8 Path Independence and Antiderivatives

Question: When is a contour integral (from z_1 to z_2) independent of the path γ?

> Note that this question was historically well founded by the desire to justify the calculation of certain real integrals by deforming the path of integration into the complex plane.

Answer: When the integrand f has an antiderivative F.

Definition:

Let $V \subseteq \mathbb{C}$ be open and $f : V \to \mathbb{C}$. We say that $F : V \to \mathbb{C}$ is an antiderivative (or primitive) of f if F is a holomorphic function and $F' = f$.

Proposition (Fundamental Theorem of Calculus for Contour Integrals):

Let $V \subseteq \mathbb{C}$ be open, $f : V \to \mathbb{C}$, and suppose f has an antiderivative F on V. Then for any contour $\gamma : [a,b] \to V$ with $\gamma(a) = z_1$ and $\gamma(b) = z_2$ we have

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

Proof:

Given any contour $\gamma : [a,b] \to \mathbb{C}$ we compute, by the chain rule and the fundamental theorem of calculus,

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt = \int_a^b (F \circ \gamma)'(t)\,dt$$

$$= F(\gamma(b)) - F(\gamma(a)),$$

as required. $\square$

> Note that if f has an antiderivative and γ is closed then $\int_\gamma f(z)\,dz = 0$.

Exercise: Prove a converse to this result. That is, prove the following proposition:

Proposition

Let $V \subseteq \mathbb{C}$ be open and $f : V \to \mathbb{C}$ a continuous function. Suppose that the contour integrals of f do not depend on the path (but only the endpoints), equivalently, suppose

$$\int_\gamma f(z)\,dz = 0$$

for all closed contours γ in V. Then f admits an antiderivative F on V.

> **Hint:** Pick a point $z_0 \in V$ and define $F : V \to \mathbb{C}$ by
>
> $$F(z) = \int_{\gamma_{z_0,z}} f(w)\,dw$$
>
> where $\gamma_{z_0,z}$ is any contour going from z_0 to z in V. The proof that F is holomorphic and $F' = f$ can be done following the proof of Proposition 2.3 below.

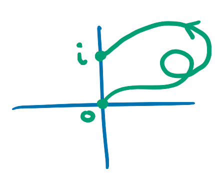
> Note that this result is an analog of the second part of the fundamental theorem of calculus in real variables.

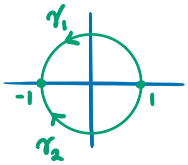
In short, for f to have an antiderivative it is necessary and sufficient for the contour integrals of f to be independent of path.

Examples

① Let $\gamma : [a,b] \to \mathbb{C}$ be any contour with $\gamma(a) = 0$, $\gamma(b) = i$.

Then $\displaystyle\int_\gamma z\,dz = \frac{z^2}{2}\Big|_0^i = \frac{i^2}{2} - \frac{0^2}{2} = \frac{-1}{2}$.





② Let $\gamma_1, \gamma_2 : [0,\pi] \to \mathbb{C}$ be given by $\gamma_1(t) = e^{it}$, $\gamma_2(t) = e^{-it}$. Then

$$\int_{\gamma_1} \bar{z}\,dz = \int_0^\pi \overline{e^{it}} \cdot ie^{it}\,dt = \int_0^\pi e^{-it} \cdot ie^{it}\,dt$$
$$= \int_0^\pi i\,dt = i\pi,$$

but

$$\int_{\gamma_2} \bar{z}\,dz = \int_0^\pi \overline{e^{-it}} \cdot (-ie^{-it})\,dt = -\int_0^\pi e^{it} \cdot ie^{-it}\,dt$$
$$= -\int_0^\pi i\,dt = -i\pi.$$

★ Local Antiderivatives of Holomorphic Functions

Question: What about holomorphic functions? Does every holomorphic function have an antiderivative? (Equivalently, are contour integrals of holomorphic functions always path independent?)

Answer: No, not every holomorphic function has an antiderivative, e.g., $\int_{\partial \mathbb{D}} \frac{1}{z}\,dz = 2\pi i$ so $\frac{1}{z}$ cannot have an antiderivative on any domain that contains the unit circle. But holomorphic functions do always have local antiderivatives.

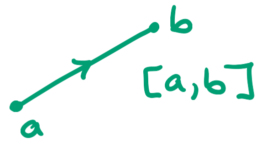
In fact, holomorphic functions always have antiderivatives on any simply connected open set (any open set without "holes"). This is stronger than the existence of local antiderivatives since any disc is simply connected.

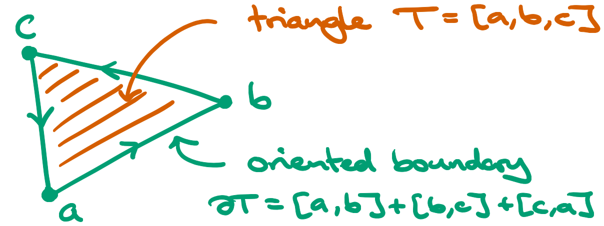
↳ For example, $\mathrm{Log}(z)$, the principal branch of the logarithm, is an antiderivative of $\frac{1}{z}$ on the cut plane $\mathbb{C}\setminus(-\infty,0]$ (a simply connected set).

It turns out that in order to prove the local existence of antiderivatives for holomorphic functions it is sufficient to prove that contour integrals are the same for two paths taken around a triangle  when the function is holomorphic inside and on the triangle.

Definition

If $a, b \in \mathbb{C}$ then $[a,b]$ denotes the oriented line segment from a to b; we also think of $[a,b]$ as the contour $\gamma : [0,1] \to \mathbb{C}$ given by $\gamma(t) = (1-t)a + tb$.





A (solid, oriented) triangle $T = [a,b,c]$ in $\mathbb{C}$ is the convex hull of its three vertices $a, b, c \in \mathbb{C}$ together with an orientation coming from the order of its vertices: $T = [a,b,c]$ is positively oriented if a, b, c are arranged counterclockwise around the (topological) boundary of T, otherwise T is negatively oriented.

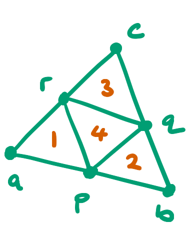
The (oriented) boundary of a triangle $T = [a,b,c]$ is the chain

$$\partial T = [a,b] + [b,c] + [c,a].$$

$$[a,b,c] = [b,c,a] = [c,a,b]$$

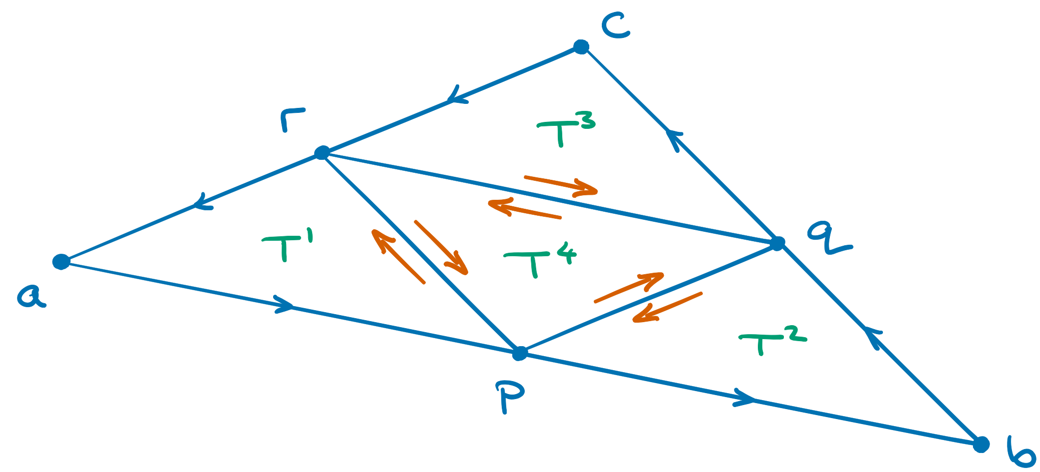
$$[b,a,c] = -[a,b,c] \quad \text{(reversed orientation)}$$

$$\partial[b,a,c] = -\partial[a,b,c] \quad (\text{—— '' ——})$$



Notation: Given $T = [a,b,c]$ we define T^1, T^2, T^3, T^4 by taking the midpoints $p = \frac{a+b}{2}$, $q = \frac{b+c}{2}$, $r = \frac{c+a}{2}$ and setting

$$T^1 = [a,p,r], \quad T^2 = [b,q,p], \quad T^3 = [c,r,q], \quad T^4 = [p,q,r].$$



Note: As chains $\partial T = \partial T^1 + \partial T^2 + \partial T^3 + \partial T^4$.

The following result of Goursat was a major breakthrough when it was first established, allowing Goursat to put the "Cauchy-Riemann theory" of complex analysis on a firm foundation.

Theorem 2.2 (Goursat's Theorem: Cauchy's Theorem for Triangles):
Let $V \subseteq \mathbb{C}$ be open and $f: V \to \mathbb{C}$ a holomorphic function. If T is a (solid, oriented) triangle in V then

$$\int_{\partial T} f(z)\,dz = 0.$$

Why is this result so significant?

- Cauchy only proved "Cauchy's theorem" and the "Cauchy Integral Formula" under the additional assumption that the holomorphic function f was C^1 (this can easily be done, e.g., using Green's theorem and the Cauchy-Riemann equations, as is standard in undergraduate complex variables); holomorphic functions are in fact always C^1 (indeed, they are infinitely diff^{ble}) as can be seen from the Cauchy Integral Formula, but Cauchy could not prove this since he assumed f was C^1 in the proof of his integral formula (so his argument would be circular).
- Goursat resolved this problem. We will show that:

Goursat's Theorem $\Rightarrow$ Holomorphic Functions have Local Antiderivatives $\Rightarrow$ Local Version of Cauchy's Theorem $\Rightarrow$ Cauchy Integral Formula in a Disc $\Rightarrow$ Holomorphic Functions are Analytic

What is so clever about Goursat's proof is that it only applies the complex differentiability of f at one (carefully chosen) point, thus avoiding any need to be concerned with the continuity of the derivative. Note that we do not know where the special point will be (and after we have proved the result we can see that it could have been any point in T) so we really do need that f is complex differentiable everywhere.

Proof of Goursat's Theorem:

Let T be a triangle in V. Subdividing T into T^1, T^2, T^3, T^4 as above, we have $\int_{\partial T} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial T^j} f(z)\,dz$. Choose k such that $\left|\int_{\partial T^k} f(z)\,dz\right| = \max_{1\le j\le 4} \left|\int_{\partial T^j} f(z)\,dz\right|$ and define T_1 to be T^k.

Note that after the theorem has been proved we will know that $\int_{\partial T^j} f(z)\,dz = 0$ for all $j = 1,2,3,4$. So the T^k we picked here could have been any of them. Right now though, all we know is that we can pick a k such that $\left|\int_{\partial T^k} f(z)\,dz\right|$ is the largest possible, and that is what we do.

By the choice of T_1 we have

$$\left|\int_{\partial T} f(z)\,dz\right| = \left|\sum_{j=1}^{4} \int_{\partial T^j} f(z)\,dz\right|$$

$$\le \sum_{j=1}^{4} \left|\int_{\partial T^j} f(z)\,dz\right| \le 4\left|\int_{\partial T_1} f(z)\,dz\right|.$$

We now repeat this argument with T_1 in place of T: choose k such that $\left|\int_{\partial T_1^k} f(z)\,dz\right| = \max_{1\le j\le 4}\left|\int_{\partial T_1^j} f(z)\,dz\right|$ and set $T_2 = T_1^k$. Then

$$\left|\int_{\partial T_1} f(z)\,dz\right| \le 4\left|\int_{\partial T_2} f(z)\,dz\right|$$

and hence

$$\left|\int_{\partial T} f(z)\,dz\right| \le 4^2\left|\int_{\partial T_2} f(z)\,dz\right|.$$

We continue this process inductively to define a sequence of triangles T_n with $T_{n+1} \in \{T_n^1, T_n^2, T_n^3, T_n^4\}$ and

(*) $$\left| \int_{\partial T} f(z)\,dz \right| \leq 4^n \left| \int_{\partial T_n} f(z)\,dz \right|$$

for all n.

- Note that if we can show that $\left|\int_{\partial T_n} f(z)\,dz\right|$ goes to zero faster than 4^{-n} we will be done!
- Note that 4^n is the number of triangles we would have if we subdivided T into 4 triangles n times.
- Note that we have constructed the sequence T_n such that $T_{n+1} \subseteq T_n$ as sets.

Writing $\operatorname{diam}(S) = \sup\{|z_1 - z_2| : z_1, z_2 \in S\}$ for $S \subseteq \mathbb{C}$ we have

$$\operatorname{diam}(T_n) = 2^{-n} \operatorname{diam}(T).$$

Since the triangles T_n are closed and nested $(T \supseteq T_1 \supseteq T_2 \supseteq T_3 \dots)$ and $\operatorname{diam}(T_n) \to 0$ there is a unique point $z_0 \in V$ such that

$$\{z_0\} = \bigcap_{n=1}^{\infty} T_n.$$

To see this simply take a point $z_n \in T_n$ for each n and observe that (z_n) is Cauchy and hence converges to a point z_0 which must lie in $\bigcap_{n=1}^{\infty} T_n$ by the closedness of the T_n. It is then easy to show that $\bigcap_{n=1}^{\infty} T_n$ contains no other points (since $\operatorname{diam}(T_n) \to 0$).

E is holomorphic but we won't need this.

We now write $f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z)$, where the error term $E(z)$ is a continuous function on V satisfying $\lim_{z \to z_0} \frac{E(z)}{|z - z_0|} = 0$. Note that the linear polynomial $\alpha + \beta(z - z_0) = f(z_0) + f'(z_0)(z - z_0)$ has an antiderivative $\alpha(z - z_0) + \frac{\beta}{2}(z - z_0)^2$ and so

(**) $$\int_{\partial T_n} f(z)\,dz = \int_{\partial T_n} E(z)\,dz \quad \text{for each } n.$$

Aim: show that $4^n \int_{\partial T_n} E(z)\,dz \to 0$ as $n \to \infty$.

For each n let $M_n = \sup_{z \in \partial T_n} |E(z)|$; if c is the length of ∂T, then ∂T_n has length $2^{-n} c$ and hence

(***) $$\left| \int_{\partial T_n} E(z)\,dz \right| \le \underbrace{c\,2^{-n}}_{\text{Length}(\partial T_n)} M_n .$$

Since $\operatorname{diam}(T_n) = 2^{-n} \operatorname{diam}(T)$ and $\lim_{z \to z_0} \frac{E(z)}{|z - z_0|} = 0$ we have

(****) $$\lim_{n \to \infty} \frac{M_n}{2^{-n} \operatorname{diam}(T)} = 0. \qquad \left(\text{So } \lim_{n\to\infty} \frac{M_n}{2^{-n}} = \lim_{n \to \infty} 2^n M_n = 0. \right)$$

One way to see this is to take $z_n \in \partial T_n$ such that $|E(z_n)| = M_n$ (possible since E is continuous) and note that $|z_n - z_0| \le \operatorname{diam}(T_n) = 2^{-n} \operatorname{diam}(T)$ so $\frac{E(z)}{|z-z_0|} \to 0$ as $z \to z_0$ implies $\frac{|E(z_n)|}{|z_n - z_0|} \to 0$ as $n \to \infty$ and hence $\frac{M_n}{2^{-n}\operatorname{diam}(T)} \to 0$ $\left(\frac{1}{2^{-n}\operatorname{diam}(T)} \le \frac{1}{|z_n - z_0|} \right)$.

Hence, by (*), (**) and (***) we have (for any n)

$$\left| \int_{\partial T} f(z)\,dz \right| \le 4^n \left| \int_{\partial T_n} f(z)\,dz \right| = 4^n \left| \int_{\partial T_n} E(z)\,dz \right|$$
$$\le c\,2^n M_n ,$$

and by (****) we have $\frac{M_n}{2^{-n}} = 2^n M_n \to 0$ as $n \to \infty$; thus

$$\left| \int_{\partial T} f(z)\,dz \right| \le \lim_{n\to\infty} c\,2^n M_n = 0,$$

i.e.

$$\int_{\partial T} f(z)\,dz = 0.$$

□

What was the big idea?

Repeatedly subdivide and then use that f is well approximated by a complex linear polynomial at small scales.

Exercise

Explain in your own words how the above proof would fail if we only assumed, say, that f was real diff$^{\text{ble}}$ instead of complex diff$^{\text{ble}}$ on V.

We can now show that holomorphic functions admit local antiderivatives.

> In fact, we will show that holomorphic functions admit antiderivatives on any convex set. (Any disc is a convex set so this implies the local existence of antiderivatives.) Really, holomorphic functions always admit antiderivatives on any simply connected open set (these include the convex open sets). For now, we are replacing the topological condition (V simply connected) with a simple geometric condition (V convex).

Definition

A set $S \subseteq \mathbb{C}$ is convex if the line segment $[a,b]$ is contained in S whenever $a,b \in S$.

Proposition 2.3

Let $V \subseteq \mathbb{C}$ be a convex open set and suppose $f: V \to \mathbb{C}$ is a continuous function with the property that $\int_{\partial T} f(z)\,dz = 0$ for every triangle T in V. Then there exists a function $F: V \to \mathbb{C}$ such that $F' = f$.

Proof:

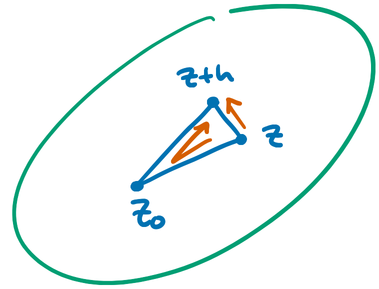
Fix any point $z_0 \in V$ and define $F: V \to \mathbb{C}$ by

$$F(z) = \int_{[z_0, z]} f(w)\,dw.$$

Now fix any $z \in V$ and take $r > 0$ s.t. $\Delta_r(z) \subseteq V$. Since V is convex, if $a,b,c \in V$ then the triangle $[a,b,c]$ lies in V. Hence for any $h \in \Delta_r(0)$ we have $\int_{\partial[z_0, z, z+h]} f(w)\,dw = 0$, i.e.

$$F(z+h) - F(z) = \int_{[z, z+h]} f(w)\,dw.$$

Hence for $h \in \Delta_r(0)\setminus\{0\}$



$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{[z,z+h]} f(w)\,dw.$$

We want to show that the left hand side of the above display tends to $f(z)$ as $h \to 0$. Since $\int_{[z,z+h]} f(z)\,dw = f(z)\cdot h$ this is equivalent to showing that

$$\lim_{h\to 0} \frac{1}{h}\int_{[z,z+h]} (f(w)-f(z))\,dw = 0.$$

We now show that

$$\lim_{h\to 0} \frac{1}{h}\int_{[z,z+h]} (f(w)-f(z))\,dw = 0.$$

Let $\varepsilon > 0$. Since f is continuous we may choose $\delta > 0$ such that $|f(w)-f(z)| < \varepsilon$ whenever $|z-w| < \delta$. Hence if $0 < |h| < \delta$ we have $|f(w)-f(z)| < \varepsilon$ for all $w \in [z,z+h]$ and thus

$$\left|\frac{1}{h}\int_{[z,z+h]} (f(w)-f(z))\,dw\right| \leq \frac{1}{|h|}\cdot\underbrace{\varepsilon\cdot|h|}_{\text{"ML"}} = \varepsilon.$$

This proves the result. $\square$

We therefore have the following two immediate corollaries:

Corollary (Holomorphic Functions Admit Local Antiderivatives):
Let $V \subseteq \mathbb{C}$ be open and $f: V \to \mathbb{C}$ a holomorphic function. If $U \subseteq V$ is an open convex subset (in particular, if $U \subseteq V$ is a disc) then f has an antiderivative on U.

Theorem 2.4 (Cauchy's Theorem for Convex Sets):
Let $V \subseteq \mathbb{C}$ be a convex open set and $f: V \to \mathbb{C}$ a holomorphic function. Then for any closed contour (or chain) γ in V we have

$$\int_\gamma f(z)\,dz = 0.$$

Proof of Theorem 2.4:

By Goursat's theorem $\int_{\partial T} f(z)\,dz = 0$ for any triangle T in V. Hence by Proposition 2.3 f has an antiderivative F on V. The conclusion then follows from the fundamental theorem of calculus for contour integrals. $\square$

Remark: Note that $\mathbb{C}\setminus\{0\}$ is not convex. Cauchy's theorem is not valid for $\mathbb{C}\setminus\{0\}$ as we have seen by computing

$$\int_{\partial \mathbb{D}} \frac{1}{z}\,dz = 2\pi i.$$

(This is actually a good thing, as the above formula leads us to the Cauchy Integral Formula.)

> Note that this is not the end of the story for Cauchy's theorem. Later we will prove:
>
> **Theorem** (Cauchy's Theorem for Simply Connected Sets): Let $V \subseteq \mathbb{C}$ be a simply connected open set and $f: V \to \mathbb{C}$ a holomorphic function. Then for any closed contour (or chain) γ in V we have
>
> $$\int_\gamma f(z)\,dz = 0.$$
>
> Here by a simply connected set we mean a set that "has no holes" (we will define this precisely later; often a simply connected set is also required to be connected, which makes sense given the name, but this does not matter for Cauchy's theorem.
>
> **Note:**
>
> - $\mathbb{C}\setminus\{0\}$ is not simply connected.
> - any convex set is simply connected (why?).

The topological condition "V is simply connected" is also a necessary condition for the conclusion of Cauchy's theorem to hold; later we will prove the following:

Theorem:

Let $V \subseteq \mathbb{C}$ be open. Suppose $\int_\gamma f(z)\,dz = 0$ for all holomorphic functions $f: V \to \mathbb{C}$ and all closed contours γ in V. Then V is simply connected.

Hence "Cauchy's theorem for simply connected sets" is in some sense a "full" version of Cauchy's theorem. We see, however, that one can prove versions of Cauchy's theorem for non simply connected sets so long as one does not allow all closed contours/chains γ (but only those which are oriented boundaries of "2-chains" in V, i.e. only those which are homologous to a constant path).

For now we are happy to have proved Cauchy's theorem for convex sets since this will already allow us to prove many strong results.