\zeta|^2$ of $\mathscr{K}\otimes\overline{\mathscr{K}}=\mathcal{E}(-3,-3)$ determined by volume normalization, the Tanaka-Webster connection respects the CR invariant identification of TM/H with $\mathcal{E}_{\mathbb{R}}(1,1)$; see [12, 17]. Another way of saying this is that $\nabla\boldsymbol{\theta}=0$. A similar argument shows that ∇ preserves the CR Levi form, $\nabla\boldsymbol{L}=0$. Hence, the Tanaka-Webster connection of θ respects all of the CR invariant identifications made above. We therefore make use of CR densities whenever convenient.

We will need to commute derivatives of weighted tensor fields, for this we need to know the curvature of the CR density bundles. Let τ be a section of $\mathcal{E}(w,w')$. From (2.4) and (2.5) one easily obtains (cf. [17, Proposition 2.2]) that

$$\nabla_1\nabla_{\bar 1}\tau-\nabla_{\bar 1}\nabla_1\tau+i\boldsymbol{h}_{1\bar 1}\nabla_0\tau=\frac{w-w'}{3}R\boldsymbol{h}_{1\bar 1}\,\tau\,; \tag{2.8}$$

$$\nabla_1\nabla_0\tau-\nabla_0\nabla_1\tau-A^{\bar 1}{}_1\nabla_{\bar 1}\tau=\frac{w-w'}{3}(\nabla_{\bar 1}A^{\bar 1}{}_1)\tau. \tag{2.9}$$

2.2. **CR invariants.** The local calculus on CR manifolds associated with the CR Cartan connection is discussed in more detail in Section 7 of [11]. For now it suffices to recall some basic definitions and formulae in terms of pseudohermitian calculus. The *Cartan umbilical tensor* Q of (M,H,J) is a (weighted) CR invariant, whose vanishing in a neighborhood is necessary and sufficient for (M,H,J) to be locally equivalent to the induced CR structure on the unit sphere in $\mathbb{C}^2$. As in [7], given a choice of contact form θ we interpret the umbilical tensor Q as an endomorphism of H, written locally as

$$Q=iQ_1{}^{\bar 1}\theta^1\otimes Z_{\bar 1}-iQ_{\bar 1}{}^1\theta^{\bar 1}\otimes Z_1. \tag{2.10}$$

By [7, Lemma 2.2] the component Q_{11} of Cartan's tensor is given by

$$Q_{11}=-\frac{1}{6}\nabla_1\nabla_1R-\frac{i}{2}RA_{11}+\nabla_0A_{11}+\frac{2i}{3}\nabla_1\nabla^1A_{11}, \tag{2.11}$$

where we have taken the opposite sign convention. If $\hat\theta=e^{\Upsilon}\theta$ is another contact form, then $\hat Q=e^{-2\Upsilon}Q$, so that Q may be thought of more invariantly as a weighted section of $\mathrm{End}(H)$. More precisely, Q may be thought of as a CR invariant section of $\mathrm{End}(H)\otimes\mathcal{E}_{\mathbb{R}}(-2,-2)$, the dependency on the contact form θ only being introduced when we use θ to trivialize $\mathcal{E}_{\mathbb{R}}(1,1) = TM/H$. Note that this means Q_{11} is a section of $\mathcal{E}_{11}(-1,-1)$. The Bianchi identity for the curvature of the CR Cartan connection is equivalent to the following Bianchi identity for Q,

$$\mathrm{Im}(\nabla^1\nabla^1 Q_{11} - iA^{11}Q_{11}) = 0, \tag{2.12}$$

which may also be seen as a direct consequence of (2.6). The *CR obstruction density* is given by

$$\mathcal{O} = \frac{1}{3}(\nabla^1\nabla^1 Q_{11} - iA^{11}Q_{11}). \tag{2.13}$$

The CR obstruction density $\mathcal{O}$ is again a (weighted) CR invariant. If $\hat{\theta} = e^{\Upsilon}\theta$ is another contact form, then $\hat{\mathcal{O}} = e^{-3\Upsilon}\mathcal{O}$, so that $\mathcal{O}$ defines a CR invariant section of $\mathcal{E}_{\mathbb{R}}(-3,-3)$.

2.3. **Complex solutions to the infinitesimal automorphism equation.** Central to our considerations will be the operator

$$\nabla_1\nabla_1 + iA_{11} : \mathcal{E}(1,1) \to \mathcal{E}_{11}(1,1) \tag{2.14}$$

whose formal adjoint, with respect to the canonical CR invariant weight $(2,2)$ volume form on M, is the operator $\nabla^1\nabla^1 - iA^{11} : \mathcal{E}_{11}(-1,-1) \to \mathcal{E}(-3,-3)$ appearing in (2.13). Here, in a standard abuse of notation, we are denoting a bundle and its sheaf of sections by the same symbol. It is well known that a CR manifold M possesses an infinitesimal CR automorphism if and only if there is a real solution to the *infinitesimal automorphism equation*

$$\nabla_1\nabla_1 f + iA_{11}f = 0,$$

in which case the infinitesimal symmetry is given by the contact Hamiltonian vector field with potential f, $V_f = fT + if^1Z_1 - if^{\bar{1}}Z_{\bar{1}}$ where $f^1 = \nabla^1 f$ and $f^{\bar{1}} = \nabla^{\bar{1}} f$; see, e.g., [11]. For our purposes, it is the existence of complex solutions of this equation that will turn out to be essential, and this is intimately related with the question of local embeddability of the CR manifold (M,H,J) in $\mathbb{C}^2$.

Proposition 2.1. *Let (M,H,J) be a strictly pseudoconvex CR manifold and $p \in M$. The following are equivalent:*

(i) *(M,H,J) is embeddable in a neighborhood of p.*
(ii) *There is a nonvanishing complex solution u of the equation*

$$\nabla_1\nabla_1 u + iA_{11}u = 0$$

on weight $(1,1)$ densities in a neighborhood of p.

This proposition follows from the following three lemmas.

Lemma 2.2. *Let (M,H,J) be a strictly pseudoconvex CR manifold and θ a contact form for H. Let f be a complex function on M and let $V = fT + if^1Z_1$, where T is the Reeb vector field of θ, Z_1 spans $T^{1,0}M$, and $f^1 = \nabla^1 f$. Then*

$$\mathcal{L}_V Z_{\bar{1}} = -i(\nabla_{\bar{1}}\nabla^1 f - iA_{\bar{1}}{}^1 f)Z_1 \mod T^{0,1}M.$$

In particular, if $\nabla_{\bar{1}}\nabla_{\bar{1}} f - iA_{\bar{1}\bar{1}}f = 0$, then $\mathcal{L}_V\Gamma(T^{0,1}M) \subset \Gamma(T^{0,1}M)$.

Remark 2.3. Here $\Gamma(T^{0,1}M)$ denotes the space of smooth sections of $T^{0,1}M$, and $\mathcal{L}_V$ the Lie derivative with respect to V.

Proof. Let $(\theta, \theta^1, \theta^{\bar 1})$ be the admissible coframe dual to $(T, Z_1, Z_{\bar 1})$. Using the structure equation (2.2) and $\theta^1(V) = if^1$ we have

$$\begin{aligned}\mathcal{L}_V\theta^1 &= V \lrcorner\, \mathrm{d}\theta^1 + \mathrm{d}(V \lrcorner\, \theta^1)\\ &= fA^1{}_{\bar 1}\theta^{\bar 1} + i(f^1{}_{\bar 1}\theta^{\bar 1} + f^1{}_1\theta^1 + f^1{}_0\theta).\end{aligned}$$

So $\theta^1(\mathcal{L}_V Z_{\bar 1}) = -(\mathcal{L}_V\theta^1)(Z_{\bar 1}) = -i(\nabla_{\bar 1}\nabla^1 f - iA_{\bar 1}{}^1 f)$. Moreover, $\mathcal{L}_V\theta = V \lrcorner\, \mathrm{d}\theta + \mathrm{d}(V \lrcorner\, \theta) = -\boldsymbol{h}_{1\bar 1} f^1\theta^{\bar 1} + \mathrm{d}f$, so that $\theta(\mathcal{L}_V Z_{\bar 1}) = -(\mathcal{L}_V\theta)(Z_{\bar 1}) = 0$, which gives the desired result. □

Lemma 2.4. *Let (M, H, J) be a strictly pseudoconvex CR manifold, and let V be a complex vector field on M with the property that $\mathcal{L}_V\Gamma(T^{0,1}M) \subset \Gamma(T^{0,1}M)$. Then $V = fT + if^1 Z_1$ mod $T^{0,1}M$ where $f^1 = \nabla^1 f$ and $u = \overline{f}$ satisfies the equation $\nabla_1\nabla_1 u + iA_{11}u = 0$.*

Proof. This is a straightforward calculation along the lines of the previous proof. □

The following lemma of Jacobowitz [22] now establishes Proposition 2.1.

Lemma 2.5. [22, Proposition 2.1] *Let (M, H, J) be a strictly pseudoconvex CR manifold. The following are equivalent:*

(i) *(M, H, J) is embeddable in a neighborhood of the point p.*
(ii) *There exists a vector field V with $\mathcal{L}_V\Gamma(T^{0,1}M) \subset \Gamma(T^{0,1}M)$ and $V \notin T^{1,0}M \oplus T^{0,1}M$ in a neighborhood of p.*

3. Existence of global complex solutions to the infinitesimal automorphism equation

While the space of real solutions to the infinitesimal automorphism equation on an embedded strictly pseudoconvex CR manifold is finite dimensional, and generically consists only of the zero solution, the space of complex solutions is always infinite dimensional (corresponding to the infinite dimensional space of ambient holomorphic vector fields). Let $M \subset \mathbb{C}^2$ be a strictly pseudoconvex hypersurface and θ a contact form for M. Given a section $X \in \Gamma(T^{1,0}\mathbb{C}^2|_M)$, expressed in terms of an admissible frame by

$$X = X^0\xi + X^1 Z_1$$

with $\xi = \frac{1}{2}(T - iJT)$ where here J denotes the standard complex structure on $\mathbb{C}^2$, we define the complex vector field X_M on M by

$$X_M = X^0 T + X^1 Z_1. \tag{3.1}$$

Note that the vector field X_M depends on the choice of θ, but is well defined modulo $T^{0,1}M$.

Lemma 3.1. *Let $M \subset \mathbb{C}^2$ be a strictly pseudoconvex hypersurface and θ a contact form for M. If X is a holomorphic vector field defined on a neighborhood of M then $X_M = fT + if^1 Z_1$ where $u = \overline{f}$ satisfies*

$$\nabla_1\nabla_1 u + iA_{11}u = 0.$$

Proof. Let ϕ_t denote the flow of $\operatorname{Re} X$, and let $\psi_t = \phi_t|_M$ be the resulting parametrized deformation of $M \subset \mathbb{C}^2$ (see [11, Section 4.3]), which is trivial since the ϕ_t are biholomorphic. Then the variational vector field (interpreted as a section of $T_{(1,0)}M = \mathbb{C}TM/T^{0,1}M$) is given by $\dot\psi = X_M$ mod $T^{0,1}$. Hence, by Lemma 4.5 of [11], we have $X_M = fT + if^1 Z_1$ where $f = \theta(X_M)$ and the infinitesimal deformation tensor φ_{11} is given by $\varphi_{11} = i(\nabla_1\nabla_1 u + iA_{11}u)$ with $u = \overline{f}$. But ψ_t is a trivial deformation, so $\varphi_{11} = 0$. The result follows. □

It follows from Lemma 2.2 and Lemma 3.1 that if X is a holomorphic vector field then $V = X_M$ satisfies $\mathcal{L}_V \Gamma(T^{0,1}M) \subset \Gamma(T^{0,1}M)$.

Lemma 3.2. *Let $M \subset \mathbb{C}^2$ be a strictly pseudoconvex hypersurface. Suppose u is a weight $(1,1)$ complex solution to $\nabla_1\nabla_1 u + iA_{11}u = 0$. Then there is an ambient $(1,0)$-vector field $X = a^1\frac{\partial}{\partial z^1} + a^2\frac{\partial}{\partial z^2}$ along M with CR coefficients a^1, a^2, such that $u = \overline{\boldsymbol{\theta}(X_M)}$.*

Proof. Let θ be a contact form for M and define $V = fT + if^1 Z_1$ where $f = \overline{u}$, which we now think of as a complex function on M. Let X be the ambient $(1,0)$-vector field along M determined by $X_M = V$, where X_M is defined as in (3.1). Then $X = a^1\frac{\partial}{\partial z^1} + a^2\frac{\partial}{\partial z^2}$ where

$$a^k = dz^k(X) = dz^k(X_M) = f\mathrm{d}z^k(T) + if^1\mathrm{d}z^k(Z_1),$$

for $k = 1, 2$. The result can then be gleaned from the proof of [11, Lemma 4.5], but we include a proof here for the readers convenience. Noting that $\mathrm{d}z^k(T) = Tz^k$ and $\mathrm{d}z^k(Z_1) = Z_1 z^k$, we compute

$$\begin{aligned} Z_{\bar 1}a^k &= (Z_{\bar 1}f)Tz^k + fZ_{\bar 1}Tz^k + i(Z_{\bar 1}f^1)Z_1z^k + if^1Z_{\bar 1}Z_1z^k \\ &= (Z_{\bar 1}f)Tz^k + f[Z_{\bar 1}, T]z^k + i(Z_{\bar 1}f^1)Z_1z^k + if^1[Z_{\bar 1}, Z_1]z^k. \end{aligned}$$

From the structure equations (2.1) and (2.2) it is straightforward to compute that

$$[Z_{\bar 1}, Z_1] = ih_{1\bar 1}T + \omega_1{}^1(Z_{\bar 1})Z_1 - \omega_{\bar 1}{}^{\bar 1}(Z_1)Z_{\bar 1}, \quad \text{and} \quad [Z_{\bar 1}, T] = A^1{}_{\bar 1}Z_1 - \omega_{\bar 1}{}^{\bar 1}(T)Z_{\bar 1}.$$

We therefore obtain

$$\begin{aligned} Z_{\bar 1}a^k &= (Z_{\bar 1}f - h_{1\bar 1}f^1)Tz^k + (iZ_{\bar 1}f^1 + fA^1{}_{\bar 1} + if^1\omega_1{}^1(Z_{\bar 1}))Z_1z^k \\ &= i(\nabla_{\bar 1}\nabla^1 f - iA_{\bar 1}{}^1 f)Z_1z^k = 0 \end{aligned} \tag{3.2}$$

for $k = 1, 2$. □

Remark 3.3. From the first line of (3.2) above and the independence of the vectors T and Z_1 it follows that in the statement of Lemma 3.1 it is sufficient for the $(1,0)$-vector field to be defined only on M and have CR coefficients a^1 and a^2.

Remark 3.4. By Proposition 2.1 and Lemma 3.2 if (M, H, J) is an abstract CR 3-manifold, and u is a complex solution of the infinitesimal automorphism equation which does not vanish at some point, then locally M can be realized as a hypersurface in $\mathbb{C}^2$ such that $f = \bar u$ is the complex Reeb component of a holomorphic vector field. More precisely, if (M, H, J) is an abstract CR 3-manifold, and u is a complex solution of the infinitesimal automorphism equation with $u(p) \neq 0$, then there is a CR embedding ψ from a neighborhood U of p into $\mathbb{C}^2$ and a holomorphic vector field X on the pseudoconvex side of $\psi(U)$ extending smoothly to $\psi(U)$ such that $\boldsymbol{\theta}(X_{\psi(U)}) = \bar u$.

4. CR sections of the complexified adjoint tractor bundle

4.1. **Tractor calculus.** Here we briefly recall the setup for the CR tractor calculus in 3-dimensions, as in [11]. See, e.g., [17, 11] for more details. The *standard tractor bundle* $\mathcal{T}$ can be identified with the set of equivalence classes of pairs $(\theta, (\sigma, \mu^1, \rho))$, where θ is a contact form and $(\sigma,\mu^1,\rho)\in\mathcal{E}(0,1)\oplus\mathcal{E}^1(-1,0)\oplus\mathcal{E}(-1,0)$, under the equivalence relation: $(\theta,(\sigma,\mu^1,\rho))\sim(\hat\theta,(\hat\sigma,\hat\mu^1,\hat\rho))$ if $\hat\theta=e^{\Upsilon}\theta$ and

$$\begin{pmatrix}\hat\sigma\\ \hat\mu^1\\ \hat\rho\end{pmatrix}=\begin{pmatrix}1&0&0\\ \Upsilon^1&1&0\\ -\frac{1}{2}(\Upsilon^1\Upsilon_1-i\Upsilon_0)&-\Upsilon_1&1\end{pmatrix}\begin{pmatrix}\sigma\\ \mu^1\\ \rho\end{pmatrix} \tag{4.1}$$

where $\Upsilon_1=\nabla_1\Upsilon$, $\Upsilon^1=\boldsymbol{h}^{1\bar 1}\Upsilon_{\bar 1}$ with $\Upsilon_{\bar 1}=\nabla_{\bar 1}\Upsilon$, and $\Upsilon_0=\nabla_0\Upsilon$. If $(\theta,(\sigma,\mu^1,\rho))\sim(\hat\theta,(\hat\sigma,\hat\mu^1,\hat\rho))$ then one easily checks that

$$2\hat\sigma\hat\rho+\hat\mu^1\hat\mu_1=2\sigma\rho+\mu^1\mu_1,$$

which defines by polarization a signature $(2,1)$ Hermitian bundle metric h on $\mathcal{T}$, called the *tractor metric*. We will adopt the abstract index notation $\mathcal{E}^A$ for $\mathcal{T}$, and $\mathcal{E}^{\bar A}$ for $\overline{\mathcal{T}}$, using capitalized Latin letters from the start of the alphabet for our abstract indices. The tractor metric h is then written as $h_{A\bar B}$. Decomposing $\mathcal{E}^A$ with respect to any choice of contact form θ, we have

$$h_{A\bar B}=\begin{pmatrix}0&0&1\\ 0&\boldsymbol{h}_{1\bar 1}&0\\ 1&0&0\end{pmatrix}. \tag{4.2}$$

The map $\mathcal{E}(-1,0)\to\mathcal{E}^A$ given by

$$\rho\mapsto\begin{pmatrix}0\\ 0\\ \rho\end{pmatrix}$$

corresponds to a canonical section $\boldsymbol{Z}^A$ of $\mathcal{E}^A\otimes\mathcal{E}(1,0)$, known as the *canonical tractor*. Bearing in mind that $\boldsymbol{Z}^A$ is a weighted tractor, we may write

$$\boldsymbol{Z}^A\overset{\theta}{=}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

for any contact form θ. The canonical tractor also induces a canonical projection $\mathcal{E}^A\to\mathcal{E}(0,1)$ taking v^A to $\sigma=h_{A\bar B}v^A\boldsymbol{Z}^{\bar B}$. This corresponds to the obvious projection

$$\begin{pmatrix}\sigma\\ \mu^1\\ \rho\end{pmatrix}\mapsto\sigma.$$

The standard tractor bundle carries a canonical connection ∇ called the *(normal) tractor connection*. In order to describe this, we define the higher order pseudohermitian curvatures

$$T_1=\frac{1}{12}(\nabla_1 R-4i\nabla^1A_{11}), \tag{4.3}$$

a section of $\mathcal{E}_1(-1,-1)$, and the real $(-2,-2)$ density

$$S=-(\nabla^1T_1+\nabla^{\bar 1}T_{\bar 1}+\frac{1}{16}R^2-A^{11}A_{11}). \tag{4.4}$$

With respect to a choice of contact form θ the tractor connection on a section $v^A \overset{\theta}{=} (\sigma, \mu^1, \rho)$ is then given by

$$\nabla_1 v^A \overset{\theta}{=} \begin{pmatrix} \nabla_1\sigma \\ \nabla_1\mu^1 + \rho + \frac{1}{4}R\sigma \\ \nabla_1\rho - iA_{11}\mu^1 - \sigma T_1 \end{pmatrix}, \tag{4.5}$$

$$\nabla_{\bar{1}} v^A \overset{\theta}{=} \begin{pmatrix} \nabla_{\bar{1}}\sigma - \mu_{\bar{1}} \\ \nabla_{\bar{1}}\mu^1 - iA_{\bar{1}}{}^1\sigma \\ \nabla_{\bar{1}}\rho - \frac{1}{4}R\mu_{\bar{1}} + \sigma T_{\bar{1}} \end{pmatrix}, \tag{4.6}$$

and

$$\nabla_0 v^A \overset{\theta}{=} \begin{pmatrix} \nabla_0\sigma - \frac{i}{12}R\sigma + i\rho \\ \nabla_0\mu^1 + \frac{i}{6}R\mu^1 - 2i\sigma T^1 \\ \nabla_0\rho - \frac{i}{12}R\rho - 2iT_1\mu^1 - iS\sigma \end{pmatrix}. \tag{4.7}$$

One can verify that these formulae give rise to a well-defined CR invariant connection on $\mathcal{E}^A$ which preserves the tractor metric $h_{A\bar{B}}$ [4, 17].

The tractor curvature κ is a 2-form valued in (trace free skew-Hermitian) endomorphisms of the standard tractor bundle. Given a choice of contact form θ, κ may be decomposed into three components $\kappa_{1\bar{1}A}{}^B$, $\kappa_{10A}{}^B$, and $\kappa_{\bar{1}0A}{}^B$, defined by

$$\begin{aligned} \nabla_1\nabla_{\bar{1}}v^B - \nabla_{\bar{1}}\nabla_1 v^B + i\boldsymbol{h}_{1\bar{1}}\nabla_0 v^B &= \kappa_{1\bar{1}A}{}^B v^A; \\ \nabla_1\nabla_0 v^B - \nabla_0\nabla_1 v^B - A^{\bar{1}}{}_1\nabla_{\bar{1}}v^B &= \kappa_{10A}{}^B v^A; \\ \nabla_{\bar{1}}\nabla_0 v^B - \nabla_0\nabla_{\bar{1}}v^B - A^1{}_{\bar{1}}\nabla_1 v^B &= \kappa_{\bar{1}0A}{}^B v^A \end{aligned}$$

for any section v^A of $\mathcal{E}^A$ (the tractor connection is coupled with the Tanaka-Webster connection of θ in order to define the iterated covariant derivatives). By definition the component $\kappa_{1\bar{1}A}{}^B$ of the tractor curvature is a CR invariant, i.e. it does not depend on the choice of θ. However, a straightforward calculation shows that $\kappa_{1\bar{1}A}{}^B = 0$. The vanishing of $\kappa_{1\bar{1}A}{}^B$ implies that $\kappa_{10A}{}^B$ and $\kappa_{\bar{1}0A}{}^B$ are CR invariant (this phenomenon is special to 3-dimensional CR structures, cf. [10]). A straightforward calculation using the above formulae for the tractor connection, the formulae (2.8) and (2.9) for the curvature of the density line bundles, and the definitions of T_1 and S, gives (cf. [17])

$$\kappa_{10A}{}^B v^A \overset{\theta}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y_1 & iQ_{11} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^1 \\ \rho \end{pmatrix} \tag{4.8}$$

where Q_{11} is given by (2.11), and (cf. [11])

$$Y_1 = -i\nabla^1 Q_{11}. \tag{4.9}$$

The CR invariance of Q_{11} then follows immediately from the CR invariance of $\kappa_{10A}{}^B$ and the transformation law (4.1). Since the tractor connection preserves the tractor metric we have $\kappa_{\bar{1}0A}{}^B = -h_{A\bar{D}}h^{B\bar{C}}\overline{\kappa_{10C}{}^D}$, giving

$$\kappa_{\bar{1}0A}{}^B v^A \overset{\theta}{=} \begin{pmatrix} 0 & 0 & 0 \\ iQ_{\bar{1}}{}^1 & 0 & 0 \\ -Y_{\bar{1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^1 \\ \rho \end{pmatrix} \tag{4.10}$$

where $Y_{\bar{1}} = \overline{Y_1}$.

The tractor curvature κ satisfies the Bianchi identity, $d^\nabla\kappa = 0$, which can be written in terms of the components as $\nabla_1 \kappa_{\bar{1}0A}{}^B - \nabla_{\bar{1}} \kappa_{10A}{}^B = 0$, i.e.

$$\nabla^1 \kappa_{10A}{}^B = \nabla^{\bar{1}} \kappa_{\bar{1}0A}{}^B.$$

Fixing a background contact form θ, a direct calculation using (4.9) shows that

$$\nabla^1 \kappa_{10A}{}^B \overset{\theta}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i(\nabla^1\nabla^1 Q_{11} - iA^{11}Q_{11}) & 0 & 0 \end{pmatrix}.$$

By (2.13) we therefore obtain:

Lemma 4.1. [11, Lemma 7.1] *Let (M, H, J) be a CR 3-manifold. Then the CR obstruction density $\mathcal{O}$ vanishes if and only if $\nabla^1 \kappa_{10A}{}^B = 0$ (equivalently $\nabla^{\bar{1}} \kappa_{\bar{1}0A}{}^B = 0$).*

4.2. **The complexified adjoint tractor bundle.** The *adjoint tractor bundle* $\mathcal{A}$ is the bundle of trace free skew-Hermitian endomorphisms of $\mathcal{T}$. The *complexified adjoint tractor bundle* is therefore the bundle of trace free endomorphisms of $\mathcal{T}$. A section s of the complexified adjoint tractor bundle may be written with respect to a choice of contact form θ as

$$s_A{}^B \overset{\theta}{=} \begin{pmatrix} \mu & \upsilon_1 & iu \\ \nu^1 & \chi - \mu & -\xi^1 \\ i\lambda & -\eta_1 & -\chi \end{pmatrix}.$$

The section $s_A{}^B$ defines a real adjoint tractor if and only if u and λ are real and $\xi_{\bar{1}}$, χ, $\eta^{\bar{1}}$ equal $\overline{\upsilon_1}$, $\overline{\mu}$, $\overline{\nu^1}$ respectively. The component $iu = s_A{}^B \boldsymbol{Z}^A \boldsymbol{Z}_B$ does not depend on the choice of θ. We observe for later use that the canonical weighted tractor $\boldsymbol{Z}_A \boldsymbol{Z}^B$ may be expressed in matrix form (noting the form of the tractor metric (4.2)) as

$$\boldsymbol{Z}_A \boldsymbol{Z}^B \overset{\theta}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.11}$$

for any contact form θ.

4.3. **Solving a prolonged system of ∂_b-equations.** Let (M, H, J) be a CR 3-manifold and suppose u is a real weight $(1,1)$ density solving the infinitesimal automorphism equation $\nabla_1\nabla_1 u + iA_{11}u = 0$. Then there exists a section $s_A{}^B$ of the adjoint tractor bundle with $s_A{}^B \boldsymbol{Z}^A \boldsymbol{Z}_B = iu$ satisfying $\nabla_1 s_A{}^B = u\kappa_{10A}{}^B$ and $\nabla_{\bar{1}} s_A{}^B = u\kappa_{\bar{1}0A}{}^B$ [3, 11]. The following gives a holomorphic generalization of this result.

Proposition 4.2. *Suppose u is a complex solution of the CR invariant equation $\nabla_1\nabla_1 u + iA_{11}u = 0$ on weight $(1,1)$ densities. Then there exist trace free sections $s_A{}^B$ of $\mathcal{E}_A{}^B$ with $s_A{}^B \boldsymbol{Z}^A \boldsymbol{Z}_B = iu$ satisfying $\nabla_1 s_A{}^B = u\kappa_{10A}{}^B \mod \boldsymbol{Z}_A \boldsymbol{Z}^B$.*

Remark 4.3. The sections $s_A{}^B$ given by this proposition depend on one free parameter, namely a section ν^1 of $\mathcal{E}^1(-1,-1)$. We expect that for a specific choice of ν^1 the section $s_A{}^B$ constructed in the proof indeed satisfies $\nabla_1 s_A{}^B = u\kappa_{10A}{}^B$ without the need to mod out by $\boldsymbol{Z}_A \boldsymbol{Z}^B$. This would require a substantial calculation however, and the precise equality is not needed for our main results.

Proof. We shall mimic what happens in the case where u is real. We start by setting ${s_A}^B \boldsymbol{Z}^A \boldsymbol{Z}_B = iu$, so that

$$ {s_A}^B \stackrel{\theta}{=} \begin{pmatrix} \mu & \upsilon_1 & iu \\ \nu^1 & \chi-\mu & -\xi^1 \\ i\lambda & -\eta_1 & -\chi \end{pmatrix}. $$

If ${s_A}^B$ is given as above then, with respect to θ, $\nabla_1 {s_A}^B$ equals

$$ \begin{pmatrix} \nabla_1\mu - \frac{1}{4}R\upsilon_1 + iuT_1 & \nabla_1\upsilon_1 - A_{11}u & i\nabla_1 u - \upsilon_1 \\ \nabla_1\nu^1 + i\lambda + \frac{1}{4}R(2\mu-\chi) - \xi^1 T_1 & * & -\nabla_1\xi^1 - (2\chi-\mu) + \frac{i}{4}Ru \\ i\nabla_1\lambda - iA_{11}\nu^1 + \frac{1}{4}R\eta_1 - (\mu+\chi)T_1 & -\nabla_1\eta_1 - i(2\chi-\mu)A_{11} - \upsilon_1 T_1 & -\nabla_1\chi + \eta_1 + iA_{11}\xi^1 - iuT_1 \end{pmatrix} $$

where the entry marked $*$ is determined by the trace-freeness. In order for the top right entry $i\nabla_1 u - \upsilon_1$ to vanish we must choose $\upsilon_1 = i\nabla_1 u$. The fact that u solves the holomorphic infinitesimal automorphism equation $\nabla_1\nabla_1 u + iA_{11}u = 0$ then gives us that $\nabla_1\upsilon_1 - A_{11}u = 0$. As in the case where u is real [11] we define μ by $\mu = \frac{1}{3}(\nabla_0 u - \nabla^1\upsilon_1 - \frac{i}{4}uR)$. A straightforward calculation, which is formally the same as in the case where u is real, gives that $\nabla_1\mu - \frac{1}{4}R\upsilon_1 + iuT_1 = 0$. We fix χ and η_1 by setting $-\nabla_1\xi^1 - (2\chi-\mu) + \frac{i}{4}Ru = 0$ and $-\nabla_1\chi + \eta_1 + iA_{11}\xi^1 - iuT_1 = 0$. Using that $2\chi - \mu = -\nabla_1\xi^1 + \frac{i}{4}Ru$ and $\eta_1 = \nabla_1\chi - iA_{11}\xi^1 + iuT_1$, where ξ^1 is as yet undetermined, the bottom middle entry becomes

$$ \begin{aligned} \frac{1}{2}\nabla_1^3\xi^1 + 2iA_{11}\nabla_1\xi^1 + i\xi^1\nabla_1 A_{11} - \frac{i}{8}(u\nabla_1^2 R + 2\nabla_1 R\,\nabla_1 u + R\,\nabla_1^2 u) \\ - \frac{1}{6}\left(\nabla_1^2\nabla_0 u - i\nabla_1^2\nabla^1\nabla_1 u - \frac{i}{4}(u\nabla_1^2 R + 2\nabla_1 R\,\nabla_1 u + R\,\nabla_1^2 u)\right) \\ - \frac{i}{12}u(\nabla_1^2 R - 4i\nabla_1\nabla^1 A_{11}) + \frac{1}{4}A_{11}Ru - \frac{i}{6}\nabla_1 u(\nabla_1 R - 4i\nabla^1 A_{11}) \end{aligned} \tag{4.12} $$

where ∇_1^k means ∇_1 to the kth power. We need to show that ξ^1 can be chosen such that (4.12) is equal to iuQ_{11}. We first simplify the part of this expression which does not involve ξ^1. The idea is to commute derivatives in the expression $\nabla_1^2\nabla_0 u - i\nabla_1^2\nabla^1\nabla_1 u$ to obtain terms involving $\nabla_1^2 u$, and use that $\nabla_1^2 u = -iA_{11}u$. Using the commutation formulae (2.4), (2.5), (2.8) and (2.9) we have

$$ \nabla_1^2\nabla^1\nabla_1 u = \nabla_1\nabla^1\nabla_1^2 u - i\nabla_0\nabla_1^2 u - iA_{11}\nabla^1\nabla_1 u + i\nabla^1 A_{11}\nabla_1 u - \nabla_1 R\nabla_1 u - R\,\nabla_1^2 u, $$

and

$$ \nabla_1^2\nabla_0 u = \nabla_0\nabla_1^2 u + (\nabla_1\nabla^1 u + \nabla^1\nabla_1 u)A_{11} + \nabla^1 u\nabla_1 A_{11} - \nabla_1 u\nabla^1 A_{11}. $$

Thus,

$$ \nabla_1^2\nabla_0 u - i\nabla_1^2\nabla^1\nabla_1 u = -i\nabla_1\nabla^1\nabla_1^2 u + A_{11}\nabla_1\nabla^1 u + \nabla^1 u\nabla_1 A_{11} + iR\,\nabla_1^2 u + i\nabla_1 R\,\nabla_1 u. $$

Substituting $\nabla_1^2 u = -iA_{11}u$, we obtain

$$ \nabla_1^2\nabla_0 u - i\nabla_1^2\nabla^1\nabla_1 u = -\nabla_1 u\nabla^1 A_{11} - u\nabla_1\nabla^1 A_{11} + R\,A_{11}u + i\nabla_1 R\,\nabla_1 u. \tag{4.13} $$

Substituting (4.13) and $\nabla_1^2 u = -iA_{11}u$, (4.12) becomes

$$ \frac{1}{2}\nabla_1^3\xi^1 + 2iA_{11}\nabla_1\xi^1 + i\xi^1\nabla_1 A_{11} - \frac{i}{6}u\nabla_1^2 R - \frac{i}{2}\nabla_1 R\nabla_1 u - \frac{1}{2}\nabla_1 u\nabla^1 A_{11} - \frac{1}{6}u\nabla_1\nabla^1 A_{11}. \tag{4.14} $$

Again using the case where u is real as a guide, we set $\xi^1 = -i\nabla^1 u$. Let $L\xi^1$ denote $\frac{1}{2}\nabla_1^3\xi^1 + 2iA_{11}\nabla_1\xi^1 + i\xi^1\nabla_1 A_{11}$. Then

$$L\xi^1 = \frac{1}{2}\nabla_1^3\xi^1 + 2iA_{11}\nabla_1\xi^1 + i\xi^1\nabla_1 A_{11} = -\frac{i}{2}\nabla_1^3\nabla^1 u + 2A_{11}\nabla_1\nabla^1 u + \nabla^1 u\nabla_1 A_{11}.$$

We also have the commutation formula

$$\begin{aligned}\nabla_1^3\nabla^1 u = \nabla_1\nabla^1\nabla_1^2 u - 2i\nabla_0\nabla_1^2 u - iA_{11}\nabla_1\nabla^1 u \\ - 2iA_{11}\nabla^1\nabla_1 u - R\,\nabla_1^2 u - \nabla_1 R\,\nabla_1 u + 2i\nabla_1 u\nabla^1 A_{11} - i\nabla^1 u\nabla_1 A_{11},\end{aligned}$$

which yields

$$\begin{aligned}L\xi^1 = -\frac{i}{2}\nabla_1\nabla^1\nabla_1^2 u - \nabla_0\nabla_1^2 u - A_{11}\nabla^1\nabla_1 u + \frac{i}{2}R\,\nabla_1^2 u \\ + \frac{i}{2}\nabla_1 R\,\nabla_1 u + \nabla_1 u\nabla^1 A_{11} + \frac{3}{2}A_{11}\nabla_1\nabla^1 u + \frac{1}{2}\nabla^1 u\nabla_1 A_{11}.\end{aligned}$$

Substituting $\nabla_1^2 u = -iA_{11}u$, we obtain

$$\begin{aligned}L\xi^1 = -\frac{1}{2}\nabla_1\nabla^1(A_{11}u) + i\nabla_0(A_{11}u) - A_{11}\nabla^1\nabla_1 u + \frac{1}{2}R\,A_{11}u \\ + \frac{i}{2}\nabla_1 R\,\nabla_1 u + \nabla_1 u\nabla^1 A_{11} + \frac{3}{2}A_{11}\nabla_1\nabla^1 u + \frac{1}{2}\nabla^1 u\nabla_1 A_{11},\end{aligned}$$

so

$$\begin{aligned}L\xi^1 = -\frac{1}{2}u\nabla_1\nabla^1 A_{11} + iu\nabla_0 A_{11} + iA_{11}\nabla_0 u \\ + A_{11}\nabla_1\nabla^1 u - A_{11}\nabla^1\nabla_1 u + \frac{1}{2}R\,A_{11}u + \frac{i}{2}\nabla_1 R\,\nabla_1 u + \frac{1}{2}\nabla_1 u\nabla^1 A_{11}.\end{aligned}$$

Since $\nabla_1\nabla^1 u - \nabla^1\nabla_1 u = -i\nabla_0 u$, we obtain

$$L\xi^1 = -\frac{1}{2}u\nabla_1\nabla^1 A_{11} + iu\nabla_0 A_{11} + \frac{1}{2}R\,A_{11}u + \frac{i}{2}\nabla_1 R\,\nabla_1 u + \frac{1}{2}\nabla_1 u\nabla^1 A_{11}. \tag{4.15}$$

Substituting (4.15) into (4.14) we obtain

$$u\left(-\frac{i}{6}\nabla_1^2 R + \frac{1}{2}A_{11}R + i\nabla_0 A_{11} - \frac{2}{3}\nabla_1\nabla^1 A_{11}\right), \tag{4.16}$$

which equals iuQ_{11}. We now choose λ so that $\nabla_1\nu^1 + i\lambda + \frac{1}{4}R(2\mu-\chi) - \xi^1 T_1 = 0$, where ν_1 is arbitrary. Recalling the matrix expression for $\boldsymbol{Z}_A\boldsymbol{Z}^B$ given in (4.11), the result follows. □

Remark 4.4. As a side remark, we note that in the embedded case one can find actual CR holomorphic sections of the complexified adjoint tractor bundle. Recall that a contact form θ on a CR 3-manifold (M, H, J) is pseudo-Einstein [15, 6] if it is locally volume normalized by a closed $(2,0)$-form. If θ is volume normalized with respect to a global closed $(2,0)$-form which admits a global cube root σ, then by [5, Proposition 4.6] there exists a corresponding global CR holomorphic section of the cotractor bundle I_A with $I_A\boldsymbol{Z}^A = \sigma$. If (M, H, J) is compact and embedded in $\mathbb{C}^2$, then any pseudo-Einstein structure gives rise to such a section I_A. In this case, one can easily see that there exist 3 pointwise linearly independent CR holomorphic cotractors I_A^1, I_A^2, I_A^3 (indeed, it suffices to take the cotractors corresponding to σ, $e^{f_1}\sigma$, $e^{f_2}\sigma$ where f_1 and f_2 are suitable linear functions on $\mathbb{C}^2$). One therefore obtains a dual basis of CR holomorphic tractors J_1^A, J_2^A, J_3^A. Combining these one obtains global CR holomorphic sections $I_A^j J_k^B$ of $\mathcal{E}_A{}^B = \mathrm{End}(\mathcal{T})$ and corresponding CR anti-holomorphic sections $\hat{s}^{(j,k)}{}_A{}^B = J_A^j I_k^B$, where $j, k = 1, 2, 3$.

5. Proofs of main results

Proof of Theorem 1.7. We imitate the proof of Theorem 1.3 of [11], with Proposition 4.2 replacing the role of Lemma 7.3 in [11]. For the reader's convenience we include the details here. Fix a contact form θ and let ∇ denote the Tanaka-Webster connection of θ coupled with the CR tractor connection. By Lemma 4.1, since $\mathcal{O} = 0$ we have $\nabla^{\bar{1}}\kappa_{\bar{1}0A}{}^B = 0$. If u is a weight $(1,1)$ complex solution of $\nabla_1\nabla_1 u + iA_{11}u = 0$, then by Proposition 4.2 there exists a trace free section $s_A{}^B$ of $\mathcal{E}_A{}^B$ with $s_A{}^B \boldsymbol{Z}^A \boldsymbol{Z}_B = iu$ satisfying $\nabla_1 s_A{}^B = u\kappa_{10A}{}^B + r_1 \boldsymbol{Z}_A \boldsymbol{Z}^B$ where $r_1 \in \Gamma(\mathcal{E}_1(-1,-1))$. Note that $s_A{}^C s_B{}^A \nabla^{\bar{1}}\kappa_{\bar{1}0C}{}^B = s_A{}^C (\nabla^{\bar{1}}\kappa_{\bar{1}0B}{}^A) s_C{}^B$ is a density of weight $(-2,-2)$ so can be invariantly integrated. Integrating by parts we obtain

$$
\begin{aligned}
0 = \int s_A{}^C (\nabla^{\bar{1}}\kappa_{\bar{1}0B}{}^A) s_C{}^B &= -\int \left((\nabla^{\bar{1}} s_A{}^C)\, \kappa_{\bar{1}0B}{}^A\, s_C{}^B + s_A{}^C\, \kappa_{\bar{1}0B}{}^A\, \nabla^{\bar{1}} s_C{}^B \right) \\
&= -\int u h^{1\bar{1}} \left(\kappa_{10A}{}^C\, \kappa_{\bar{1}0B}{}^A\, s_C{}^B + s_A{}^C\, \kappa_{\bar{1}0B}{}^A\, \kappa_{10C}{}^B \right)
\end{aligned} \tag{5.1}
$$

where we used that $\boldsymbol{Z}_A \kappa_{\bar{1}0B}{}^A = 0$ and $\kappa_{\bar{1}0B}{}^A \boldsymbol{Z}^B = 0$. Now, as in the proof of Theorem 1.3 in [11], by (4.8) and (4.10) we have that $h^{1\bar{1}}\kappa_{\bar{1}0B}{}^A\, \kappa_{10C}{}^B = 0$ and

$$
h^{1\bar{1}}\kappa_{10A}{}^C\, \kappa_{\bar{1}0B}{}^A \overset{\theta}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -|Q|^2 & 0 & 0 \end{pmatrix}
$$

where $|Q|^2 = Q_{11}Q^{\bar{1}\bar{1}}$. Hence (5.1) simplifies to

$$
\int u h^{1\bar{1}} \kappa_{10A}{}^C\, \kappa_{\bar{1}0B}{}^A\, s_C{}^B = 0
$$

and we have

$$
\begin{aligned}
h^{1\bar{1}}\kappa_{10A}{}^C\, \kappa_{\bar{1}0B}{}^A\, s_C{}^B &\overset{\theta}{=} \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -|Q|^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu & \upsilon_1 & iu \\ \nu^1 & \chi - \mu & -\xi^1 \\ i\lambda & -\eta_1 & -\chi \end{pmatrix} \\
&= \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\mu|Q|^2 & -\upsilon_1|Q|^2 & -iu|Q|^2 \end{pmatrix} = -iu|Q|^2.
\end{aligned}
$$

We conclude that $\int u^2|Q|^2 = 0$. □

Corollary 5.1. *Let (M, H, J) be a compact strictly pseudoconvex CR 3-manifold which is obstruction flat. For any two weight $(1,1)$ complex solutions u_1, u_2 of the infinitesimal automorphism equation and any anti-CR function f, we have*

$$
\int_M f u_1 u_2 |Q|^2 = 0.
$$

Proof. Note that fu_1 is also a complex solution of the infinitesimal automorphism equation, so without loss of generality we may assume $f = 1$. Since $u_1 + u_2$ is also a complex solution, the result follows from Theorem 1.7. □

Remark 5.2. Note that in the proof of Theorem 1.7 it is possible to replace the integrand ${s_A}^C\,(\nabla^{\bar{1}}{\kappa_{\bar{1}0B}}^A)\,{s_C}^B$ by ${s_A}^C\,(\nabla^{\bar{1}}{\kappa_{\bar{1}0B}}^A)\,{\hat{s}^{(j,k)}}{}_C{}^B$ where ${\hat{s}^{(j,k)}}{}_C{}^B$ is as in Remark 4.4. This leads directly to a result which turns out to be a special case of Corollary 5.1.

Proof of Theorem 1.6. Note that the condition $(\operatorname{Im} u)^2 < (\operatorname{Re} u)^2$ almost everywhere is equivalent to $\operatorname{Re} u^2 > 0$ almost everywhere. Hence, Theorem 1.7 implies $|Q|^2 = 0$ and therefore (M, H, J) is locally spherical. □

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