9. The Cauchy Integral Formula in a Disc

Warm up:

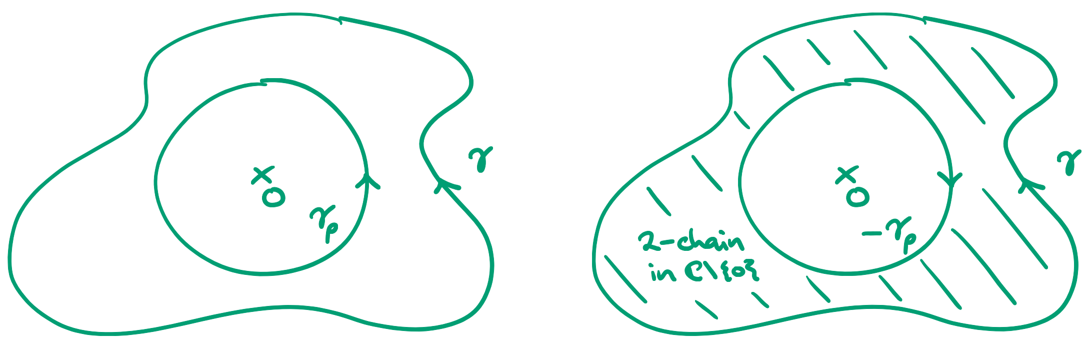
For $\rho > 0$ let $\gamma_\rho : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma_\rho(t) = \rho e^{it}$.

Recall that

$$\int_{\gamma_\rho} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\rho e^{it}} \cdot i\rho e^{it}\, dt = \int_0^{2\pi} i\, dt = \underline{\underline{2\pi i}}.$$

The answer is independent of ρ.

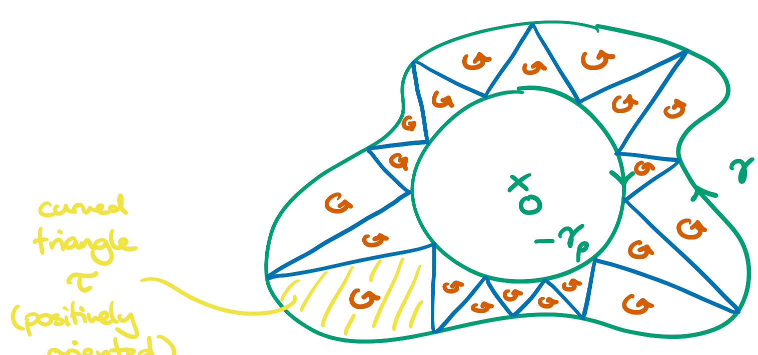
Q: Does it matter which path γ we take going once counterclockwise around 0 in $\mathbb{C}\setminus\{0\}$ when we compute $\int_\gamma \frac{1}{z}\, dz$?



A: No. We will still get $2\pi i$.

Hueristic Argument (we will give a proof using winding numbers later):

Break up into (curved) triangles and subdivide these if necessary* until one can apply Cauchy's theorem to each of the small (curved) subtriangles.



* No need to further subdivide in this picture.

By Cauchy's theorem for convex sets (the "local Cauchy theorem")
$\int_{\partial\tau} \frac{1}{z}\,dz = 0$ for any "small" curved triangle τ in $\mathbb{C}\setminus\{0\}$.
Here "small" just means small enough to be contained in a convex subset of $\mathbb{C}\setminus\{0\}$ (e.g. in a disc that doesn't contain 0).
Adding up the integrals over all of the curved subtriangles we get

$$\int_{\gamma-\gamma_\rho} \frac{1}{z}\,dz = \sum_{\substack{\text{small}\\\text{curved}\\\text{triangles}\\\tau_j}} \int_{\partial\tau_j} \frac{1}{z}\,dz = \sum 0 = 0.$$

local version of Cauchy's theorem

Hence

$$\int_{\gamma} \frac{1}{z}\,dz = \int_{\gamma_\rho} \frac{1}{z}\,dz.$$

Q: What if we ask the same question for some other function $g: \mathbb{C}\setminus\{0\} \to \mathbb{C}$ that is also holomorphic?

A: We still get $\int_{\gamma} g(z)\,dz = \int_{\gamma_\rho} g(z)\,dz$ by the same reasoning. In particular, we see that $\int_{\gamma_\rho} g(z)dz$ will be independent of ρ.

Warm up continued:

Let $f: \mathbb{C} \to \mathbb{C}$ be a holomorphic function and for each $\rho > 0$ let $\gamma_\rho : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma_\rho(t) = \rho e^{it}$.

Since (we claim) $\int_{\gamma_\rho} \frac{f(z)}{z}\,dz$ is independent of ρ, we can calculate it by taking the limit as $\rho \to 0^+$ (the limit of a constant function of ρ).

$$\lim_{\rho\to 0^+} \int_{\gamma_\rho} \frac{f(z)}{z}\,dz = \lim_{\rho\to 0^+} \int_0^{2\pi} \frac{f(\rho e^{it})}{\rho e^{it}} \cdot i\rho e^{it} dt = \lim_{\rho\to 0^+} i\int_0^{2\pi} f(\rho e^{it})\,dt.$$

Since $f(\rho e^{it}) \to f(0)$ uniformly in $t \in [0, 2\pi]$ as $\rho \to 0^+$ we may exchange the order of the limit and integral to get:

$$\lim_{\rho \to 0^+} \int_{\gamma_\rho} \frac{f(z)}{z}\, dz = i \int_0^{2\pi} f(0)\, dt = 2\pi i \cdot f(0).$$

Hence

$$f(0) = \lim_{\rho \to 0} \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{z}\, dz.$$

Note:

- By the above heuristic discussion we should be able to drop the limit in the above formula, i.e. we should have
$$f(0) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{z}\, dz$$
for any $\rho > 0$.

- Moreover, we should be able to replace γ_ρ by any closed contour γ in $\mathbb{C} \setminus \{0\}$ that "winds once around 0, counterclockwise" to get
$$f(0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z}\, dz.$$
→ This would mean that the value of f at 0 can be obtained from the values of f on any closed contour winding around 0!

- Such a result will clearly also hold if f is only assumed to be holomorphic in some disc $\Delta_r(0)$, provided $\rho < r$ and γ lies in $\Delta_r(0) \setminus \{0\}$.

It is not hard to see that the above discussion can be "translated" to give a way of computing the value of $f(z)$ at any point in z.

→ This leads to the Cauchy Integral Formula!

Exercise: Fix $z \in \mathbb{C}$, $R > 0$. Show that if $f : \Delta_R(z) \to \mathbb{C}$ is a continuous function then

$$f(z) = \lim_{\rho \to 0^+} \int_{\gamma_\rho} \frac{f(w)}{w-z}\,dw$$

where $\gamma_\rho : [0, 2\pi] \to \mathbb{C}$ is given by $\gamma_\rho(t) = z + \rho e^{it}$.

> **Note:**
> $$\int_{\gamma_\rho} \frac{f(w)}{w-z}\,dw = \int_0^{2\pi} \frac{f(z+\rho e^{it})}{z+\rho e^{it}-z} \cdot i\rho e^{it}\,dt$$
> $$= i\int_0^{2\pi} f(z+\rho e^{it})\,dt$$

Theorem 2.5 (The Cauchy Integral Formula for Discs):
Let $V \subseteq \mathbb{C}$ be open and $f : V \to \mathbb{C}$ holomorphic. Suppose $\overline{\Delta_r(z_0)} \subseteq V$. Then

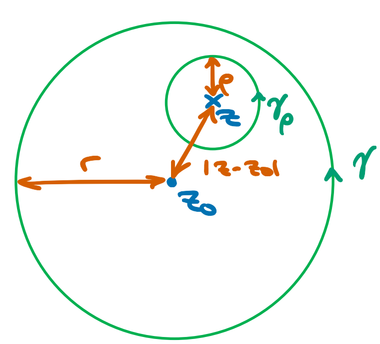
$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta_r(z_0)} \frac{f(w)}{w-z}\,dw \qquad \text{for any } z \in \Delta_r(z_0).$$

> Here $\partial\Delta_r(z_0)$ is the positively oriented boundary of $\Delta_r(z_0)$, which can be taken to be the contour $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + re^{it}$ (up to equivalence of chains).

Proof:

see picture

Fix a point $z \in \Delta_r(z_0)$. Let $d = \mathrm{dist}(z, \partial\Delta_r(z_0)) = r - |z - z_0|$ $(d > 0)$.
Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = z_0 + re^{it}$, and for each $\rho \in (0, d)$ let $\gamma_\rho : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma_\rho(t) = z + \rho e^{it}$.



Let $V' = V \setminus \{z\}$ and define $g : V' \to \mathbb{C}$ by

$$g(w) = \frac{f(w)}{w-z} \qquad \text{for } w \in V'.$$

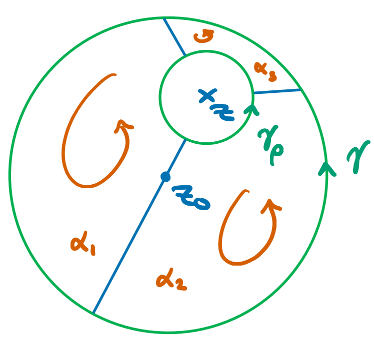
Claim: For any $\rho \in (0,d)$ we have
$$\int_\gamma g(w)\,dw = \int_{\gamma_\rho} g(w)\,dw$$
$$\left(\text{i.e. } \int_\gamma \frac{f(w)}{w-z}\,dw = \int_{\gamma_\rho} \frac{f(w)}{w-z}\,dw \right).$$

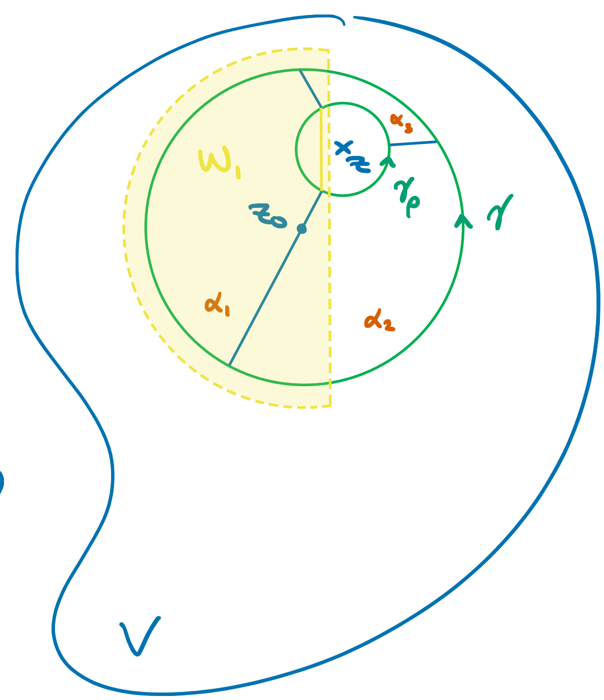
Later we will see this as simply an application of the homology version of Cauchy's theorem. →

Proof of claim:

Fix $\rho \in (0,d)$. By dividing the region between γ^* and γ_ρ^* up into 3 subregions (using 3 straight rays emanating from z which are at an angle of $\frac{2\pi}{3}$ from each other) and taking the oriented boundaries of these regions we may write the chain $\gamma - \gamma_\rho$ as a sum of closed contours
$$\gamma - \gamma_\rho = \alpha_1 + \alpha_2 + \alpha_3,$$
where the closed contours α_j $(j=1,2,3)$ each lie in a convex open subset W_j of V.

It is important that $\rho \in (0,d)$ is fixed (but arbitrary) at this step (as the sets W_j must depend upon ρ). →





Hence $\int_{\alpha_j} g(w)\,dw = 0$ $(j=1,2,3)$

and thus
$$\int_{\gamma-\gamma_\rho} g(w)\,dw = 0.$$

This proves the claim.

Hence for each $\rho \in (0, d)$ we have

Independent of ρ. $\rightarrow$

$$\int_\gamma \frac{f(w)}{w-z}\,dw = \int_\gamma g(w)\,dw$$

$$= \int_{\gamma_\rho} g(w)\,dw$$

$$= \int_{\gamma_\rho} \frac{f(w)}{z-w}\,dw$$

$$= \int_0^{2\pi} \frac{f(z+\rho e^{it})}{\cancel{z}+\rho e^{it}-\cancel{z}} \cdot i\rho e^{it}\,dt$$

$$= i\int_0^{2\pi} f(z+\rho e^{it})\,dt.$$

$\leftarrow$ Must be independent of ρ!

Since $f(z+\rho e^{it}) \to f(z)$ uniformly for $t \in [0, 2\pi]$ as $\rho \to 0^+$ we have

$$\int_\gamma \frac{f(w)}{w-z}\,dw = \lim_{\rho\to 0^+} \int_\gamma \frac{f(w)}{w-z}\,dw = \lim_{\rho \to 0^+} i\int_0^{2\pi} f(z+\rho e^{it})\,dt$$

$$= i\int_0^{2\pi} f(z)\,dt$$

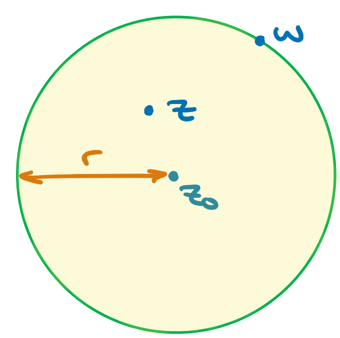
$$= \underline{2\pi i \cdot f(z)}.$$

This proves the result. $\square$

Note that the formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta_r(z_0)} \frac{f(w)}{w-z}\,dw$$

holds for <u>any</u> $z \in \Delta_r(z_0)$.



$\rightsquigarrow$ The value of the holomorphic function $f(z)$ for any $z \in \Delta_r(z_0)$ is determined by its values on the boundary $\partial\Delta_r(z_0)$ ($f(w)$ for $w \in \partial\Delta_r(z_0)$)!

This does not happen for real differentiable functions $f:[a,b]\to\mathbb{R}$ (unless they are linear) !

Note also that for each $w\in\partial\Delta_r(z_0)$ the integrand $\frac{f(w)}{w-z}$ is an analytic function of z (for $z\neq w$). In particular, all of these functions (integrands) are analytic on $\Delta_r(z_0)$. We now use this to show that $f(z)$ is represented by a power series in $\Delta_r(z_0)$.

__Theorem 2.6__

Let $V\subseteq\mathbb{C}$ be open and $f:V\to\mathbb{C}$ holomorphic. Suppose $\overline{\Delta_r(z_0)}\subseteq V$. Then there exists a sequence of complex numbers (c_n) such that

$$f(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n \quad \text{for every } z\in\Delta_r(z_0)$$

(and hence the series converges uniformly on compact subsets of $\Delta_r(z_0)$). Moreover, the coefficients are given by the formula

$$c_n=\frac{1}{2\pi i}\int_{\partial\Delta_r(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}\,dw. \qquad \left(=\frac{f^{(n)}(z_0)}{n!}\right)$$

__Proof__:

To simplify the notation in the proof we assume, without loss of generality, that $z_0=0$. By the Cauchy Integral Formula

$$f(z)=\frac{1}{2\pi i}\int_{\partial\Delta_r(0)}\frac{f(w)}{w-z}\,dw \quad \text{for } z\in\Delta_r(z_0)=\Delta_r(0).$$

For $z\in\Delta_r(0)$ and $w\in\Delta_r(0)$ we have $\left|\frac{z}{w}\right|=\frac{|z|}{r}<1$, so

$$\frac{1}{w-z}=\frac{1}{w}\cdot\frac{1}{1-\frac{z}{w}}=\frac{1}{w}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n.$$

Geometric Series.

Moreover, if $z\in\Delta_r(0)$ is fixed, then the series $\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n$ converges uniformly in $w\in\partial\Delta_r(0)$ (by direct comparison with $\sum_{n=0}^{\infty}\left(\frac{|z|}{r}\right)^n$).

This justifies interchanging the order of the sum and integral in the following: Take $z \in \Delta_r(0)$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Delta_r(0)} \frac{f(w)\,dw}{w - z}$$

$$= \frac{1}{2\pi i} \int_{\partial \Delta_r(0)} f(w) \cdot \left(\sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}} \right) dw$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial \Delta_r(0)} \frac{f(w)}{w^{n+1}} \, dw \right) z^n$$

$$= \sum_{n=0}^{\infty} c_n z^n$$

where the c_n are as given in the statement of the theorem. Since $z \in \Delta_r(0)$ was arbitrary, the result follows. $\square$

> Note: the function $\frac{1}{w-z}$ of $(w, z) \in \mathbb{C} \times \mathbb{C}$ with $w \neq z$ is known as the "Cauchy kernel" (the term "kernel" is being used in the sense that it is something you integrate against; in this case we are integrating our holomorphic function $f(w)$ against $\frac{1}{w-z}$ over $\partial \Delta_r(z_0)$ and obtaining $2\pi i f(z)$ for any $z \in \Delta_r(z_0)$). The above proof can be seen as saying that $f(z)$ "inherits" the property of analyticity from the kernel $\frac{1}{w-z}$ which is clearly analytic.

Corollary

Let $V \subseteq \mathbb{C}$ be open. A function $f : V \to \mathbb{C}$ is analytic if and only if it is holomorphic.

Corollary (Holomorphic functions are infinitely complex differentiable):
Let $V \subseteq \mathbb{C}$ be open and $f : V \to \mathbb{C}$ holomorphic.
Then f is infinitely complex differentiable.
In particular, f' is holomorphic.

This allows us to prove a very useful converse to Goursat's theorem:

Corollary 2.8 (Morera's Theorem):

Let $V \subseteq \mathbb{C}$ be open and $f: V \to \mathbb{C}$ continuous. If $\int_{\partial T} f(z)dz = 0$ for every triangle T in V, then f is holomorphic.

Proof:

It is enough to prove that f is holomorphic in every convex open subset of V (since these cover V). Take any open convex subset W of V. By Proposition 2.3 f has an antiderivative on W, i.e. there exists $F: W \to \mathbb{C}$ holomorphic such that $F' = f|_W$. But F is infinitely complex differentiable, so $f|_W = F'$ is holomorphic. $\square$

> Note that we could have proved the same result under the stronger assumption that "$\int_\gamma f(z)dz = 0$ for all closed contours γ in V" but this would obviously give us a weaker and less practically useful theorem.

As an easy consequence of Morera's theorem we have:

Corollary 3.0:

Let $V \subseteq \mathbb{C}$ be open and let $f_n: V \to \mathbb{C}$ be a sequence of holomorphic functions that converges to $f: V \to \mathbb{C}$ uniformly on compact subsets of V. Then f is holomorphic.

> **Note:**
>
> - This is a highly nontrivial & unexpected result!
> - For real differentiable functions on open sets this does not happen! E.g., $\sqrt{x^2 + \frac{1}{n^2}}$ is a sequence of differentiable (even C^∞, in fact, even C^ω) functions on $\mathbb{R}$ but $\sqrt{x^2+\frac{1}{n^2}} \xrightarrow{\text{unif.}} |x|$ on $\mathbb{R}$ as $n \to \infty$.

Proof of Corollary 3.0:

Let T be any triangle in V. By Goursat's theorem $\int_{\partial T} f_n(z)\,dz = 0$ for every n, and since $f_n \to f$ uniformly on ∂T (compact) we have $\int_{\partial T} f(z)\,dz = \lim_{n\to\infty} \int_{\partial T} f_n(z)\,dz = 0$. Since T was arbitrary we conclude by Morera's theorem that f is holomorphic. $\square$

Question: If $f_n \to f$ uniformly on compact sets (& f_n holo^m) does $f_n' \to f'$ uniformly on compact sets?

$\rightsquigarrow$ We will see that the answer is yes.

(Hence $f_n'' \to f''$ unif. on cpct sets, etc, by induction.)

Recall that if f is holomorphic on $\overline{\Delta_r(z_0)}$ then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \quad \text{for } z \in \Delta_r(z_0)$$

with

$$c_n = \frac{1}{2\pi i} \int_{\partial \Delta_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw = \frac{f^{(n)}(z_0)}{n!}.$$

To avoid notational confusion in what follows we restate this last formula as a lemma:

Lemma:

Let $V \subseteq \mathbb{C}$ be open and $f: V \to \mathbb{C}$ a holomorphic function. If $\overline{\Delta_\rho(z)} \subseteq V$ then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial \Delta_\rho(z)} \frac{f(w)}{(w-z)^{n+1}}\,dw.$$

From the lemma we can see that if holomorphic functions f_n converge uniformly to f on some set, then (for any k) the sequence $f_n^{(k)}$ will converge uniformly to $f^{(k)}$ on a slightly smaller set (slightly smaller so that there is a small disc $\Delta_\rho(z)$ around each point z in the smaller set contained in the original set.

↰ We'll come back to this shortly.

Corollary 3.1 (Cauchy Integral Formula for Derivatives):

Let $V \subseteq \mathbb{C}$ be open and $f: V \to \mathbb{C}$ holomorphic. Suppose $\overline{\Delta_r(z_0)} \subseteq V$.

Then, for any $n = 0, 1, 2, 3, \ldots$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial \Delta_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\, dw \qquad \text{for all } z \in \Delta_r(z_0).$$

Note: The $n=0$ case is just the usual Cauchy Integral Formula.

Proof:

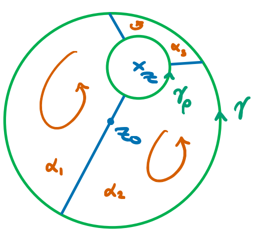
Fix $n \in \{0, 1, 2, 3, \ldots\}$. Fix $z \in \Delta_r(z_0)$.

Let $d = \text{dist}(z, \partial \Delta_r(z_0)) = r - |z - z_0|$ $(d > 0)$.

Take any $\rho \in (0, d)$ and define $\gamma_\rho : [0, 2\pi] \to V$ by

$\gamma_\rho(t) = z + \rho e^{it}$.

Arguing as in the proof of the Cauchy Integral Formula we have



$$\int_{\partial \Delta_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\, dw = \int_{\partial \Delta_\rho(z)} \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

The result then follows by the lemma stated above. $\square$

The advantage of the formula $f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial \Delta_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\, dw$ over the formula $f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial \Delta_\rho(z)} \frac{f(w)}{(w-z)^{n+1}}\, dw$ is that it (the former) holds for all $z \in \Delta_r(z_0)$ and the contour that one integrates over does not depend on z.

Exercise

Corollary 3.1 can also be proved by differentiating under the integral sign. Check the details of this. That is:

1. Apply $\frac{\partial}{\partial z}$ (n times) to both sides of the Cauchy Integral Formula and check that a form of the Leibniz integral rule lets you bring the $\frac{\partial}{\partial z}$ inside the integral.
2. Check that doing this gives you the Cauchy Integral Formula for Derivatives as stated above (easy!).

Note that we really only need $|f| \le M$ on $\partial\Delta_r(z_0)$.

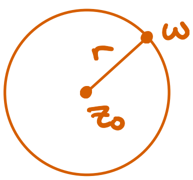
Corollary 3.2 (Cauchy's Estimates):

Let $V \subseteq \mathbb{C}$ be open and $f: V \to \mathbb{C}$ holomorphic.
If $\overline{\Delta_r(z_0)} \subseteq V$ and $|f| \le M$ in $\overline{\Delta_r(z_0)}$ then

$$|f^{(n)}(z_0)| \le \frac{M n!}{r^n} \qquad \text{for } n = 0, 1, 2, 3, \ldots.$$

Proof:

Just apply the "ML Inequality" to the Cauchy Integral Formula for Derivatives:



$|w - z_0| = r$

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_{\partial\Delta_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\, dw \right|$$

$$\le \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot \text{length}(\partial\Delta_r(z_0))$$

$$= \frac{M n!}{2\pi r^{n+1}} \cdot 2\pi r = \frac{M n!}{r^n}.$$

$\square$

As commented above, we can use the Cauchy Integral Formula for Derivatives (or Cauchy's Estimates) to get uniform convergence of the sequence $f_n^{(k)}$ (to $f^{(k)}$) provided we have uniform convergence of the sequence of holomorphic functions f_n (to f) on a slightly larger set.

Since we usually work with "uniform convergence on compact subsets" this poses no problem, as shown by the following two lemmas.

Lemma 3.3:

Let $V \subseteq \mathbb{C}$ be open and $K \subseteq V$ compact. Then

$$\text{dist}(K, \mathbb{C}\setminus V) \stackrel{\text{def}}{=} \inf\{|z-w| : z \in K, w \in \mathbb{C}\setminus V\} > 0.$$

| That is, $\exists \rho > 0$ s.t. $|z-w| > \rho$ for all $z \in K$ and $w \in \mathbb{C}\setminus V$.

Proof (by contradiction):

Suppose not. Then there exist sequences (z_n) in K and (w_n) in $\mathbb{C}\setminus V$ s.t.

$$|z_n - w_n| < \frac{1}{n} \qquad (n = 1, 2, 3, \dots).$$

Since K is compact, (z_n) has a convergent subsequence (z_{n_j}) with limit $z \in K$. Since $(z_n - w_n) \to 0$, however, we must also have $w_n \to z$. But $\mathbb{C}\setminus V$ is closed and hence $z = \lim\limits_{n\to\infty} w_n \in \mathbb{C}\setminus V$, contradicting that $z \in K \subseteq V$. We conclude that $\operatorname{dist}(K, \mathbb{C}\setminus V)$ must be positive. $\square$

Lemma 3.4:

Suppose $K \subseteq \mathbb{C}$ is compact and $\rho > 0$. Then

$$K' = \bigcup_{z \in K} \overline{\Delta_\rho(z)}$$

is also compact.

Proof:

We prove sequential compactness: Let (z_n) be a sequence in K'. For each n write $z_n = \alpha_n + \beta_n$ where $\alpha_n \in K$ and $\beta_n \in \overline{\Delta_\rho(0)}$. Since K is compact there exists a subsequence (α_{n_j}) of (α_n) that converges to a point $\alpha \in K$. Since $\overline{\Delta_\rho(0)}$ is compact there is a subsequence $(\beta_{n_{j_k}})$ of (β_{n_j}) that converges to $\beta \in \overline{\Delta_\rho(0)}$. Hence

$$z_{n_{j_k}} = \alpha_{n_{j_k}} + \beta_{n_{j_k}} \longrightarrow \alpha + \beta \in K'.$$

$\square$

Proposition 3.5

Let $V \subseteq \mathbb{C}$ be open and $f_n : V \to \mathbb{C}$ a sequence of holomorphic functions converging to $f : V \to \mathbb{C}$ uniformly on compact subsets of V. Then $f_n' \to f'$ uniformly on compact subsets of V.

! Hence, by induction, $f_n^{(k)} \to f^{(k)}$ uniformly on subsets of V $(k \in \mathbb{N})$.

Proof:

Fix any compact subset $K \subseteq V$. We show that $f_n' \to f'$ uniformly on K. Let $\varepsilon > 0$.

By Lemmas 3.3 and 3.4 $\exists \rho > 0$ s.t. $K' = \bigcup_{z \in K} \overline{\Delta_\rho(z)}$ is a compact subset of V. Hence $f_n \to f$ uniformly on K', so $\exists N$ s.t.

$$|f_n(z) - f(z)| < \underline{\underline{\varepsilon\rho}} \qquad \text{for all } z \in K', n > N.$$

leave this blank and then determine it at end

Therefore, if $n > N$ and $z \in K$ then $|f_n - f| < \underline{\underline{\varepsilon\rho}}$ on $\overline{\Delta_\rho(z)}$

leave blank until end

so by Cauchy's estimates (applied to estimate the first derivative of $f_n - f$):

choose $\varepsilon\rho$ here to end up with ε (and fill in blanks above)

$$|(f_n - f)'(z)| \leq \frac{\sup_{w \in \overline{\Delta_\rho(z)}} |(f_n - f)(w)|}{\rho} < \frac{\varepsilon\rho}{\rho} = \varepsilon$$

for all $z \in K$ and $n > N$.

That is, $f_n' \to f'$ uniformly on K. $\square$

Corollary:

Let $V \subseteq \mathbb{C}$ be open and $f_n : V \to \mathbb{C}$ a sequence of holomorphic functions converging to $f : V \to \mathbb{C}$ uniformly on compact subsets of V. Then, for any k, the sequence of $(k+1)$-tuples of functions $(f_n, f_n', \ldots, f_n^{(k)})$ converges to $(f, f', \ldots, f^{(k)})$ uniformly on compact subsets of V.

Note: The corollary is just saying that $f_n, \ldots, f_n^{(k)}$ can simultaneously be made close to $f, \ldots, f^{(k)}$ (on compact sets).

Precisely: for any $K \subseteq V$ compact and any $\varepsilon > 0$ $\exists N$ s.t.

$$|f_n^{(\ell)}(z) - f^{(\ell)}(z)| < \varepsilon \quad \text{for all } z \in K,\ n > N \text{ and } \ell \in \{0, 1, \ldots, k\}.$$

The upshot of all this is that we don't need to talk about C^k convergence of sequences of holomorphic functions, only C^0 convergence* (i.e. uniform convergence) and C^0 convergence on compact subsets.

$\rightsquigarrow$ Complex analysis is <u>less</u> technical than real analysis!

* C^0 convergence (uniform convergence) and uniform convergence on compact subsets are the natural notions of convergence for C^0 (i.e. continuous) functions. Surprisingly, it turns out that they are the natural notions of convergence for holomorphic functions too!